A Suzuki type fixed point theorems for a generalized hybrid maps on a partial Hausdorff metric spaces

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Abstract: In this paper, we obtain a Suzuki type fixed point theorems for generalized hybrid pairs of single-valued and multi-valued maps on partial Hausdorff metric space. Our results generalize, extend, unify several known results in existing literature.

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1. Introduction

In 1969, Nadle[1] proved a multi-valued version of Banach contraction principle in Hausdorff metric space.

Indeed, the fixed theorems for multi-valued maps are useful in control theory and have been used in solving many problems in economics and game theory. Hybrid contractive condition, that is, contractive condition involving single-valued and multi-valued maps are the further addition to metric fixed point theory and its applications[2,3,4,5,6,7] .

In 1994 Matthews[8] introduced the partial metric spaces as a part of the study of denotational semantics of data flow net works. Also the partial metric spaces play an important role in constructing models in the theory of computation[9,10,11,12,13,14].

In a partial metric space, the distance of a point in the self may not zero. Recently, Aydi et al.[15] initiated the concept of a partial Hausdorff metric and obtained an analogue of Nadler's fixed point theorem in partial metric space.

Our results extend, unify and complement a multi-valued of related fixed point theorem for metric space and extend them in partial Hausdorff metric space.

2. Preliminaries

We recall some definitions and notions of partial metric spaces and Hausdorff metric spaces.

Definition 2.1[8] : A partial metric on a non-empty set X is a function $p: X \times X \to \Re^+$ (where $\Re^+$ denotes the set of all non-negative real numbers) such that for all $x, y, z \in X$, the following conditions are satisfied:

(P1) $x = y$ if and only if
$$p(x,x) = p(y,y) = p(x,y)$$
(indistancy implies equality)

(P2) $p(x,y) = p(y,x)$ (symmetry)

(P3) $p(x,x) \le p(x,y)$ (small self-distances and non negativity)

(P4) $p(x,z) \le p(x,y) + p(y,z) - p(y,y)$ (triangularity)

A pair (X, p) is called partial metric space and p is a partial metric on X .For each partial metric p on X , the function $d_p : X \times X \to \Re^+$ defined by $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ is a (usual) metric on X .

Definition 2.2[8] : Let (X, p) be a partial metric space

(i) A sequence $(x_n)_{n\in N}$ in a partial metric space (X,p) converges to a point $x\in X$ if $\lim_{n\to\infty} p(x,x_n)=p(x,x)$.

(ii) A sequence $(x_n)_{n\in N}$ in a partial metric space (X,p) is called Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists (and is finite).

(iii) A partial metric space (X,p) is said to be complete if every Cauchy sequence $(x_n)_{n\in N}$ in X converges, with respect to $\tau(p)$, to a point $x\in X$ such that,

$$p(x,x)=\lim_{n,m\to\infty} p(x_n,x_m).$$

Lemma 2.3[8]: Let (X,p) be the partial metric space, then

(i) A sequence $\{x_n\}_{n\in N}$ in a partial metric space (X,p) is a Cauchy sequence if and only if it is a Cauchy in a metric space (X,d_p),

(ii) A partial metric space (X,p) is complete if and only if a metric space (X,d_p) is complete .Moreover,

$$\lim_{n\to\infty} d_p(x,y)=0 \Leftrightarrow$$

$$p(x,x)=\lim_{n\to\infty} p(x,x_n)=\lim_{n,m\to\infty} p(x_n,x_m)$$

(iii) A subset A of the partial metric space (X,p) is closed if whenever $\{x_n\}$ is a sequence in A such that $\{x_n\}$ converges to some $x\in X$,then $x\in A$.

Lemma 2.4[9]: Let (X,p) be a partial metric space, then

A. If $p(x,y)=0$, then $x=y$

B. If $x\neq y$, then $p(x,y)>0$.

Compatible with[15,16], Let (X,p) be the partial metric space (P.M.S) for short and let $CB^p(X)$ the family of all nonempty closed and bounded subsets of the partial metric space (X,p), induced by the partial metric p. Note that the Closedness is taken from (X,τ_p) (τ_p is the topology induced by p) and boundedness is given as follows: A is a bounded subset in (X,p) if there is $x_0\in X$ and $M\geq 0$ such that for all $a\in A$, we have $a\in B_p(x_0,M)$, that is,

$$p(x_0,a)<p(a,a)+M.$$

For $A,B\in CB^p(X)$ $x\in X$,

$\delta_p: CB^p(X)\times CB^p(X)\to\Re^+$ define

$$p(x,A)=\inf\{p(x,a);a\in A\},$$

$$p(A,B)=\inf\{p(x,y);x\in A,y\in B\}$$

$$\delta_p(A,B)=\sup\{p(a,B);a\in A\},$$

$$\delta_p(B,A)=\sup\{p(b,A);b\in B\},$$

$$H_p(A,B)=\max\{\delta_p(A,B),\delta_p(B,A)\},$$

$$d_p(x,A)=\inf\{d_p(x,a),a\in A\}.$$

Not that $p(x,A)=0\Rightarrow d_p(x,A)=0$.

Lemma 2.5[8]: Let (X,p) be a partial metric space and A be any non-empty subset of X, then $a\in\bar{A}$ if and only if $p(a,A)=p(a,a)$.

Proposition 2.6[15]: Let (X,p) be a partial metric space. For any $A,B,C\in CB^p(X)$, we have the following:

(i) $\delta_p(A,A)=\sup\{p(a,a),a\in A\}$;

(ii) $\delta_p(A,A)\leq\delta_p(A,B)$;

(iii) $\delta_p(A,B)=0$ implies that $A\subseteq B$;

(iv) $\delta_p(A,B)\leq\delta_p(A,C)+\delta_p(C,B)-\inf_{c\in C} p(c,c)$

Proposition 2.7[17]: Let (X,p) be a partial metric space. For all $A,B,C\in CB^p(X)$, we have

(h1) $H_p(A,A)\leq H_p(A,B)$;

(h2) $H_p(A,B)=H_p(B,A)$;
(h3) $H_p(A,B)\le H_p(A,C)+$
$H_p(C,B)-\inf_{c\in C} p(c,c)$;
(h4) $H_p(A,B)=0\Rightarrow A=B$.The mapping $H_p: CB^p(X)\times CB^p(X)\to\Re^+$ is called the partial Hausdorff metric induced by p. Every Hausdorff metric is a partial Hausdorff metric but the converse is not true[15].

Lemma 2.8[15]: Let (X,p) be a partial metric space. $A,B\in CB^p(X)$ and $h>1$, then for any $a\in A$, there exists $b(a)\in B$ such that $p(a,b)\le hH_p(A,B)$.

Let (X,p) be a partial metric space and $T: X\to CB^p(X)$ be a multi-valued mapping define:

$$M_p(x,y)=\max\{p(x,y),p(x,Tx), p(y,Ty),\frac{1}{2}(p(x,Ty)+p(y,Tx))\}$$

for all $x,y\in X$. (1)

Also let $\psi:[0,1)\to(0,1]$ be a non-increasing function defined by

$$\psi(r)=\begin{cases}1, & if\quad 0\le r<\frac{1}{2},\\ 1-r, & if\quad \frac{1}{2}\le r<1.\end{cases} \quad (2)$$

Theorem 2.9[17]: Let (X,p) be a complete partial metric space, $T: X\to CB^p(X)$ be a multi-valued mapping and $\psi:[0,1)\to(0,1]$ be the non-increasing function defined by (2), if there exists $r\in[0,1)$ such that T satisfies the condition

$$\psi(r)p(x,Tx)\le p(x,y)\Rightarrow H_p(Tx,Ty)\le rM_p(x,y)$$

for all $x,y\in X$, then, T has a fixed point in X, that is, there exists a point $z\in X$ such that $z\in Tz$.

Definition 2.10[16]: Let (X,p) be a partial metric space and $f: X\to X$ and $T: X\to CB^p(X)$ A point $x\in X$ is said to be

(i) a fixed point of f if $x=fx$,
(ii) a fixed point of T if $x\in Tx$,
(iii) a coincidence point of a pair (f,T) if $fx\in Tx$,
(iv) a common fixed point of the pair (f,T) if $x=fx\in Tx$.

we denote the set of coincidence point of the pair (f,T) by $C(f,T)$.
Motivated by the work[16].

Definition 2.11[16]: Let (X,p) be a partial metric space and $f: X\to X$ and $T: X\to CB^p(X)$. The pair (f,T) is called

(i) commuting if $Tfx=fTx$ for all $x,y\in X$
(ii) weakly compatible if the pair (f,T) commute at their coincidence point, that is, $Tfx=fTx$ whenever $fx\in Tx$,
(iii) IT-commuting[7] at $x\in X$ if $fTx\in Tfx$.

Now we introduce the following definition:

Definition 2.12 Let (X,p) be a partial metric space and $f,g: X\to X$ and $T: X\to CB^p(X)$ be a single-valued and multi-valued maps, respectively. The hybrid pairs (f,T) and (g,T) are said to satisfy Fisher Type Ciric-Suzuki generalized hybrid contraction condition if there exist $r\in[0,1)$ such that

$\psi(r)\min\{p(fx,Tx),p(gy,Ty)\}$
$\le p(fx,gy)$

(3)

Implies $H_p(Tx,Ty)\le rM_p(x,y)$, where

$M_p(x,y)=r\max\{p(fx,gy),p(fx,Tx),$
$p(gy,Ty),\frac{1}{2}[p(Tx,gy)+p(Ty,fx)]\}.$

3. Main Results

Theorem 3.1 Let (X,p) be a complete partial metric space. Assume that $T:X\to CB^p(X)$ and $f,g:X\to X$ satisfies the condition (3) and $TX\subseteq fX\cap gX$. If one of the subsets TX, fX *or* gX is a complete subspace of X then,

(i) $C(f,T)\ne\phi$,that is there exist $z\in X$ such that, $fz\in Tz$,

(ii) $C(g,T)\ne\phi$, that is there exist $z_1\in X$ such that, $gz_1\in Tz_1$.

(iii) f and T have a common fixed point provided f and T are IT-commuting just at coincidence point z and fz is a fixed point of f, that is, $f(fz)=fz$,

(iv) g and T have a common fixed point provided g and T are IT-commuting just at coincidence point z and gz_1 is a fixed point of g, that is $g(gz_1)=gz_1$.

(v) T, f and g have a common fixed point provided that (iii) and (iv) are true.

Proof: Let $h=\frac{1}{\sqrt{r_1}}$, where r_1 is a real number such that $0\le r<r_1<1$ and $u_0\in X$ since, $TX\subseteq gX$ there exist a point u_1 such that $y_0=gu_1\in Tu_0$ and since $TX\subseteq fX$, there exist $u_2\in X$ such that $y_1=fu_2\in Tu_1$. By lemma (2.8)

$p(y_1,y_0)=p(fu_2,gu_1)\le hH_p(Tu_0,Tu_1)$
.

Similarly, there exist $y_2=gu_3\in Tu_2$ such that

$p(y_1y_2)=p(fu_2gu_3)\le hH_p(Tu_0,Tu_2)$.

Inductively, we find a sequence $\{y_n\}$ in X such that

$y_{2n}=gu_{2n+1}\in Tu_{2n}$,

$y_{2n+1}=fu_{2n+2}\in Tu_{2n+1}$.

If $y_{2n}=y_{2n+1}$ for some $n\in N$, then

$fu_{2n}=y_{2n}=y_{2n+1}=gu_{2n+1}\in Tu_{2n}$.

This implies f and T have a coincident point. Similarly g and T have a coincident point. Suppose, further, that $y_{2n}\ne y_{2n+1}$ for all $n\in N$.

We claim that

$p(y_{2n},y_{2n-1})<p(y_{2n-1},y_{2n-2})$

(3.1)

Assume, if

$p(gu_{2n-1},Tu_{2n-1})\ge p(fu_{2n},Tu_{2n})$

i.e. $p(y_{2n-2},y_{2n-1})\ge p(y_{2n-1},y_{2n})$,

then

$\varphi(r)\min\{p(fu_{2n},Tu_{2n}),p(gu_{2n-1},Tu_{2n-1})\}$
$=\varphi(r)\min\{p(y_{2n-1},y_{2n}),p(y_{2n-2},y_{2n-1})\}$
$\le p(y_{2n-1},y_{2n})$

$p(y_{2n},y_{2n-1})=p(fu_{2n},gu_{2n+1})$
$\le hH_p(Tu_{2n},Tu_{2n-1})$
$\le\sqrt{r_1}\max\{p(fu_{2n},gu_{2n-1}),p(fu_{2n},Tu_{2n}),$
$p(gu_{2n-1},Tu_{2n-1}),\frac{1}{2}[p(gu_{2n-1},Tu_{2n})$
$+p(fu_{2n},Tu_{2n-1})]\}$
$\le\sqrt{r_1}\max\{p(y_{2n-2},y_{2n-1}),p(y_{2n-1},y_{2n}),$
$p(y_{2n-2},y_{2n-1}),\frac{1}{2}[p(y_{2n-2},y_{2n})$
$+p(y_{2n-1},y_{2n-1})]\}$
$\le\sqrt{r_1}\max\{p(y_{2n-2},y_{2n-1}),p(y_{2n-1},y_{2n}),$
$\frac{1}{2}[p(y_{2n-2},y_{2n-1})+p(y_{2n},y_{2n-1})]\}$

by P4

$$\frac{1}{2}\left[p(y_{2n-2},y_{2n})+p(y_{2n-1},y_{2n-1})\right]$$
$$\leq\frac{1}{2}\left[p(y_{2n-2},y_{2n-1})+p(y_{2n-1},y_{2n})\right]$$
and
$$\frac{1}{2}\left[p(y_{2n-2},y_{2n-1})+p(y_{2n-1},y_{2n})\right]$$
$$\leq\max\left\{p(y_{2n-2},y_{2n-1}),p(y_{2n-1},y_{2n})\right\}$$
Hence
$$p(y_{2n},y_{2n-1})\leq\sqrt{r_1}\max\{p(y_{2n-2},y_{2n-1}), p(y_{2n-1},y_{2n})\}$$
$$\leq\sqrt{r_1}\,p(y_{2n-2},y_{2n-1})<p(y_{2n-2},y_{2n-1}).$$

This yields (3.1).
Similarly assume if $p(y_{2n-1},y_{2n})\geq p(y_{2n-2},y_{2n-1})$, then
$$\psi(r)\min\{p(y_{2n-1},y_{2n}),p(y_{2n-2},y_{2n-1})\}$$
$\leq p(y_{2n-2},y_{2n-1})$. Therefore,
$$p(y_{2n},y_{2n-1})\leq\sqrt{r_1}\,p(y_{2n-1},y_{2n-2})$$
(3.2) by (3.1) and (3.2) we have
$$p(y_{n+1},y_n)\leq\sqrt{r_1}\,p(y_n,y_{n-1})$$
$$\leq(\sqrt{r_1})^n p(y_0,y_1)$$
for every $n\in N$.
This show $\lim_{n\to\infty}p(y_n,y_{n+1})=0$.
Since,
$$p(y_n,y_n)+p(y_{n+1},y_{n+1})\leq 2p(y_n,y_{n+1})$$
by P4, So we obtain,
$$\lim_{n\to\infty}p(y_n,y_n)=0,\lim_{n\to\infty}p(y_{n+1},y_{n+1})=0 \quad (3.3)$$
Now, for $m>n\geq1$, we have
$$d_p(y_n,y_{n+m})=2p(y_n,y_{n+m})-p(y_n,y_n)-p(y_{n+m},y_{n+m})$$
$$\leq 2[p(y_n,y_{n+1})+\ldots+p(y_{n+m-1},y_{n+m})]$$
$$\leq 2[\left(\sqrt{r_1}\right)^n+\ldots+\left(\sqrt{r_1}\right)^{n+m-1}]p(y_0,y_1),$$
$$\leq 2\frac{(\sqrt{r_1})^n}{1-\sqrt{r_1}}p(y_0,y_1)\to 0 \quad as\ n\to\infty$$
It follow that $\{y_n\}$ is Cauchy sequence.

Assume that gX , is complete subspace of X , then by lemma (2.3) (gX,d_p) is a complete metric space.
Notice that the sequence $\{y_{2n}\}$ contained in gX and has a limit in gX call it u .
Let $z\in f^{-1}u$. Then $u=fz$, the subsequence $\{y_{2n+1}\}$ also converge to u .
And let $z_1\in g^{-1}u$. Then $u=gz_1$.
Therefore, $\lim_{n\to\infty}d_p(y_n,u)=0$ implies,
$$p(u,u)=\lim_{n\to\infty}p(y_n,u)=\lim_{n\to\infty}p(y_n,y_m). \quad (3.4)$$
Now, since $\{y_n\}$ is a Cauchy sequence in (gX,d_p) and (fX,d_p), then we have $\lim_{m,n\to\infty}d_p(y_n,y_m)=0$ and so
$$\lim_{m,n\to\infty}2p(y_n,y_m)-\lim_{m\to\infty}p(y_m,y_m)-\lim_{n\to\infty}p(y_n,y_n)=0$$
It follow from (3.4) that,
$$\lim_{m,n\to\infty}p(y_n,y_m)=0$$
$$p(u,u)=\lim_{n\to\infty}p(y_n,u)=\lim_{n\to\infty}p(y_n,y_n)=0$$
Now we claim that
$$p(fz,Ty)\leq r\max\{p(fz,gy),p(gy,Ty) \quad (3.5)$$
For all $gy\in X-\{fz\}$ and
$$p(gz_1,Ty)\leq r\max\{p(gz_1,fy),p(fy,Ty) \quad (3.6)$$
for all $fy\in X-\{gz_1\}$. If $fz=gy$, then
$$p(fz,Ty)\leq r\max\left\{p(fz,gy),p(gy,Ty)\right\}$$
and $p(gy,Ty)=0$,this give $d_p(gy,Ty)=0$,
which implies, $gy\in Ty$ and we are done, and similarly if $gz_1=fy$ we have $fy\in Ty$.
Since $fu_{2n}\to fz$, there exist $n_0\in N$ such that, $p(fu_{2n},fz)\leq\frac{1}{3}p(fz,gy)$ for $gy\neq fz$ and all $n\geq n_0$, also since ,

$gu_{2n+1} \to fz$, there exist $n_1 \in N$ such that,

$$p(gu_{2n+1}, fz) \le \frac{1}{3} p(fz, gy)$$

for all $gy \ne fz$ and all $n \ge n_1$.then

$\psi(r)\, p(fu_{2n}, Tu_{2n}) \le p(fu_{2n}, Tu_{2n})$

$\le p(fu_{2n}, gu_{2n+1})$

$\le p(fu_{2n}, fz) + p(fz, gu_{2n+1}) - p(fz, fz)$

$\le \frac{2}{3} p(fz, gy) = p(fz, gy) - \frac{1}{3} p(fz, gy)$

$\le p(fz, gy) - p(fu_{2n}, fz)$

$\le p(fu_{2n}, gy) - p(fz, fz)$

$\le p(fu_{2n}, gy)$ Therefore,

$\psi(r)\, p(fu_{2n}, Tu_{2n}) \le p(fu_{2n}, gy)$ (3.7)

Now if $p(fu_{2n}, Tu_{2n}) \le p(gy, Ty)$ or

$p(gy, Ty) \le p(fu_{2n}, Tu_{2n})$ In each case by (3.7) and (3) we have ,

$p(fu_{2n+1}, Ty) \le H_p(Tu_{2n}, Ty)$

$\le r \max\{ p(fu_{2n}, gy), p(fu_{2n}, Tu_{2n}),$

$p(gy, Ty), \frac{1}{2}[p(fu_{2n}, Ty) + p(gy, Tu_{2n})]\}$

$\le r \max\{ p(fu_{2n}, fz) + p(fz, gy) - p(fz, fz),$

$p(fu_{2n}, fu_{2n+1}), p(gy, Ty),$

$\frac{1}{2}[p(fu_{2n}, fz) + p(fz, Ty) - p(fz, fz)$

$+ p(gy, fz) + p(fz, fu_{2n+1}) - p(fz, fz)]\}$

$\le r \max\{ p(fu_{2n}, fz) + p(fz, gy),$

$p(fu_{2n}, fu_{n+1}), p(gy, Ty),$

$\frac{1}{2}[p(fu_{2n}, fz) + p(fz, Ty) + p(gy, fz)$

$+ p(fz, fu_{2n+1})]\}$

Taking limit as $n \to \infty$, we obtain

$p(fz, Ty) \le r \max\{ p(fz, gy), p(gy, Ty),$

$\frac{1}{2}[p(fz, Ty) + p(gy, fz)]\}$

$\le r \max\{ p(fz, gy), p(gy, Ty)\}$

This yield (3.5)

Similarly, we can prove (3.6);

Now we shall prove that $fz \in Tz$.

There is two cases

(1) when $0 \le r < \frac{1}{2}$.

Suppose $fz \notin Tz$. Then as in Dhompongsa & Yinglawesittikul (2009)

Let $ga \in Tz$ be such that

$$2rp(ga, fz) < p(Tz, fz).$$

Since, $ga \in Tz$ implies $ga \ne fz$,

we have from (3.5) and (3.6)

$$p(fz, Ta) \le r \max\{ p(fz, ga), p(ga, Ta)\} \quad (3.8)$$

$\psi(r)\, p(fz, Tz) \le p(fz, Tz) \le p(fz, ga)$

$\psi(r) \min\{ p(fz, Tz), p(ga, Ta)\}$

$$\le p(fz, ga) \quad (3.9)$$

Therefore, by (3)

$p(ga, Ta) \le H_p(Tz, Ta) \le r \max\{ p(fz, ga),$

$p(fz, Tz), p(ga, Ta),$

$\frac{1}{2}[p(fz, Ta) + p(ga, Tz)]\}$

$\le r \max\{ p(fz, ga), p(ga, Ta)\}$. This implies,

$p(ga, Ta) \le H_p(Tz, Ta) \le rp(fz, ga)$ So by P4 and (3.8), $p(fz, Ta) \le rp(fz, ga)$.

Thus by the assumptions

$p(fz, Tz) \le p(fz, Ta) + p(Ta, Tz) - p(Ta, Ta)$

$\le p(fz, Ta) + H_p(Tz, Ta)$

$\le r \max\{ p(fz, ga), p(ga, Ta)\} + H_p(Tz, Ta)$

$\le rp(fz, ga) + rp(fz, ga) \le 2rp(fz, ga)$

$< p(fz, Tz)$ Which is contradicting to $fz \notin Tz$. Hence $fz \in Tz$, that is, $C(f, T) \ne \phi$.

(2) when $\frac{1}{2} \le r < 1$

$H_p(Tz, Ty) \le r \max\{ p(fz, gy), p(fz, Tz),$

$$p(gy, Ty), \frac{1}{2}[p(fz, Ty) + p(gy, Tz)]\} \quad (3.10)$$

Assume that $fz \neq gy$. For each $n \in N$, there exist $z_n \in Ty$ such that

$$p(fz, z_n) \leq p(fz, Ty) + \tfrac{1}{n} p(fz, gy) \quad (3.11)$$

And consequently we have,

$$p(gy, Ty) \leq p(gy, z_n)$$
$$\leq p(gy, fz) + p(fz, z_n) - p(fz, fz) \quad (3.12)$$
$$\leq p(gy, fz) + p(fz, Ty) - p(fz, fz) + \frac{1}{n} p(fz, gy)$$

By using (3.8) and (3.12) we have

$$p(gy, Ty) \leq p(fz, gy) + r \max\{p(fz, gy), p(gy, Ty) + \tfrac{1}{n} p(fz, gy) \quad (3.13)$$

If $p(fz, gy) \geq p(gy, Ty)$ then (3.13) gives

$$p(gy, Ty) \leq p(fz, gy) + rp(fz, gy) + \tfrac{1}{n} p(fz, gy)$$
$$= \left(1 + r + \tfrac{1}{n}\right) p(fz, gy)$$

Taking $n \to \infty$,

$p(gy, Ty) = (1 + r) p(fz, gy)$ thus

$$\psi(r) p(gy, Ty) = (1 - r) p(gy, Ty)$$
$$\leq \tfrac{1}{1+r} p(gy, Ty)$$
$$\leq \left(1 + \tfrac{1}{(1+r)n}\right) p(gy, fz) \leq p(gy, fz)$$

Taking limit as $n \to \infty$, we get

$\psi(r) p(gy, Ty) \leq p(fz, gy)$ then

$$\psi(r) \min\{p(fz, Tz), p(gy, Ty)\} \leq p(fz, gy)$$

and by (3) with $x = z$ we get (3.10)

$$H_p(Tz, Ty) \leq r \max\{p(fz, gy), p(fz, Tz), p(gy, Ty), \tfrac{1}{2}\left[p(gy, Tz) + p(fz, Ty)\right]\}$$

If $p(fz, gy) < p(gy, Ty)$

then (3.13) gives

$$p(gy, Ty) \leq p(fz, gy) + rp(gy, Ty) + \frac{1}{n} p(fz, gy)$$

That is,

$$(1 - r) p(gy, Ty) \leq \left(1 + \tfrac{1}{n}\right) p(fz, gy).$$

Taking limit as $n \to \infty$, we get

$$\psi(r) p(gy, Ty) \leq p(fz, gy).$$

Then

$$\psi(r) \min\{p(fz, Tz), p(gy, Ty)\} \leq p(fz, gy)$$

and by (3) we get (3.10)

Now taking $y = u_{2n+1}$ in (3.10) and passing to the limit, we obtain $p(fz, Tz) \leq rp(fz, Tz)$,

since $r < 1$

$$p(fz, Tz) = 0 = p(fz, fz)$$

Which implies

$$d_p(fz, Tz) \leq 2p(fz, Tz) = 0$$

Hence by (lemma 2.4) we have $fz \in Tz$.

Analogously, $gz \in Tz$.

Thus (i) and (ii) are completely proved.

Since fz is a fixed point of f , T and f are IT-commuting.

$fz = ffz \in fTz \subseteq Tfz$. This show $u = fz$ is a common fixed point of the pair (f, T) . Analogously T and g have a common fixed point $u = gz_1$.

This prove (iii) and (iv). Now (v) is immediate

Corollary 3.2 (Theorem 2.1,[17]): Let (X, p) be a complete partial metric space $T : X \to CB^p(X)$ be a multi-valued mapping and $\psi : [0,1) \to (0,1]$ be non-increasing function defined by (2), if there exists $0 \leq r < 1$ such that T satisfies the condition

$$\psi(r) p(x, Tx) \leq p(x, y)$$
$$\Rightarrow H_p(Tx, Ty) \leq r M_p(x, y)$$

Where,

$$M_p(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \tfrac{1}{2} p\left[(x, Ty) + p(y, Tx)\right]\} \quad (3.14)$$

For all $x, y \in X$. Then T has a fixed point, that is, there exist a point $z \in X$ such that $z \in Tz$.

Proof: Theorem 3.1 with $f = g = I_X$ (identity mapping on X).

Corollary 3.3 (Theorem 2.1,[12]): Let

(X,p) be a complete partial metric space $T:X \to CB^p(X)$ be a multi-valued mapping and $\psi:[0,1)\to(0,1]$ be non-increasing function defined by (2), if there exists $0 \le r < 1$ such that T satisfies the condition

$$\psi(r)\,p(x,Tx) \le p(x,y) \Rightarrow H_p(Tx,Ty) \le r\max\{p(x,y), p(x,Tx), p(y,Ty)\} \qquad (3.15)$$

For all $x,y \in X$. Then T has a fixed point, that is, there exist a point $z \in X$ such that $z \in Tz$.

Proof: It comes from corollary 3.2 since (3.15) implies (3.14).

Corollary 3.4: Let (X,p) be a complete partial metric space $T:X \to CB^p(X)$ be a multi-valued mapping and $\psi:[0,1)\to(0,1]$ be non-increasing function defined by (2), if there exists $0 \le r < 1$ such that T satisfies the condition

$$\psi(r)\,p(x,Tx) \le p(x,y) \Rightarrow H_p(Tx,Ty) \le r\max\{p(x,y), \frac{p(x,Tx)+p(y,Ty)}{2}, \frac{p(x,Ty), p(y,Tx)}{2}\} \quad (3.16)$$

For all $x,y \in X$. Then T has a fixed point, that is, there exist a point $z \in X$ such that $z \in Tz$

Proof: It comes from theorem 3.1 since (3.16) implies (3.14) with $f = g = I_X$.

Corollary 3.5: Let (X,p) be a complete partial metric space $T:X \to X$ be a single-valued mapping and $\psi:[0,1)\to(0,1]$ be non-increasing function defined by (2), if there exists $0 \le r < 1$ such that T satisfies the condition

$$\psi(r)\,p(x,Tx) \le p(x,y) \Rightarrow p(Tx,Ty) \le r\,M_p(x,y),$$

for all $x,y \in X$. Then T has a unique fixed point, that is, there exist a unique point $z \in X$ such that $z = Tz$.

Corollary 3.6 (Theorem 3.2[15]): Let (X,p) be a complete partial metric space. If $T:X \to CB^p(X)$ is a multi-valued mapping such that for all $x,y \in X$, we have,

$$H_p(Tx,Ty) \le k\,p(x,y) \qquad (3.17)$$

Where $k \in (0,1)$. Then T has a fixed point.

Proof: It comes from Corollary 3.5 with $f = I_X$ and since, (3.17) implies (3.16).

Now, we give an example to illustrate our main results. In this example there is a partial Hausdorff metric and a generalized map satisfying the hypothesis of our main result but do not satisfy the generalized contractive in relation to the usual metric

Example (3.7) let $X = \{0, \tfrac{1}{2}, 1\}$ and $p: X \times X \to R^+$ defined by $p(0,0) = p(\tfrac{1}{2},\tfrac{1}{2}) = 0$, $p(1,1) = \tfrac{1}{3}$, $p(0,\tfrac{1}{2}) = \tfrac{1}{4}$, $p(0,1) = \tfrac{2}{5}$, $p(\tfrac{1}{2},1) = \tfrac{11}{5}$, and $p(x,y) = p(y,x)$ for all $x,y \in X$.

Then p is a partial metric on X. Let $\psi(r)$ defined by condition (2) and define $T:X \to CB^p(X)$ by

$T(0) = T(\tfrac{1}{2}) = \{0\}$ and $T(1) = \{0, \tfrac{1}{2}\}$,

therefore we get

$\max\{p(x,Tx);\ x \in X\} = \tfrac{2}{5}$ and

$\min\{p(x,Tx);\ x \in X\ /\{0\}\}$

$= \min\{p(x,y): x,y \in X \text{ and } x \neq y\} = 1/4$

Note that for any $r \geq 1/6$,we have $\psi(r) \leq 5/6$ and then

$\psi(r)\, p(x,Tx) \leq p(x,y)$ for all $x,y \in X$ with $x \neq y$. Put $r = 5/6$ and so $\psi(r) = 1/6$. Consequently we have

$H_p(T0, T\,1/2) = p(0,0) = 0 \leq rM_p(0, 1/2)$,

$H_p(T0, T1) = p(0, 1/2) = 1/4 \leq 1/3$

$= rp(0,1) \leq rM_p(0,1)$,

$H_p(T\,1/2, T1) = p(0, 1/2) = 1/4 \leq 11/18$

$= rp(1/2, 1) \leq rM_p(1/2, 1)$,

and similarly

$H_p(Tx,Ty) \leq r\,M_p(x,y)$ also hold for $x = y$ .Hence, for all $x,y \in \{0, 1/2, 1\}$

We have $H_p(Tx,Ty) = 0$,

Hence $\psi(r)\, p(x,Tx) \leq p(x,y)$

$\Rightarrow H_p(Tx,Ty) \leq r\,M_p(x,y)$.

Thus all condition of corollary (3.2) are satisfied and $x = 0$ is the only fixed point of T

On the other hand, the metric $d_p : X \times X \to \Re^+$ induced by partial metric p is given by

$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$.

$d_p(0,0) = d_p(1/2,1/2) = d_p(1,1) = 0$

$d_p(0,1/2) = d_p(1/2,0) = 1/2$,

$d_p(1/2,1) = d_p(1,1/2) = 17/15$,

$d_p(0,1) = d_p(1,0) = 7/15$.

Now we show that corollary 3.2 is not applicable (in the case of a metric induced by a partial metric p).

For $x = 0$ *and* $y = 1$. *We have*,

$H(T0,T1) = H(T\{0\}, T\{0,1/2\})$

$= \max\{\sup\{d_p(\{0\},\{0,1/2\})\},$

$\{\sup\{d_p(\{0,1/2\},\{0\})\}\}$

$= \max\{\sup\{d_p(0,0)\}, \{\sup\{d_p(0,0),$

$d_p(1/2,0)\}\}$

$= \max\{0, 1/2\} = 1/2$,and

$Md_p(0,1) = \max\{d_p(0,1), d_p(0,T0), d_p(1,T1),$

$1/2[d_p(0,T0) + d_p(1,T1)]\}$

$= \max\{7/15, 0, 7/5, 1/2[0 + 7/15]\}$

$= 7/15 < 1/2$

Thus, for any $0 \leq r < 1$ we have, $H(T0,T1) \neq rMd_p(0,1)$.

References

[1] Nadler, S. Jr.(1969), Multi-valued contraction mappings, Pacific J. Math., Vol. 30, 475-488.

[2] Dhompongsa S. and Yinglaweesittikul, H. (2009), fixed points for multi-valued mappings and the metric completeness, Fixed Point Theory and Applications, Vol. 2009, ID 972395.

[3] Hashim, Amal M. (2010), Coincidence and fixed point theorems for nonexpansive type hybrid maps, Bull of Allahabad mathematical society, Vol. 25, part 2, 209-222.

[4] Hashim, Amal M. and. Abbas, Shereen J, (2013) Some fixed points of single-valued maps and multi-valued maps with their continuity, Basrah Journal of science, vol. 31 No. IA76-86.

[5] Singh S.L. and. Hashim Amal M. (2004), New coincidence and fixed point of reciprocally continuous and compatible hybrid maps, J. Nature. Phys. Sci. Vol. 18(2), 97-104

[6] Singh S.L. and Mishra S.N.(2001), Coincidences and fixed points of non - self hybrid contractions, Journal of Mathematical Analysis and Applications, Vol. 256(2): 486-497.

[7] Singh S.L. and. Mishra S..N (1991-1994), Nonlinear Hybrid contractions, Journal of Natural and Physical Sciences Vol. 5-8 191-206.

[8] Matthews, S. G. (1994). Partial metric topology, Proc. 8th Summer Conference on General Topology and Applications, Ann. New York, Acad. Sci, Vol. 728, 183-197.

[9] Alun, I. and Simsek H. (2008), Some fixed point theorem on dualistic partial metric spaces, J. Adv. Math. Stud. 1, 1-8.

[10] Altun, I., Sola F. and Simsek H. (2010), Generalized contractions on partial metric space, Topology Appl. 157 :2778-2785.

[11] Ciric Lj., Samet B., Aydi H. and Verto C.(2011), Common fixed point of generalized contraction on partial metric spaces and an application, Appl. Math. Compute, 218, 2398-2406.

[12] Damjananovic B. and Doric, D. (2011) Multi-valued generalizations of the Kannan fixed point theorem, Filomat 25(1):125-131.

[13] Hashim, Amal M. (2016), Fixed points of generalized weakly contractive maps in partial metric spaces, Jnanabha, Vol. 46, 155-166.

[14] Kaewcharoen A. and Yuying, T. (2013), Coincidence points and fixed point theorems for multi-valued mappings on partial metric spaces, International Journal of Pure and Applied Mathematics, Vol. 89(4), 531-546.

[15] Aydi, H., Abbas M. and Verto C.(2012), Partial Hausdorff metric and Nadler's fixed point theorem on partial metric space, Topology Appl. 159, 3234-3242.

[16] Abbas, M. and Ali B. (2013), Fixed point of Suzuki-Zamfirescu hybrid contractions in partial metric spaces via partial Hausdorff metric, Fixed Point Theory and Applications 21 :1-6.

[17] Abbas, M., Ali B.and Vetro C. (2013), A Suzuki type fixed point theorem for a generalized multivalued mapping on partial Hausdorff metric spaces, Topology and its Application, 160 553-563.

مبرهنات النقطه الصامده من النوع سازوكي للدوال الهجينيه المعممه على الفضاء المتري الهاوزدرفي الجزئي

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المستخلص

في هذا البحث حصلنا على مبرهنات النقطه الصامده من النوع سازوكي للدوال الهجينيه المعممه (زوج من داله احاديه وداله متعددة القيم) في الفضاء المتري الهاوزدورفي الجزئي .

لقد تم تعميم توسيع وتوحيد العديد من النتائج في الموجودة في الدراسات السابقه.