Comparing Different Estimators of two Parameters Kumaraswamy Distribution

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Abstract

This paper deals with comparing different methods, to estimate scale parameter (λ), and shape parameter (θ), these estimators are, maximum likelihood, moment estimator, and the maximum likelihood of ordered observation. The comparison has been done through simulation using different sample size and different set of initial values of (λ, θ), then comparing the results using statistical measure mean square error (MSE).

Keywords : Kumaraswamy Distribution, θ shape parameter, λ scale parameter, mean square error (MSE).

الخلاصة

فكرة هذا البحث هي المقارنة بين طرائق مختلفة لتقدير معلمة القياس (λ) و معلمة الشكل (θ), هذه المقدرات هي مقدرات الامكان الاعظم (MLE) , مقدرات العزوم ,ومقدرات الامكان الاعظم لمشاهدات مرتبة . المقارنة تمت عن طريق المحاكاة باستعمال حجوم عينات مختلفة وقيم معلمات اولية مختلفة (λ, θ) , مقارنة النتائج تمت عن طريق المعايير الاحصائية متوسط مربعات الخطأ (MSE)

الكلمات المفتاحية: توزيع كومارساوي، معامل شكل θ ، معامل مستوي لامدا، معدل الخطأ التربيعي.

1. Introduction

The Kumaraswamy distribution is similar to the Beta distribution but has thekey advantage of a closed-form cumulative distribution function . Poondni Kumaraswamy was a leading Indian engineer and hydrologist.Kumaraswamy, introduced the distribution for variables that are lower and upper bounded. The Kumaraswamy distribution is very similar to the Beta distribution, but has the important advantage of an invertible closed form cumulative distribution function. Kumaraswamy (1976,1978)has showed that the well-known probability distribution functions such as the normal, log-normal, and developed a new probability density function known as the sine power probability density function. Furthermore , Kumaraswamy (1980) developed a more general probability density function for double bounded random processes, which is known as Kumaraswamy's distribution. Also, this distribution could be appropriate in situations where scientists use probability distributions which have infinite lower and/or upper bounds to fit data, when in reality the bounds are finite. [Mostafa Mohie Eldin , (**2014**) , Samir K. Safi, (**2013**)]

2- Kumaraswamy Distribution summary

In probability and statistics the Kumaraswamy Distribution is a family of continues probability distribution defined on the interval [0,1] it is similar to the beta distribution but much sampler to use especially in simulation studies due to the sample close form of both its probability density function and cumulative Distribution function . this distribution was originally proposed by **Poondi Kumaraswamy** for variable that are lower and upper pounded. [Gauss M. Cordeiro, (**2012**)] , I. Elbatal, (2013)]

The probability density function for Kumaraswamy Distribution is:-

$$f_T(t) = \theta\, \lambda\, t^{\lambda-1}\left(1-t^{\lambda}\right)^{\theta-1} \qquad 0<t<1 \quad \lambda,\theta>0 \quad \ldots(1)$$

The cumulative Distribution function Kumaraswamy Distribution is:-

$$F_T(t,\theta) = 1-\left(1-t^{\lambda}\right)^{\theta} \qquad \ldots.(2)$$

Reliability function Kumaraswamy Distribution is:-

$R(t) = \left(1 - t^{\lambda}\right)^{\theta}$
Hazard function Kumaraswamy Distribution is:-
$H(t) = \frac{\lambda\,\theta\,t^{\lambda-1}}{1-t^{\lambda}}$
Moment generating function Kumaraswamy Distribution is:
$E(t^r) = \theta\lambda \int_0^1 t^{r+\lambda-1}\left(1 - t^{\lambda}\right)^{\theta-1} dt$
Let;

$$1 - t^{\lambda} = z \qquad 1 - z = t^{\lambda} \qquad t = (1 - z)^{\frac{1}{\lambda}}$$

$$-dz = \lambda t^{\lambda} dt \qquad dt = \frac{dz}{(1 - z)^{\frac{\lambda-1}{\lambda}}}$$

$$E(t^r) = \theta\lambda \int_0^1 t^{r+\lambda-1}\,(1 - z)^{\frac{r+\lambda-1}{\lambda}} z^{\theta-1} \frac{dz}{(1 - z)^{\frac{\lambda-1}{\lambda}}}$$

$$E(t^r) = \theta\lambda \int_0^1 (1 - z)^{\frac{r}{\lambda}+1-\frac{1}{\lambda}+\frac{1}{\lambda}-1}\, z^{\theta-1} dz$$

$$= \theta\lambda \int_0^1 (1 - z)^{\frac{r}{\lambda}} z^{\theta-1} dz$$

$$= \theta\lambda\, Beta\left(\frac{r}{\lambda} + 1, \theta\right)$$

The formula of r^{th} moment about origin
$E(t^r) = \theta\lambda\, \frac{\Gamma\left(\frac{r}{\lambda}+1\right)\Gamma(\theta)}{\Gamma\left(\frac{r}{\lambda}+1+\theta\right)}$

3. Methods of Estimation
3.1 Maximum Likelihood Method

This method depend on maximizing the $(\log L)$, which is the likelihood function to obtain, $(\hat{\theta}_{MLE}, \hat{\lambda}_{MLE})$. [Marcelo, B. Silva, Luz M. Zea and Gauss (2013)]

$$L = \theta^n \lambda^n \prod_{i=1}^{n} t_i^{\lambda-1} \prod_{i=1}^{n} \left(1 - t_i^{\lambda}\right)^{\theta-1}$$

$$\log L = n \log\theta + n\,\log\lambda + (\lambda - 1)\sum_{i=1}^{n} \log t_i + (\theta - 1)\sum_{i=1}^{n} \left(1 - t_i^{\lambda}\right)$$

$$\frac{\partial \log L}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log\left(1 - t_i^{\lambda}\right)$$

$$\frac{\partial \log L}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log t_i + (\theta - 1)\sum_{i=1}^{n} \frac{1}{\left(1 - t_i^{\lambda}\right)}\left[- t_i^{\lambda}(\log t_i)\right]$$

The θ estimation using maximum likelihood method is : -
$\hat{\theta}_{MLE} = -\,\frac{n}{\sum_{i=1}^{n}\log\left(1 - t_i^{\lambda}\right)}$

$$\frac{n}{\hat{\lambda}} = (\theta - 1)\sum_{i=1}^{n} \frac{t_i^{\lambda}(\log t_i)}{\left(1 - t_i^{\lambda}\right)} - \sum_{i=1}^{n} \log t_i$$

The λ estimation using maximum likelihood method is : -

$\hat{\lambda}_{MLE} = \frac{n}{k}$

Where; $k = (\theta - 1)\sum_{i=1}^{n} \frac{t_i^{\lambda}(\log t_i)}{(1 - t_i^{\lambda})} - \sum_{i=1}^{n} \log t_i$

Which is an implicit function of (λ) can be solved numerically.

3.2 Moment Estimators

The moment method depended in estimation for equal the moment population and moment distribution . [Jones, M.C., (2009)], Pascoa, A.R.M., Ortega,(2011), Saulo, H., J. Le˜ao and M. Bourguignon, (2012)]

Since the formula for r^{th} moment is;

$$E(t^r) = \theta\lambda \frac{\Gamma\left(\frac{r}{\lambda} + 1\right)\Gamma(\theta)}{\Gamma\left(\frac{r}{\lambda} + 1 + \theta\right)}$$

When ($r = 1$);

$$E(t) = \bar{t}$$

$\theta\lambda \frac{\Gamma\left(\frac{1}{\lambda}+1\right)\Gamma(\theta)}{\Gamma\left(\frac{1}{\lambda}+1+\theta\right)} = \bar{t}$

$$E(t^2) = \frac{\sum_{i=1}^{n} t_i^2}{n}$$

$\theta\lambda \frac{\Gamma\left(\frac{2}{\lambda}+1\right)\Gamma(\theta)}{\Gamma\left(\frac{2}{\lambda}+1+\theta\right)} = \frac{\sum_{i=1}^{n} t_i^2}{n}$

From equation $\theta\lambda \frac{\Gamma\left(\frac{1}{\lambda}+1\right)\Gamma(\theta)}{\Gamma\left(\frac{1}{\lambda}+1+\theta\right)} = \bar{t}$ we have;

$$\bar{t}\Gamma\left(\frac{1}{\lambda} + 1 + \theta\right) = \theta\lambda\, \Gamma\left(\frac{1}{\lambda} + 1\right)\Gamma(\theta)$$

$$\bar{t}\Gamma\left(\frac{1}{\lambda} + 1 + \theta\right) = \lambda\, \Gamma\left(\frac{1}{\lambda} + 1\right)\Gamma(\theta + 1)$$

$\hat{\lambda}_{MOM} = \frac{\bar{t}\Gamma\left(\frac{1}{\lambda}+1+\theta\right)}{\Gamma\left(\frac{1}{\lambda}+1\right)\Gamma(\theta+1)}$

which is an implicit function.

3.3 L – Moment

The estimation parameters by this method depend on equating (β_r) with (b_r) where (β_r) is population moments and defined by;[B. E. Mohammed, (2014), Jones, M.C., (2009)]

$\beta_r = \int_0^{\infty} x\, F^r(x) f(x) dx$

While (b_r) is sample moment.

$$b_r = \frac{1}{n\, C_r^{n-1}} \sum_{i=1}^{n} C_r^{i-1} x_{(i)}$$

Now for the $p.d.f$ given in **(1)**, and C.D.F given in (**2**)**,** we have to solve the integral;

$$\beta_r = \int_0^1 t\left[1 - (1 - t^{\lambda})^{\theta}\right]^r (1 - t^{\lambda})^{\theta-1} dt$$

$= \theta\lambda \int_0^1 t^{\lambda}\left[1 - (1 - t^{\lambda})^{\theta}\right]^r (1 - t^{\lambda})^{\theta-1} dt$

Let;

$$z = (1 - t^{\lambda})^{\theta}$$

$$z^{\frac{1}{\theta}} = (1 - t^{\lambda})$$

$$t^{\lambda} = 1 - z^{\frac{1}{\theta}}$$
$$t^{\lambda-1} = \left(1 - z^{\frac{1}{\theta}}\right)^{\frac{\lambda-1}{\lambda}}$$
$$\lambda\, t^{\lambda-1} dt = -\frac{1}{\theta}\, z^{\frac{1}{\theta}-1}\, dz$$
$$f(t) = \theta\lambda t^{\lambda-1}(1 - t^{\lambda})^{\theta-1} \qquad 0 < t < 1$$

3.4 Method of Ordered Maximum Likelihood

Assume that the estimated parameter say (θ), lies between (x_m, x_{m+1}) observation, then the likelihood function for ordered observation is; [M. Q. Shahbaz, S. Shahbaz, N. S. Butt, (2012), Tiago VianaFlor de Santana, (2012)]

$$L(x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{m} f(x_i) \prod_{i=m+1}^{n} f(x_i)$$
$$L = \theta^m \lambda^m \prod_{i=1}^{m} t_i^{\lambda-1} \prod_{i=m+1}^{n} \left(1 - t_i^{\lambda}\right)^{\theta-1}$$
$$\ln L = m \ln\theta + m \ln\lambda + (\lambda - 1)\sum_{i=1}^{m} \ln t_i + (\theta - 1)\sum_{i=m+1}^{n} \ln\left(1 - t_i^{\lambda}\right)$$
$$\frac{\partial \ln L}{\partial\theta} = \frac{m}{\theta} + \sum_{i=m+1}^{n} \ln\left(1 - t_i^{\lambda}\right)$$
$$\hat{\theta}_{ORM} = -\frac{m}{\sum_{i=m+1}^{n} \ln\left(1 - t_i^{\lambda}\right)}$$

which is an implicit function of (λ).

$$\frac{\partial \ln L}{\partial\lambda} = \frac{m}{\lambda} + \sum_{i=1}^{m} \ln t_i + (\theta - 1)\sum_{i=m+1}^{n} \frac{-t_i^{\lambda} \log t_i}{\left(1 - t_i^{\lambda}\right)}$$
$$\frac{m}{\hat{\lambda}} + \sum_{i=1}^{m} \ln t_i - (\theta - 1)\sum_{i=m+1}^{n} \frac{t_i^{\lambda} \log t_i}{\left(1 - t_i^{\lambda}\right)} = 0$$

Solved numerically to find ($\hat{\lambda}_{ORM}$)

4 – Simulation Experiment

Here we apply simulation procedure to estimate the parameters (λ, θ), and reliability function by two methods, and show these effected due to change in;

a. sample size number.

b. mixing proportion parameter.

$\boldsymbol{Table\ (3.1)\ where\ \theta = 0.5\ \ \lambda = 1}$

n	Method	θ	$MSE(\theta)$	λ	$MSE(\lambda)$
20	L-Moment	0.955112	0.209692	0.461239	0.292907
	Moment	0.501365	0.017783	1.004498	0.004123
	MLE	0.554525	0.027697	1.205558	0.281685
	Order-MLE	1.744078	7.445257	2.3798	4.905077
BEST		Moment		Moment	
40	L-Moment	0.955782	0.208924	0.446744	0.307355
	Moment	0.526162	0.009035	0.999795	0.001966
	MLE	0.541361	0.011295	1.061269	0.076141
	Order-MLE	1.099021	0.567448	1.730006	0.98441
BEST		Moment		Moment	
80	L-Moment	0.950973	0.203871	0.452554	0.300427
	Moment	0.503172	0.00478	1.000778	0.001096
	MLE	0.516134	0.005518	1.037584	0.042222
	Order-MLE	1.007235	0.347639	1.696073	0.685507
BEST		Moment		Moment	
100	L-Moment	0.950502	0.203391	0.453378	0.299351
	Moment	0.502516	0.003782	0.996765	0.000867
	MLE	0.50903	0.003658	1.021135	0.027657
	Order-MLE	0.972873	0.287612	1.624723	0.531574
BEST		MLE		Moment	

$Table\ (3.2)\ where\ \theta = 0.5\ \ \lambda = 1.5$					
n	Method	θ	$MSE(\theta)$	λ	$MSE(\lambda)$
20	L-Moment	0.919113	0.177665	0.511728	0.979436
	Moment	0.483663	0.017583	1.063743	0.192726
	MLE	0.607638	0.074992	1.8079	0.61901
	Order-MLE	1.700243	7.850937	3.037394	7.591235
BEST		Moment		Moment	
40	L-Moment	0.925133	0.181619	0.511656	0.978073
	Moment	0.440129	0.009333	1.048276	0.2051
	MLE	0.506173	0.007263	1.547412	0.166171
	Order-MLE	1.13214	0.961877	2.72576	3.214077
BEST		MLE		MLE	
80	L-Moment	0.922201	0.178627	0.507466	0.985862
	Moment	0.450897	0.00547	1.05329	0.200106
	MLE	0.521152	0.005626	1.556063	0.094601
	Order-MLE	0.989827	0.338374	2.447985	1.317496
BEST		Moment		MLE	
100	L-Moment	0.917324	0.174446	0.514586	0.971635
	Moment	0.437629	0.007113	1.054973	0.198606
	MLE	0.512795	0.00415	1.601242	0.089438
	Order-MLE	0.947686	0.249405	2.529966	1.54408
BEST		MLE		MLE	

$Table\ (3.3)\ where\ \theta = 0.5\ \ \lambda = 2$

n	Method	θ	$MSE(\theta)$	λ	$MSE(\lambda)$
20	L-Moment	0.905573	0.165475	0.549391	2.106396
	Moment	0.408035	0.021622	1.081667	0.845441
	MLE	0.568185	0.039228	2.400402	1.56194
	Order-MLE	1.670558	6.817151	4.625071	21.0375
BEST		Moment		Moment	
40	L-Moment	0.902484	0.162436	0.550078	2.103106
	Moment	0.395247	0.015399	1.077775	0.85114
	MLE	0.523481	0.008282	2.144476	0.252532
	Order-MLE	1.056633	0.596125	3.564739	5.029922
BEST		MLE		MLE	
80	L-Moment	0.903977	0.163441	0.548174	2.108416
	Moment	0.394611	0.013447	1.07571	0.85469
	MLE	0.508878	0.00505	2.039691	0.160682
	Order-MLE	0.970044	0.28229	3.253557	2.200937
BEST		MLE		MLE	
100	L-Moment	0.90089	0.16088	0.552497	2.095693
	Moment	0.387385	0.014398	1.076217	0.853691
	MLE	0.511688	0.003589	2.084528	0.144271
	Order-MLE	0.938389	0.243315	3.262032	2.239815
BEST		MLE		MLE	

$Table\ (3.4)\ where\ \theta = 1\ \ \lambda = 1$					
n	Method	θ	$MSE(\theta)$	λ	$MSE(\lambda)$
20	L-Moment	1.029719	0.005635	0.34411	0.432228
	Moment	1.031392	0.074498	1.003831	0.005438
	MLE	1.117831	0.153955	1.109233	0.127259
	Order-MLE	3.530894	43.06458	1.616204	1.029472
BEST		L-Moment		Moment	
40	L-Moment	1.02212	0.001977	0.342592	0.4331
	Moment	1.064336	0.033386	1.019229	0.003054
	MLE	1.113802	0.056786	1.113247	0.055025
	Order-MLE	2.298455	3.31223	1.486882	0.481886
BEST		L-Moment		Moment	
80	L-Moment	1.025406	0.001485	0.341181	0.434372
	Moment	1.015334	0.021021	1.003253	0.001476
	MLE	1.037314	0.032126	1.031779	0.025235
	Order-MLE	1.999821	1.480717	1.374945	0.251739
BEST		L-Moment		Moment	
100	L-Moment	1.026323	0.001418	0.341459	0.434124
	Moment	0.995345	0.01395	0.99971	0.001217
	MLE	1.010776	0.019372	1.02705	0.020604
	Order-MLE	2.015641	1.528252	1.409182	0.256766
BEST		L-Moment		Moment	

$Table\ (3.5)\ where\ \theta = 2\ \ \lambda = 1$					
n	Method	θ	$MSE(\theta)$	λ	$MSE(\lambda)$
20	L-Moment	1.134469	0.755041	0.238494	0.580982
	Moment	2.148997	0.523999	1.004303	0.01044
	MLE	2.314363	0.983266	1.065838	0.091775
	Order-MLE	7.200105	133.7573	1.400988	0.503525
BEST		Moment		Moment	
40	L-Moment	1.122805	0.772849	0.24381	0.57221
	Moment	2.05751	0.159939	1.008726	0.003216
	MLE	2.189654	0.293031	1.068519	0.033512
	Order-MLE	5.417075	70.46209	1.279766	0.243612
BEST		Moment		Moment	
80	L-Moment	1.130763	0.757277	0.240175	0.577549
	Moment	2.070331	0.078345	1.006535	0.002042
	MLE	2.129714	0.136349	1.036344	0.01734
	Order-MLE	4.368802	9.737965	1.252483	0.128111
BEST		Moment		Moment	
100	L-Moment	1.130096	0.757687	0.239782	0.578099
	Moment	2.009723	0.041173	0.996865	0.00139
	MLE	2.017565	0.059263	0.997968	0.009884
	Order-MLE	3.97416	6.365081	1.205409	0.086445
BEST		Moment		Moment	

$Table\ (3.6)\ where\ \theta = 1.5\ \ \lambda = 1$					
n	Method	θ	$MSE(\theta)$	λ	$MSE(\lambda)$
20	L-Moment	1.084294	0.177467	0.28187	0.516926
	Moment	1.639368	0.209924	1.021292	0.007361
	MLE	1.782537	0.43958	1.126852	0.088388
	Order-MLE	6.301281	110.6744	1.556165	1.107074
BEST		L-Moment		Moment	
40	L-Moment	1.089524	0.170276	0.276763	0.523862
	Moment	1.565319	0.115555	1.000131	0.004055
	MLE	1.601656	0.156901	1.02652	0.047501
	Order-MLE	3.790038	18.30385	1.342664	0.341938
BEST		Moment		Moment	
80	L-Moment	1.084848	0.17356	0.282998	0.514576
	Moment	1.523624	0.039267	1.007926	0.002295
	MLE	1.531762	0.053282	1.019215	0.019077
	Order-MLE	3.017637	4.050479	1.281291	0.147922
BEST		Moment		Moment	
100	L-Moment	1.088474	0.170271	0.278866	0.520388
	Moment	1.518697	0.038172	1.002282	0.002035
	MLE	1.540083	0.050758	1.018342	0.015272
	Order-MLE	3.181599	4.199939	1.314734	0.167817
BEST		Moment		Moment	

Conclusions

1. The Moment estimator is the best one than MLE, and L – Moment and Ordered MLE
2. For certain chosen set values ($\lambda = 1, \theta = 1$), we find [MSE($\hat{\theta}$)] using L-Moment is the smallest for θ.
3. For large sample size ($n = 40{,}80{,}100$), the MLE is the best method, there is convergent between Moment and MLE.
4. In the set of parameter ($\lambda = 1, \theta = 2$) the moment method is a best estimator for sample size.

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