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Meromorphic Functions That Share One Finite Value CM or IM with Their First Derivative

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ABSTRACT

In this paper we shall prove that if a non-constant meromorphic function f and its derivative f' share the value $a(\neq 0,\infty)$ CM (IM) and if $\overline{N}\left(r,\frac{1}{f}\right)=S(r,f)$ ($\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f'}\right)=S(r,f)$), then either $f=f'$ or $f(z)=\frac{a(z-c)}{1+Ae^{-z}}$ ($f(z)=\frac{2a}{1-Ae^{-2z}}$), where $A(\neq 0)$ and c are constants. These results give improvement and extension of the following result of Gundersen: if a non-constant meromorphic function f and its derivative f' share two distinct values $0, a(\neq\infty)$ CM, then $f=f'$

Introduction and Results

In this paper, the term meromorphic will always mean meromorphic in the complex plane. We use the standard notations and results of the Nevanlinna theory (see [1] or [2], for example). In particular, $S(r,f)$ denotes any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$ except possibly for a set E of r of finite linear measure .We say that two non-constant meromorphic functions f and g share a value a IM (ignoring multiplicities), if f and g have the same a-points. If f and g have the same a-points with the same multiplicities, we say that f and g share the value a CM (counting multiplicities). Let k be a positive integer, we denote by $N_{k)}\left(r,\frac{1}{f-a}\right)$ the counting function of a-points of f with multiplicity $\leq k$ and by $N_{(k+1}\left(r,\frac{1}{f-a}\right)$ the counting function of a-points of f with multiplicity $>k$.

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In [3] G. G. Gundersen proved the following theorem:

Theorem A. Let f be a non-constant meromorphic function. If f and f' share two distinct values $0, a(\neq\infty)$ CM, then $f=f'$.

In this paper we are give two improvement and extension of Theorem A and prove the following theorems:

Theorem 1. Let f be a non-constant meromorphic function. If f and f' share the value $a(\neq 0,\infty)$ CM, and if $\overline{N}\left(r,\frac{1}{f}\right)=S(r,f)$, then either $f=f'$ or

$$f(z)=\frac{a(z-c)}{1+Ae^{-z}}, \qquad (1.1)$$

where $A(\neq 0)$ and c are constants.

Theorem 2. Let f be a non-constant meromorphic function. If f and f' share the value $a(\neq 0,\infty)$ IM, and if $\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f'}\right)=S(r,f)$, then either $f=f'$ or

$$f(z)=\frac{2a}{1-Ae^{-2z}}, \qquad (1.2)$$

where A is a nonzero constant.

Remark Theorem 1 and Theorem 2 are give improvement and extension of Theorem A, because the condition "f and f' share 0 CM" in Theorem A is exactly the condition

$$N\left(r,\frac{1}{f}\right)=N\left(r,\frac{1}{f'}\right)=0.$$

2. Proof of Theorem 1

Suppose $a=1$ (the general case follows by considering $\frac{1}{a}f$ instead of f) and $f\neq f'$. Since f and f' share 1 CM, we know that the zeros of $f-1$ are simple zeros. By the second fundamental theorem and $\overline{N}\left(r,\frac{1}{f}\right)=S(r,f)$, we have

$$T(r,f)\leq N\left(r,\frac{1}{f-1}\right)+\overline{N}(r,f)-N_0\left(r,\frac{1}{f'}\right)+S(r,f), \quad (2.1)$$

where in $N_0\left(r,\frac{1}{f'}\right)$ only zeros of f' which are not zeros of f are to be considered.

We set

$$F=\frac{1}{f}\left(\frac{f''}{f'-1}-\frac{f'}{f-1}\right). \quad (2.2)$$

From the fundamental estimate of logarithmic derivative it follows that

$$m(r,F)=S(r,f). \quad (2.3)$$

If f has a pole of order $p\geq 1$ at z_∞, by (2.2) F is holomorphic at z_∞. From this and the hypotheses of Theorem 1 we see that

$$N(r,F)=S(r,f). \quad (2.4)$$

If $F=0$, then from (2.2), we find that $f'-1=c(f-1)$, with $c(\neq 0)$ constant. From which and $\overline{N}\left(r,\frac{1}{f}\right)=S(r,f)$ we arrive at $f=f'$ which is a contradiction. Therefore $F\neq 0$ and so we deduce from (2.2), (2.3) and (2.4) that

$$m(r,f)=S(r,f). \quad (2.5)$$

Again from (2.2), if z_∞ is a pole of f of order $p\geq 2$, then z_∞ is possible a zero of F of order $p-1$. Consequently , from (2.3) and (2.4),

$$N_{(2}(r,f)\leq 2N\left(r,\frac{1}{F}\right)\leq 2T(r,F)+O(1)=S(r,f). \quad (2.6)$$

Set

$$H=\frac{f''(f-1)}{f'(f'-1)}. \quad (2.7)$$

Then from the fundamental estimate of logarithmic derivative and (2.5) it follows that

$$m(r,H)=S(r,f). \quad (2.8)$$

If f has a pole of order p at z_∞, by (2.7) z_∞ is a pole of the numerator of (2.7) with order $2(p+1)$ and a pole of the denominator of (2.7) with order $2(p+1)$. This shows that the poles of f, being not the poles of H. Also, because of f and f' share 1 CM, H is holomorphic at the zero of $f'-1$. Thus, the poles of H can occur at only the zero of f', and so that

$$N(r,H)\leq \overline{N}\left(r,\frac{1}{f'}\right). \quad (2.9)$$

Let z_∞ be a simple pole of f. By (2.7) a short calculation with Laurent series shows that $H(z_\infty)=2$. If $H=2$ then $f'-1=c(f-1)^2$, with $c(\neq 0)$ constant. Since f and f' share 1 CM, we have a contradiction. Thus we conclude $H\neq 2$, and so

$$\begin{aligned} N_{1)}(r,f) &\leq N\left(r,\frac{1}{H-2}\right) \\ &\leq T(r,H)+O(1) \\ &\leq \overline{N}\left(r,\frac{1}{f'}\right)+S(r,f) \\ &\leq \overline{N}_0\left(r,\frac{1}{f'}\right)+S(r,f), \end{aligned}$$

by (2.8) and (2.9). Combining this with (2.5) and (2.6) yields

$$T(r,f)\leq \overline{N}_0\left(r,\frac{1}{f'}\right)+S(r,f). \quad (2.10)$$

Hence, we obtain from (2.5), (2.6), (2.1), and (2.10) that

$$m\left(r,\frac{1}{f-1}\right)=S(r,f). \qquad (2.11)$$

Set

$$L=\frac{f'-f}{f(f-1)}. \qquad (2.12)$$

By using (2.11) and the hypotheses of Theorem 1 we may conclude that

$$T(r,L)=S(r,f). \qquad (2.13)$$

Equation (2.12) may also be written in the form

$$f'-1=L(f-L)\left(f+\frac{1}{L}\right), \qquad (2.14)$$

and also written

$$\frac{\left(f+\frac{1}{L}\right)'}{f+\frac{1}{L}}-\frac{1+\left(\frac{1}{L}\right)'}{f+\frac{1}{L}}=L(f-1). \qquad (2.15)$$

Since f and f' share 1 CM, we may obtain from (2.14) and (2.13)

$$N\left(r,\frac{1}{f+\frac{1}{L}}\right)=S(r,f). \qquad (2.16)$$

If $1+\left(\frac{1}{L}\right)'\neq 0$, then from (2.15), (2.13) and (2.5) we get $m\left(r,\frac{1}{f+\frac{1}{L}}\right)=S(r,f)$ from which, (2.16) and (2.13) we conclude $T(r,f)=S(r,f)$. This is impossible. Therefore $1+\left(\frac{1}{L}\right)'=0$, and so $L=\frac{1}{c-z}$, with c constant. Thus equation (2.14) may now be put in the form $\frac{d}{dz}\left[\frac{(c-z)e^z}{f(z)}\right]=-e^z$. By integration and $\overline{N}\left(r,\frac{1}{f}\right)=S(r,f)$ we get (1.1).■

3. Proof of Theorem 2

Suppose $a=1$ and $f\neq f'$. From (2.2), if z_p is a pole of f of multiplicity $p\geq 1$, then

$$F(z)=O((z-z_p)^{p-1}). \qquad (3.1)$$

If z_1 is a simple zero of $f'-1$, then from (2.2) we find that F will be holomorphic at z_1. From this, (2.2), (3.1) and hypotheses of Theorem 2 it can be seen that the poles of F can only occur at the multiple zeros of $f'-1$. That is

$$N(r,F)\leq \overline{N}_{(2}\left(r,\frac{1}{f'-1}\right). \qquad (3.2)$$

If $F=0$, then similarly as in the proof of Theorem 1, we arrive at a contradiction. Next we assume that $F\neq 0$. Thus, we get from (3.1), (3.2) and (2.3)

$$\overline{N}_{(2}(r,f)\leq N\left(r,\frac{1}{F}\right)\leq T(r,F)-m\left(r,\frac{1}{F}\right)+O(1)\leq N(r,F)+m(r,F)-m\left(r,\frac{1}{F}\right)+S(r,f)+O(1)\leq \overline{N}_{(2}\left(r,\frac{1}{f'-1}\right)-m\left(r,\frac{1}{F}\right)+S(r,f). \qquad (3.3)$$

It follows from (2.2) that

$$m(r,f)\leq m\left(r,\frac{1}{F}\right)+S(r,f). \qquad (3.4)$$

Combining (3.3) with (3.4) we obtain

$$\overline{N}_{(2}(r,f)+m(r,f)\leq \overline{N}_{(2}\left(r,\frac{1}{f'-1}\right)+S(r,f). \qquad (3.5)$$

By (2.7), we have

$$m(r,H)\leq m(r,f)+S(r,f). \qquad (3.6)$$

From (2.7), we know that if z_∞ is a pole of f of multiplicity $p\geq 1$, then

$$H(z_\infty)=\frac{p+1}{p}. \qquad (3.7)$$

Let z_1 be a zero of $f'-1$ of multiplicity $q\geq 1$. Since f and f' share 1 IM, we must have z_1 is a simple zero of $f-1$. By a simple calculation on the local expansion we see that

$$H(z_1)=q. \qquad (3.8)$$

From (3.7), (3.8) and $\overline{N}\left(r,\frac{1}{f'}\right)=S(r,f)$ we conclude that

$N(r,H)=S(r,f)$. (3.9)

It can be obtained from (3.7), (3.8), (3.9) and (3.6) that, if $H\neq 2$,

$$N_{1)}(r,f)+\overline{N}_{2)}\left(r,\frac{1}{f'-1}\right)-$$

$$N_{1)}\left(r,\frac{1}{f'-1}\right)\leq N\left(r,\frac{1}{H-2}\right)\leq T(r,H)+O(1)$$

$$\leq N(r,H)+$$

$$m(r,H)+O(1)\leq m(r,f)+S(r,f).$$

Combining this with (3.5) yields

$$\overline{N}(r,f)\leq \overline{N}_{(3}\left(r,\frac{1}{f'-1}\right)+S(r,f).$$

Hence, we get from this, the second fundamental theorem for f' and $\overline{N}\left(r,\frac{1}{f'}\right)=S(r,f)$ that

$$T(r,f')\leq \overline{N}\left(r,\frac{1}{f'}\right)+\overline{N}\left(r,\frac{1}{f'-1}\right)+$$

$$\overline{N}(r,f)+S(r,f)\leq \overline{N}\left(r,\frac{1}{f'-1}\right)+$$

$$\overline{N}_{(3}\left(r,\frac{1}{f'-1}\right)+S(r,f). \qquad (3.10)$$

Therefore

$$N_{(2}\left(r,\frac{1}{f'-1}\right)\leq \overline{N}_{(2}\left(r,\frac{1}{f'-1}\right)+$$

$$\overline{N}_{(3}\left(r,\frac{1}{f'-1}\right)+S(r,f).$$

This implies that

$$\overline{N}_{(2}\left(r,\frac{1}{f'-1}\right)=S(r,f). \qquad (3.11)$$

It is easy to see that $H\neq 1$. Thus we deduce from (3.10), (3.11), (3.8), (3.9), (3.6) and (3.5) that

$$T(r,f')\leq N_{1)}\left(r,\frac{1}{f'-1}\right)+S(r,f)\leq$$

$$N\left(r,\frac{1}{H-1}\right)+S(r,f)\leq T(r,H)+$$

$$S(r,f)=S(r,f),$$

which implies the contradiction $T(r,f) = S(r,f)$. Therefore, we have $H=2$, and integration yields

$$f'-1=c(f-1)^2, \qquad (3.12)$$

where c is a nonzero constant. We rewrite this in the form $f'= c(f-1+A)(f-1-A)$, where $A^2 = -\frac{1}{c}$. Since $\overline{N}\left(r,\frac{1}{f'}\right)=S(r,f)$ by the assumption, it follows from the second fundamental theorem for f that if $A\neq\pm 1$,

$$T(r,f)\leq \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1+A}\right)+$$

$$\overline{N}\left(r,\frac{1}{f-1-A}\right)+S(r,f)=S(r,f),$$

which is a contradiction. Therefore, we have $A=\pm 1$ and so $c=-1$. Then (3.12) reads $\frac{f'}{f-2}-\frac{f'}{f}=-2$. By integration once we conclude (1.2). ■

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دوال الميرومورفك التي لها حصة قيمة واحدة منتهية CM او IM مع مشتقتها الاولى

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الخلاصة:

في هذا البحث نحن سوف نبرهن، اذا كانت f دالة ميرومورفك غير ثابتة و مشتقتها f' لها حصة قيمة واحدة منتهية a $(\neq 0,\infty)$ CM (IM) واذا كانت $\overline{N}\left(r,\frac{1}{f}\right)=S(r,f)$ ($\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f'}\right)=S(r,f)$)، فان اما $f=f'$ او $f(z)=\frac{2a}{1-Ae^{-2z}}$ ($f(z)=\frac{a(z-c)}{1+Ae^{-z}}$) (حيث ان $A(\neq 0)$ و c ثابتان . هاتان النتيجتان هي تطوير و توسيع للنتيجة التالية العائدة الى Gundersen: اذا كانت f دالة ميرومورفك غير ثابتة و مشتقتها f' لها حصة قيمتان مختلفتان 0 و a CM $(\neq\infty)$ ، فان $f=f'$.