Processing of Near Singular Integrals in 3D Boundary Elements Method

Jaafar Khalaf Ali, Lecturer Assistant, Dept. of Mech. Engineering.
College of Engineering, University of Basrah

Abstract

In this work, the efficiency of double Gauss quadrature method, used to integrate over a rectangular element in 3D BEM, has been investigated. The efficiency of a quadrature or integration scheme is investigated by estimating the critical ratio for which the absolute relative error of the numerical integration is less than 1×10^{-6}. As small as the critical ratio is, the quadrature is more efficient. Also, special transformation techniques have been introduced and used to increase the accuracy and efficiency of double Gauss quadrature especially for near singular cases, where the source point is very close to the element under consideration. Three types of kernels were considered, weak, strong and hyper singular kernels which can be encountered in the integral equation of 3D elastodynamics BEM problems.

معالجة التكاملات شبه الاحادية في طريقة العناصر الحدودية ثلاثية البعد

جعفر خلف علي، مدرس مساعد، قسم الهندسة الميكانيكية، كلية الهندسة، جامعة البصرة

الخلاصة

تم في هذا العمل دراسة طريقة رباعية كاوس الثنائية والمستخدمة في اجراء التكاملات العددية على العناصر الحدودية مستطيلة الشكل والمستخدمة في المسائل ثلاثية الابعاد. حيث تم تحديد كفاءة التكامل العددي او الرباعية من خلال حساب النسبة الحرجة والتي هي عبارة عن حاصل قسمة اقل مسافة بين نقطة المصدر والعنصر الحدودي على اكبر بعد في ذلك العنصر بحيث يكون مقدار الخطأ النسبي للتكامل العددي المقابل للنسبة الحرجة اقل من 1×10^{-6} ، وكلما كانت النسبة الحرجة أقل فهذا يعني ان الرباعية اكثر كفاءة وقدرة على التعامل مع الحالات شبه الاحادية. كذلك تم ايجاد طريقة تكامل تعتمد على تحويل المتغيرات بواسطة دالة خاصة والتي اثبتت نجاحا في زيادة دقة الرباعية وقدرتها على التعامل مع التكاملات شبه الاحادية والتي تنتج عندما تكون نقطة المصدر قريبة جدا على العنصر الحدودي المعني. تم دراسة ثلاثة حالات من الدوال الاساسية هي الدوال الضعيفة والقوية والفائقة الاحادية والتي يمكن ان نحصل عليها في المعادلات التكاملية ذات الثلاثة ابعاد في مسائل المرونة الحركية.

Keywords: Boundary Elements Method, near-singular integrals, Gauss quadrature, variable transformation.

1. Introduction

The Boundary Element Method (BEM) or the Boundary Integral Equation (BIE) method is a powerful technique for solving partial differential equations. It requires discretization only on the boundary of the domain and, hence, reducing the dimensionality by one [1, 2]. One of the biggest challenges in the BEM is the evaluation of integrals over the discretized boundary, especially in 3D problems where the integral must be performed over an area. In fact, the accuracy and efficiency of the BEM technique depends mainly on the evaluation of these integrals, in particular, the evaluation of near singular integrals which occur when the field point is very close to the area of integration.

Lachat and Watson [3] proposed an adaptive element subdivision technique using an error estimator for the numerical integration. Later, Fiedler [4] proposed a regularization procedure, to be used together with the variable transformation approach to further weaken the near-strong singular integrals. A different approach using quadratic and cubic variable transformations in order to weaken the near singularity before applying Gauss quadrature was introduced by Telles [5].

Special weight function formulae for the 3D kernel $1/r$ were developed by Cristescu and Lobignac [6] for triangular and square planar elements. Aliabadi *et al.* [7, 8] introduced the subtraction of singularity method, which is based on series expansion of the fundamental solution, shape function and the Jacobian in 3D BEM problems. The reader who is interested in further details is referred to Ref. [9]. On the other hand, analytical integration can be found very efficient in this task, since it is not only useful for near singular cases but also when the source point lies on the element itself. However, it is limited only to the planar elements or polyhedral domains as can be seen in [10]. Hayami [11] applied the variable transformation method to evaluate near singular integrals over curved boundaries. His work was extension to previous works executed by him and other co-workers [12]-[15]. It includes using of system coordinates transformation, from Cartesian into polar coordinates, in order to weaken the singularity, then by using variable transformation for both the radius and angle, the singularity is further weakened by multiplication by the derivative, or in other terms Jacobian, of the new transformation.

In this work, the variables transformation method, called T^{β}, is applied for both the two variables of integration in rectangular-shaped elements. The singularity is efficiently weakened by the Jacobian of transformation resulting in enhanced accuracy and ability to process near singular cases. The technique can easily be extended to triangular elements. In section 2, the problem of near singular integrals will be described for 3D BEM in elastodynamics. Gauss quadrature over an area and variable transformation technique will be presented in section 3. The proposed integration scheme will be demonstrated in section 4, while investigation for different orders of singularities will be given in section 5. Finally, section 6 will be devoted to discuss the results and record some conclusions.

2. Problem Statement

Reference will be made to elastodynamics BEM problems for being the problems with the highest order of singularity. The fundamental solution of the displacement (also called Green's Function) for 3D elastodynamics problems is given by [16];

$$G_{uv}(x-y,t)=\frac{1}{4\pi\rho}\left\{\frac{t}{r^2}\left(\frac{3\mathbf{r}\otimes\mathbf{r}}{r^3}-\frac{\mathbf{I}}{r}\right)\left[H\left(t-\frac{r}{c_1}\right)-H\left(t-\frac{r}{c_2}\right)\right]+\frac{3\mathbf{r}\otimes\mathbf{r}}{r^3}\left[\frac{1}{c_1^2}\delta\left(t-\frac{r}{c_1}\right)-\frac{1}{c_2^2}\delta\left(t-\frac{r}{c_2}\right)\right]+\frac{1}{rc_2^2}\delta\left(t-\frac{r}{c_2}\right)\right\} \quad (1)$$

Where x is the source (field) point, y is any point in the space, t is the time, $\mathbf{r} = x-y$, $r = \| x-y \|$, $\mathbf{I}$ is the identity matrix, c_1, c_2 are the speeds of pressure and shear waves respectively, H is the Heaviside (unit step) function and δ is the Dirac delta function. By examining the above equation, three orders of singularities have to be dealt with, weak $1/r$, strong $1/r^2$ and hypersingular $1/r^3$ when integrating over space. Integration over time has no effect on the singularity order of the kernel. When the source point is far away enough from the element under consideration, double Gauss quadrature can be found very efficient and accurate. However, when the source point is very close to the integration element, or in other words, the nearest distance between the source point and the element is very small as compared to the largest dimension in the element, error will arise unless a special treatment is adopted especially for higher order singularities. This is due to the fact that, when r becomes very small (when trying to integrate over the points close to the source point), the resulting kernel will be unstable and, hence, must be represented by a higher degree polynomial. Therefore, a higher order Gauss quadrature must be used to implement the integration which leads to long computational time.

Introducing a dimensionless ratio R, this represents the ratio of minimum distance between the source point and the element (d) to the largest dimension (a);

$$R=\frac{d}{a} \quad (2)$$

The efficiency of any integration scheme can be investigated by examining its ability to maintain the error within a specified limit as R becomes very small. The value of R for which the relative error of numerical integration is just about to exceed the tolerance is called critical ratio for that integration scheme. The tolerance is selected as 1×10^{-6} in this work which is quite sufficient for most applications.

3. Gauss Quadrature over a Rectangle

Gauss quadrature can be combined to integrate over an area in 3D BEM [2, 17], for a rectangular element it takes the following form:

$$\int_0^1\int_0^1 f(\zeta_1,\zeta_2)\,d\zeta_1\,d\zeta_2=\sum_{n1=1}^{N_1}\sum_{n2=1}^{N_2} f(\eta_{n1},\eta_{n2})\,w_{n1}\,w_{n2} \quad (3)$$

Where ζ_1, ζ_2 are the curve-linear coordinates, N_1, N_2 are the orders of inner and outer quadrature respectively, η_k, w_k are the abscissas and weights for the period {0, 1}. Although this integration scheme is efficient and accurate when the source point is far enough from the element, it becomes erroneous for nearly singular cases unless special treatment is adopted. In this work, a special variable transformation are tested and applied to enhance the ability of double Gauss quadrature in numerical integration even when the ratio R is very small. The used transform, called T^{β} , where β is degree of the transform, takes the following form:

$$T^{\beta}(x) = (x + R)^{-1/\beta} \qquad (4)$$

Which represents the reciprocal of β-order root. Applying this transform for both curve-linear variables ζ_1 and ζ_2 yields:

$$\hat{\zeta}_k = T^{\beta}(\zeta_k) = (\zeta_k + R)^{-1/\beta}$$
$$\Rightarrow \zeta_k = \hat{\zeta}_k^{-\beta} - R$$
$$d\zeta_k = \hat{\zeta}_k^{-(1+\beta)} d\hat{\zeta}_k$$
$$so,\ \int_0^1 \int_0^1 f(\zeta_1, \zeta_2)\, d\zeta_1\, d\zeta_2 =$$
$$\int_{\hat{c}}^{\hat{d}} \int_{\hat{a}}^{\hat{b}} \hat{f}(\hat{\zeta}_1, \hat{\zeta}_2)\, d\hat{\zeta}_1\, d\hat{\zeta}_2 =$$
$$(\hat{d} - \hat{c}) \sum_{n1=1}^{N_1} (\hat{b} - \hat{a}) \sum_{n2=1}^{N_2}$$
$$\left[\hat{f}(\hat{a} + (\hat{b} - \hat{a})\eta_{n1}, \hat{c} + (\hat{d} - \hat{c})\eta_{n2}) \right]$$
$$w_{n1}\, w_{n2} \qquad (5)$$

where $\hat{f}(\hat{\zeta}_1, \hat{\zeta}_2) = f(\hat{\zeta}_1^{-\beta} - R, \hat{\zeta}_2^{-\beta} - R)$
$$\hat{\zeta}_1^{-(1+\beta)}\ \hat{\zeta}_2^{-(1+\beta)}$$
$$\hat{a} = \hat{c} = T^{\beta}(0),\ \ \hat{b} = \hat{d} = T^{\beta}(1)$$

It is worthy to mention here that the Jacobian of the transformation $\hat{\zeta}_k^{-(1+\beta)}$ yield very small value near singular points and, hence, will efficiently weaken the singularity. This technique will be used in combination with the following scheme to enhance the ability of Gauss quadrature to integrate over near singular elements.

4. The Proposed Integration Scheme

In order to study the effect of a small ratio R and to demonstrate the idea of the proposed scheme, a rectangular planar element is considered as shown in Figure (1) for the sake of simplicity. However, the procedure can also be extended to the curved elements. Let ξ be the source point and ξ_s the point on the element nearest to ξ. Hence, the minimum distance is simply $d = \|\xi - \xi_s\|$. It is clear that when the vector of minimum distance **d** is parallel to the normal from point ξ_s , or in other words the point ξ_s lies inside the element, the singularity will be higher for the same value of d. This is due to the fact that there will be more singular values for the integrand around ξ_s with worst case when ξ_s lies exactly on the center of the element. For that reason, we will consider the worst case for ξ in this study. When d is very small compared to the size of the element, element subdivision must be performed at the point ξ_s, as shown in Figure (1), when it is far enough from any edge of the element.

This subdivision is useful from two aspects. Firstly, it introduces smaller elements and, hence, higher ratio R which will result in better accuracy, and secondly, it permits the application of the proposed scheme which depends on the fact that the nearest point ξ_s is very close or lies exactly on the corner of the element.

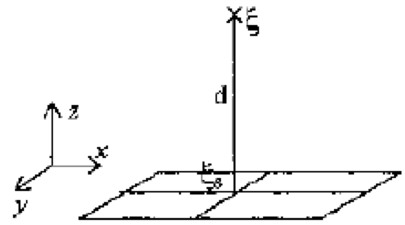


Figure (1) Rectangle Subdivision

When ξ_s is very close to any edge of the element, then there is no need to subdivide the adjacent edge. For example, when the distance between ξ_s and the edge $x = x_j$ is less than $5d$, then there is no need to subdivide the edge $y = y_j$, where (x_j, y_j) is the corner nearest to ξ_s. In order to apply the proposed scheme for the task of numerical integration, the following procedure is introduced;

(1) Find the nearest point ξ_s and the minimum distance d, this can be done by applying Newton-Raphson method.

(2) If $|x(\xi_s) - x_j| < 5d$ for any edge, then no need to subdivide in the x-direction, see Figure (2), but a new minimum distance d_2 must be considered in determination of R, defined as $d_2 = d - 0.5|x(\xi_s) - x_j|$ to compensate for loss of accuracy due to presence of more singular points around ξ_s, the same thing can be said about y-axis.

(3) In case of only one subdivision in one direction is done, consider d_2 for the two resulting sub-elements. In case of no sub-division is performed, consider $d_3 = d_2 - 0.5|y(\xi_s) - y_j|$ for the main element. In case of full sub-division, consider d for all the resulting sub-elements.

(4) Now calculate the ratio R for each (sub-)element by dividing the (corrected) minimum distance by the largest dimension in that (sub-)element. Then apply the appropriate transformation and Gauss quadrature scheme which can be known from the following discussion.

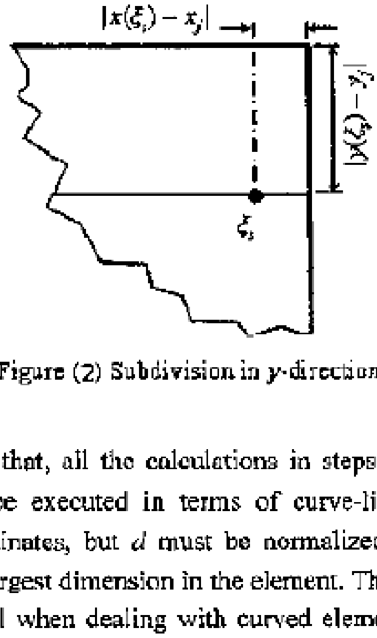


Figure (2) Subdivision in y-direction

Note that, all the calculations in steps 1-4 can be executed in terms of curve-linear coordinates, but d must be normalized by the largest dimension in the element. This is useful when dealing with curved elements. These steps will take very little execution time (less than 1%) in comparison to the upcoming processing.

5. Investigation of the Proposed Scheme

In order to investigate the efficiency of T^{β} transform as well as direct integration, a standard rectangle {(x , y) ∈ R, 0 < x , y < 1} was considered with a source point of coordinates (0, 0, *d*), i.e., ξ lies exactly on the corner of the element located at the origin. The following procedure is used to calculate the critical ratio R_c for direct and transformed integration variables with different degrees of transformation, β, for three types of singularity. Both the exact (analytical) and the numerical solutions are expressed in terms of the ratio *R* for different types of kernels, then the absolute error function E_r given in eq.(6) below is plotted against *R*. The critical value of *R* is the point at which E_r is just less than 1×10^{-6}. Figure (3) shows two cases of the plot E_r – *R* corresponding to two different quadratures. The oscillation of E_r is reasoned to the fact that Gauss Quadrature is a polynomial of the integration variables and, hence, exhibits a kind of lobe generation. If the main lobe is less than 1×10^{-6}, the critical value can be safely considered before it as shown in Figure (3a). In Figure (3b), the main lobe is higher than 1×10^{-6} so R_c must be taken beyond it.

$$E_r = \left| 1 - \frac{I_{numerical}(R)}{I_{exact}(R)} \right| \qquad (6)$$

The investigation results as well as the suggested integration scheme will be given for each type of singularity in what follows.

(a) Weak Singularity 1/r

It has been found that when $R > 1$ then direct integration is more accurate due to the fact that the Jacobian of transformation will play undesired role. Also, when $1.0 < R < 0.75$, no significant improvement in accuracy is indicated for the same quadrature order, therefore direct method is preferable in that range. But when *R* is less than 0.75, transformation method is found to be more efficient. It is clear from Table-1 which records the values of critical ratio for different quadratures that when $0.75 > R > 0.08$ the low degree transformation is more efficient than higher degree one. However, when *R* is less than 0.08, the transform T^5 is found to be more efficient than T^2. By inspecting Table-1, the following formula can be concluded about the optimum value of β ;

$$\beta = 2 + Int\,(Ln \frac{1}{R}) \qquad (7)$$

But a higher order quadrature is needed, order at least (1.5β×1.5β) or 2.25(β×β), in order to achieve the required accuracy, otherwise it will be degraded. It is clear that a quadrature less then 5×5 is not capable of handling transformation scheme and high error will arises if it is used. Increasing β beyond 5 has an effect of reducing the critical ratio, but a higher order quadrature is needed. This is obvious when comparing

T^6 and T^5 columns. Since β is 6 for the former, a quadrature of order at least ~81 is needed to achieve better accuracy which is satisfied for all the quadratures higher than 8×8, but for other quadratures, the accuracy is affected. The proposed procedure to apply numerical integration is given as follows:

1. After evaluating R from the previous section, for each sub-element, if $R > 0.5$ then using direct integration is preferable.
2. If $0.5 > R > 0.08$, using T^2 is preferable with the appropriate quadrature.
3. If $R<0.08$ then use T^5 transform with quadrature of order at least 8×8.

The drawback in transformation method is the evaluation of transformed variables in the form of power functions which consumes a lot of CPU resources. To minimize this problem, repeated multiplication of the variable by itself can be used instead of power function. For that reason, β must be kept integer as compromising between speed and accuracy. Thanks to real time multiplication in math-coprocessors, repetitive multiplication will consume very little time in most programming environments.

(b) Strong Singularity 1/r²

The critical ratios for this type of singularity are listed inTable-2. In this case, using T^2 was found to be, in general, more accurate than transforms of higher degree when $R > 0.05$. However, if $R < 0.05$, higher degree transform has advantage over T^2, but a quadrature of order at least (2β×2β) or 4(β×β) is needed to achieve the required accuracy. This is clear in Table-2 which lists the critical ratios for different integration schemes for the strong singularity. Note that T^6 has lower accuracy than T^5 when the order of quadrature is less than 12×12. The proposed procedure to apply numerical integration for strong singularity is:

1. If ($R > 0.5$) then using direct integration is preferable
2. If ($0.5 > R > 0.05$), using T^2 is preferable with the appropriate quadrature
3. If ($R < 0.05$) then use T^5 or T^6 but a quadrature of order 10×10 is needed for the former and 12×12 for the latter scheme.

(c) Hyper Singularity 1/r³

Here, T^2 transform is, generally, more accurate than other transforms when $R > 0.01$ as can be shown in Table-3. When $R < 0.01$, T^5 is found to be the most accurate transform with abrupt improvement in accuracy over T^2, but a quadrature of order at least (3β×3β) or 9(β×β) is needed to achieve the required accuracy (in this case 16×16 or higher order is required). Note that T^5 is better than T^6 when the order of quadrature is less than 18×18 which confirms the previous conclusion. The proposed procedure to apply numerical integration is:

1. If ($R > 0.6$) then using direct integration is preferable
2. If ($0.6 > R > 0.01$), using T^2 is preferable with the appropriate quadrature

3. If ($R < 0.01$) then use any one of the transforms according to the value of R but a quadrature of order at least 16×16 is needed.

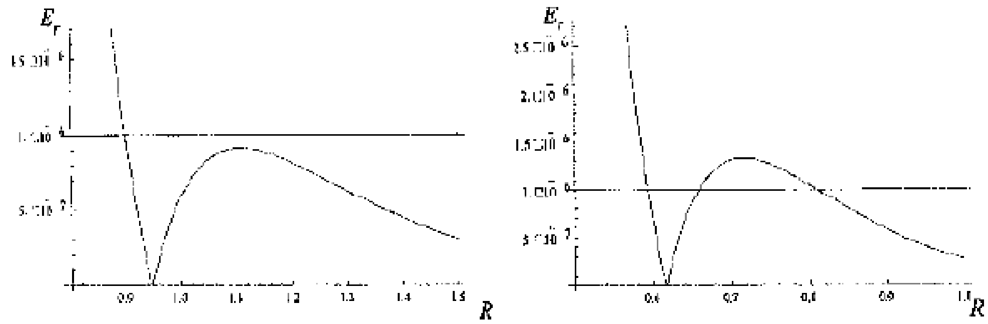
It can be shown that using high-order quadrature will significantly improve the results and reduces critical ratio. This is due to the fact that n-order Gauss quadrature can exactly integrate a polynomial of order up to (2n-1) [18].

6. Conclusions

So far, a powerful integration scheme has been introduced. This scheme involves the combination of two techniques, element subdivision and transformation of integration variables. The first technique reduces the critical ratio R and permits the application of the appropriate quadrature by comparing the evaluated ratio R with the critical ratios given in section 5. While the second technique weakens the singularity due to multiplication by the transformation Jacobian. Tables 1, 2 and 3 can be used to select the appropriate quadrature for various values of R. For large values of the ratio R, direct integration is more efficient, but when R is very small, transformation of variables is found to be superior over direct integration. A reduction in the critical ratio down to 1/500 has been obtained for the quadrature of order 18×18 in hypersingular case. In fact, the efficiency is abruptly improved with the increase of quadrature order even for kernels of high order singularity. For intermediate values of R, lower degree transformations are found to be more efficient than higher degree ones, but for very small values of R, one must use higher degree transformations due to their improved efficiency.

The main disadvantage of the transformation method, resulting from the calculation of power functions, can be avoided by using repetitive multiplication instead of power functions. For that reason β is always chosen as integer value. This is important to save CPU utility.

The proposed scheme can easily be extended to integrate over triangular-shaped elements calling that the inner period of integration can be expressed as a function of the outer abscissa [17]. Also, the proposed scheme can be used with super-hyper singularities such as $1/r^4$ and $1/r^5$, but higher order Gauss quadratures are needed in order to achieve the desired accuracy.



(a) Small Lobe (b) Large Lobe

Figure (3) Typical examples of relative error function E_r vs. the ratio R

Table-1 Critical ratios for weak singularity

Quadrature order	Critical Ratio R_c					
	Direct	T^2	T^3	T^4	T^5	T^6
3□3	1.70	2.40	2.40	2.20	2.10	2.05
4□4	1.25	1.20	1.15	0.78	0.83	0.87
5□5	0.70	0.41	0.49	0.54	0.54	0.55
6□6	0.45	0.11	0.18	0.22	0.24	0.26
7□7	0.33	0.075	0.080	0.088	0.097	0.11
8□8	0.25	0.046	0.041	0.035	0.033	0.035
10□8	0.23	0.046	0.019	0.017	0.015	0.012
10□10	0.15	0.022	0.015	0.011	0.0085	0.0080
12□12	0.10	0.011	0.0070	0.0040	0.0030	0.0027

Table-2 Critical ratios for strong singularity

Quadrature order	Critical Ratio R_c					
	Direct	T^2	T^3	T^4	T^5	T^6
3□3	2.00	3.20	3.25	3.20	3.10	3.00
4□4	1.50	1.55	1.45	1.05	1.10	1.15
5□5	0.85	0.56	0.70	0.74	0.76	0.78
6□6	0.53	0.21	0.29	0.34	0.36	0.37
7□7	0.45	0.16	0.24	0.26	0.24	0.19
8□8	0.33	0.090	0.10	0.13	0.145	0.145
9□9	0.27	0.050	0.030	0.048	0.065	0.074
10□8	0.27	0.038	0.080	0.050	0.070	0.075
10□10	0.22	0.032	0.025	0.016	0.014	0.016
12□12	0.16	0.016	0.0090	0.0085	0.0085	0.0070
16□16	0.092	0.0050	0.0022	0.0016	0.0014	0.0012

Table-3 Critical ratios for hyper singularity

Quadrature order	Critical Ratio R_c					
	Direct	T^2	T^3	T^4	T^5	T^6
3□3	2.50	3.95	3.75	3.55	3.40	3.30
4□4	1.70	1.80	1.65	1.35	1.35	1.38
5□5	1.00	0.70	0.85	0.85	0.85	0.85
6□6	0.75	0.55	0.59	0.58	0.48	0.49
7□7	0.60	0.26	0.33	0.37	0.38	0.39
8□8	0.45	0.19	0.17	0.20	0.22	0.24
10□8	0.375	0.13	0.15	0.18	0.20	0.22
10□10	0.27	0.030	0.065	0.080	0.055	0.065
12□12	0.20	0.013	0.0090	0.018	0.026	0.030
16□16	0.11	0.0029	0.0012	0.0009	0.0006	0.0008
18□18	0.086	0.0015	0.00045	0.00026	0.00020	0.00015

References

1. Brebbia, C. A., Telles, J. C. F. and Wrobel, L. C., Boundary Element Techniques: Theory and Applications in Engineering, *Springer-Verlag, Berlin, 1984.*
2. Brebbia C. A. and Dominguez J., Boundary Elements Methods A Introductory Course, *London 1989.*
3. Lachat, J. C. and Watson, J. O., Effective numerical treatment of boundary integral equations: A formulation for three-dimensional elastostatics, *Int. J. Numer. Methods Eng.*, **10** (1976), 991-1005.
4. Fiedler, C. A., New regularization method for computing interior displacements and stresses in 3D BEM, *Boundary Element Technology IX, Comp. Mech. Pub., Southampton,* 287-297, 1995.
5. Telles, J. C. F., A self-adaptive coordinate transformation for efficient numerical evaluations of general boundary element integrals, *Int. J. Numer. Methods Eng.*, **24** (1987), 959-973.
6. Cristescu, M. and Lobignac, G., Gaussian quadrature formulae for functions with singularities in 1/R over triangles and quadrangles, *Recent Advances in the Boundary Elements Method, Pentech Press, London,* 375-390, 1978.
7. Aliabadi, M. H., Hall, W. S. and Phemister, T. G., Taylor expansion for singular integrals in the boundary element method, *Int. Journal for Numerical Methods in Engineering,* 21, 2221-2236, 1985.
8. Aliabadi, M. H., Hall, W. S. and Phemister, T. G., Taylor expansion in the boundary element method for Neumann problems, *Boundary Elements VII*, 12-31 to 12-39, 1985.
9. Alabadi M. H., The Boundary Element Method, *vol. 2, Wiely 2002,* 491-520.
10. Salvadory, A., Analytical integration of hypersingular kernels in 3D BEM problems, *Computer Methods in Applied Mechanics and Engineering,* 3957-3975, 190 (2001).
11. Hayami, K., Variable Transformations for Nearly Singular Integrals in the Boundary Element Method, *publ. RIMS, Kyoto Univ. 2005*, **41**: 821-842.
12. Hayami, K., Numerical Quadrature for Nearly Singular Integrals in the Three Dimensional Boundary Element Method, *Ph.D. Thesis, The University of Tokyo, 1992.*
13. Hayami, K. and Matsumoto, H., A numerical quadrature for nearly singular boundary element integrals, *Eng. Anal. Bound. Elem.*, **13** (1994), 143-154.
14. Kunihiro, N., Hayami, K. and Sugihara, M., Automatic numerical integration of nearly singular boundary element integrals, in T. Ushijima, Z. Shi and T. Kako eds., *Adv. Numer. Math.*, Proc. 2nd Japan-

China Seminar on Numerical Mathematics, Tokyo, 1994, *Lecture Notes in Numer. Appl. Anal.*, **14** (1995), 249-252.

15. Kunihiro, N., Hayami, K. and Sugihara, M., Automatic numerical integration for the boundary element method using variable transformation and its error analysis, *Trans. Japan Soc. Ind. Appl. Math.*, **5** (1995), 101-119, (in Japanese).
16. Manolis, G. D. and Beskos, D. E., Boundary Elements Methods in elastodynamics, *Unwin Hyman, London 1988.*
17. Rathod, H. T., Gauss Legendre Quadrature over a Triangle, *J. Indian Inst., 2004*, **84**: 183-188.
18. Wandzura, S., Symmetric Quadrature Rule on a Triangle, *Computer and Mathematics with Application, Pergamon 2003.*