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The χ–Subgroups of Special linear group SL(n,q)

Afrah M. Ibraheem

Al-Mustansiriyah University -College of Education

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ABSTRACT

In this paper we find χ-subgroup for the irreducible characters χ of the Special linear group SL(n,q) when n=2,3 ..

INTRODUCTION

Let G be a finite group and χ be an irreducible character of G. DIXEN in [1] define the subgroup H of G as a χ-subgroup if there exists a linear character θ of H such that $\langle \chi_H, \theta \rangle$ = 1, where $\langle , \rangle$ is the inner product of restriction of χ to H and θ .And he using the character restriction method of χ-subgroup to construct a representation of G affording χ .In this paper we find χ-subgroups for the irreducible characters χ of the Special linear group SL(n,q) when n=2,3 and $q=p^k$ for some prime p ,k∈N
depending on some tables related by SL(n,q) .

PRELIMINARIES

Let G be a finite group ,and let **C** be a field of Complex numbers . We give in this section some concepts of general group theory and representation theory that we shall use latter, and we can found in [3] .

Definition (1.1):

Let GL(n,**C**) be the group of all non singular n×n matrices over **C** ,then **a representation** of G is a homomorphism R of G into GL(n,**C**) for some n≥1 . The number n is called **the degree** of R and is denoted by deg R.
A representation R is called **irreducible** if R(G) is an irreducible matrix group .

Definition (1.2):

Let R be a representation of G .Then **the character** χ of G afforded by R is a function of G into **C** given by χ(g)=tr(R(g)) for $g \in G$,which is the sum of the diagonal entries of R(g) .
The degree of χ is χ(1)=deg R .

* Corresponding author at: Al-Mustansiriyah University - College of Education, Iraq.E-mail address:

A character χ is **an irreducible character** if R is irreducible. And χ is called the **regular character** if χ(1)=|G| and χ(g)=0 for $g \in G$ and g≠1 .
Characters of order 1 are called **linear character**. And the function 1(g)=1 for all $g \in G$ is a linear character and is called the **principal character**, and denoted by **1**. We denote the set of all irreducible characters of G by Irr(G).

Definition (1.3):

Let ϕ and ψ be characters of G. Then

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\psi(g)}$$

is the **inner product** of ϕ and ψ .

Definition (1.4):

Let H be a subgroup of G , the **restriction** χ_H of a character χ of G to H is a character of H and we can write

$$\chi_H = \sum_{\psi \in Irr(H)} \eta_\psi \psi \quad \text{, for suitable integers}$$

η_ψ.
Note that if $\chi_H \in Irr(H)$ then $\chi \in Irr(G)$.

Definition (1.5):

If H is a subgroup of G and χ and ϕ are characters of G and H, respectively, then the **Frobenius reciprocity** Theorem shows that
$\langle \phi, \chi_H \rangle = \langle \phi^G, \chi \rangle$, ϕ^G is the induced character on G .

Lemma (1.6) : (The Frattini Argument)

Let G be a group and N be a normal subgroup of G. If P is a Sylow p-subgroup of N then G = $N_G(P)N$.

Definition (1.7) [4] :

Let F be a field ,and GL(n,F) is the group of invertible n×n matrices over F.The Special linear group SL(n,F) is the subgroup of GL(n,F) which contains all matrices in GL(n,F) of determinant one . When we replace F by the prime power p^k, for some prime p, k∈N, we have SL(n,q) ,$q=p^k$.

2. THE **χ**-SUBGROUP OF SL(2,q)

Let G =SL(2,q) ,where $q=p^k$ for some prime p ,and let

$H=\left\{\begin{pmatrix}1 & 0\\ \beta & 1\end{pmatrix}:\beta\in F_q\right\}$,$F_q$ a field with q elements .

Then H is abelian Sylow p-subgroup of G of order q. In this section we show H is a χ-subgroup for all irreducible characters of G .We shall need some tables which we can see in [2] . The following tables are the tables of values of characters of G on elements 1 and 1≠h ∈H, when q is odd and when q is even .

Table (2.1): The values of characters of SL(2,q) on elements of H when q is odd .

	1	h
1	1	1
ρ	q	0
Ψ_i	q-1	-1
θ_j	q+1	1
η_1	(q-1)/2	$(-1\pm\sqrt{\varepsilon q})/2$
η_2	(q-1)/2	$(-1\mp\sqrt{\varepsilon q})/2$
ξ_1	(q+1)/2	$(-1\pm\sqrt{\varepsilon q})/2$
ξ_2	(q+1)/2	$(-1\mp\sqrt{\varepsilon q})/2$

where $\varepsilon=(-1)^{(q-1)/2}$, 1 ≤ i ≤ (q-1)/2 and 1 ≤ j ≤ (q-3)/2.
Note that $\eta_1(h)+\eta_2(h)=-1$ and $\xi_1(h)+\xi_2(h)=1$ for all 1 ≠ h ∈H.

Table (2.2): The values of characters of SL(2,q) on elements of H when q is even .

	1	h
1	1	1
ρ	q	0
Ψ_i	q-1	-1
θ_j	q+1	1

where 1 ≤ i ≤ q/2 and 1 ≤ j ≤ (q-2)/2.

Lemma (2.3) [3] :

Let χ be an irreducible character of a group G and suppose p ∤ (|G| / χ(1)) for some prime p. Then χ(g) = 0 whenever p \ o(g). In particular if G has a Sylow subgroup H and an irreducible character χ such that |H| = χ(1) ,then χ_H is the regular character of H and so $\langle \chi_H,\varphi\rangle=1$ for each linear character φ of H.

Theorem (2.4) :

Let G = SL(2,q) for $q=p^k\geq 4$ and H be a Sylow p-subgroup of G. Then for all characters χ of G, H is a χ-subgroup.

Proof:

By lemma(2.3)the character ρ_H of degree q is the regular character of H.
Since H is abelian, all irreducible characters $\varphi_1=1,\varphi_2,\ldots,\varphi_q$ of H are linear. On the other hand $\psi_j(h)=-1$ and $\theta_i(h)=1$ for all 1 ≠ h∈H so $(\psi_j)_H=\rho_H-1$ and $(\theta_i)_H=\rho_H+1$.
Also when q is odd we have $\eta_1(h)+\eta_2(h)=-1$ and $\xi_1(h)+\xi_2(h)=1$ for all 1 ≠ h ∈H so $(\eta_1)_H+(\eta_2)_H=\rho_H-1$ and $(\xi_1)_H+(\xi_2)_H=\rho_H+1$.

Now since $\rho_H=\sum_{i=1}^{q}\varphi_i$ and q ≥4, therefore the restriction of each irreducible character of G to H has at least one linear constituent with multiplicity 1.

3. THE **χ**-SUBGROUP OF SL(3,q)

Let G= SL(3,q) ,where $q=p^k$ for some prime p .
In this section we shall show that for each irreducible character χ of G either a sylow subgroup H or a p-subgroup of order q^2 of G is a χ-subgroup.

The character tables of G are known by the work [6] ,we shall use those tables to get the values of characters on the different conjugacy classes of G which contain the elements of the Sylow subgroup H defined below .

Define $H=\left\{\begin{pmatrix}1 & 0 & 0\\ a & 1 & 0\\ b & c & 1\end{pmatrix}:a,b,c\in F_q\right\}$; F_q is a field with q elements.
Then the order of H is q^3 and H is a Sylow p-subgroup for G .

Now we found the following table in [6] that shows the structure of conjugacy classes of G which contain some elements of the Sylow p-subgroup H ,where d = gcd (3,q-1) , ε ∈GF(q) and ω is a cube root of unity.

Table (3.1) : Conjugacy classes of SL(3,q) which contain elements of the Sylow p-subgroup H for d = 1, 3.

Conjugacy class	Canonical representative	Parameters
$C_1^{(k)}$	$\begin{pmatrix} \omega^k & 0 & 0 \\ 0 & \omega^k & 0 \\ 0 & 0 & \omega^k \end{pmatrix}$	$o \le k \le (d-1)$
$C_2^{(k)}$	$\begin{pmatrix} \omega^k & 0 & 0 \\ 1 & \omega^k & 0 \\ 0 & 0 & \omega^k \end{pmatrix}$	$o \le k \le (d-1)$
$C_3^{(k,I)}$	$\begin{pmatrix} \omega^k & 0 & 0 \\ \varepsilon^I & \omega^k & 0 \\ 0 & \varepsilon^I & \omega^k \end{pmatrix}$	$o \le k, I \le (d-1)$

Note that each element of H is contained in one of the conjugacy classes $C_1^{(0)}, C_2^{(0)}$ and $C_3^{(0,I)}$ of G.

The centre $Z(H) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ z & 0 & 1 \end{pmatrix} : z \in F_q \right\}$ is an elementary abelian p-group of order q.

Using the canonical representative elements of conjugacy classes $C_1^{(0)}, C_2^{(0)}$ and $C_3^{(0,I)}$ we see that the minimal polynomials of elements of these conjugacy classes have degrees 1,2 and 3, respectively and the minimal polynomials of nontrivial elements of Z(H) have degree 2 so nontrivial elements of Z(H) are contained in the conjugacy class $C_2^{(0)}$.

Now we can see the the following lemma in [4] which gives us some properties of H.

Lemma (3.2) :

Suppose G=SL(3,q) where q is a power of the prime p. If H is a Sylow p-subgroup of G then we have:

1. H has q^2+q -1 conjugacy classes.
2. H has q^2 linear characters and q-1 non-linear characters of degree q such that their values on nontrivial elements of Z(H) are 1 and $q\omega^i$ for some $1 \le i \le p$ respectively, where ω is a primitive p^{th} root of unity.
3. If τ is an irreducible character of degree q of H then τ(x) = 0 for $x \notin Z(H)$,and $\sum_{1 \ne z \in Z(H)} \tau(z) = -q$.

Now we can see the following table in [5] which shows the values of the restriction of the irreducible characters of group SL(3,q) on the elements of Sylow subgroup H when d=1. Since d = 1, I = 0.

Table (3.3) : Values of characters of SL(3,q) on elements of H when d = 1

	$C_1^{(0)}$	$C_2^{(0)}$	$C_3^{(0,0)}$
1	**1**	**1**	**1**
ψ	**q^2 + q**	**q**	**0**
ρ	**q^3**	**0**	**0**
ζ_i	**q^2 + q +1**	**q +1**	**1**
η_j	**q^3 + q^2 + q**	**q**	**0**
ε_r	**q^3 + 2q^2 + 2q+1**	**2q +1**	**1**
μ_s	**q^3 -1**	**-1**	**-1**
ν_t	**q^3- q^2 - q +1**	**1-q**	**1**

where 1 ≤ i , j≤ q-2 , 1≤ r≤ (q^2+5q+6)/6 , 1 ≤ s≤ (q^2 - q)/2 and 1 ≤ t ≤ (q^2+q)/3 .

Lemma (3.4) :

Let G = SL(3,q) where q > 2 is a power of the prime p and let H be the Sylow p-subgroup of G and ψ be the irreducible character of degree q^2 + q of G. Then

1. $\langle \psi_H, 1 \rangle = 2$.
2. $\langle \psi_H, \tau \rangle = 1$ for each irreducible character τ of degree q of H.
3. There exist some non-principal linear characters φ and ϕ of H such that $\langle \psi_H, \varphi \rangle = 0$ and $\langle \psi_H, \phi \rangle = 1$.

Proof : Suppose that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \in H$$

is contained in the conjugacy class $C_2^{(0)}$ of G. Since each element in $C_2^{(0)}$ has a minimal polynomial of degree 2,

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$$(x-1)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ ac & 0 & 0 \end{pmatrix} = 0$$

This, together with $x \notin Z(H)$ implies a = 0 or c = 0 but not both. Therefore the number of possibilities for the elements x with above properties is 2q(q-1). The elements of Z(H) are also contained in $C_2^{(0)}$ and the values of ψ on $C_1^{(0)}, C_2^{(0)}$ and $C_3^{(0,0)}$ are q^2+q , q and 0, respectively. Thus we have

$$\langle \psi_H, 1\rangle = \frac{1}{|H|}\sum_{x\in H}\psi_H(x)1(x) = \frac{1}{q^3}(\psi_H(1) + \sum_{1\neq z\in Z(H)}\psi_H(z) + \sum_{z\notin Z(H)}\psi_H(z))$$

$$\cdots\cdots\cdots = \frac{1}{q^3}((q^2+q)+(q-1)q+2q(q-1)q) = 2.$$

Now suppose τ is an irreducible character of degree q of H. By using table(3.3) for the value of ψ on the conjugacy class $C_2^{(0)}$ of G which contains the elements of Z(H) , by lemma (3.2) we have

$$\langle \psi_H, \tau\rangle = \frac{1}{|H|}\sum_{x\in H}\psi_H(x)\overline{\tau(x)}$$

$$\cdots\cdots\cdots\cdots = \frac{1}{q^3}(\psi_H(1)\tau(1) + \sum_{1\neq z\in Z(H)}\psi_H(z)\overline{\tau(z)} + \sum_{z\notin Z(H)}\psi_H(z)\overline{\tau(z)})$$

$$\cdots\cdots\cdots\cdots = \frac{1}{q^3}((q^2+q)q - q^2 + 0) = 1.$$

Therefore for each irreducible character τ of degree q of H $\langle \psi_H, \tau\rangle = 1$ as claimed.

Now since $\langle \psi_H, \tau\rangle = 1$ for each irreducible character τ of degree q of H , hence $\psi_H = \sum_{i=1}^{q-1}\tau_i + \sum_{j=1}^{t} m_j\phi_j$ where ϕ_j are linear characters of H with multiplicity m_j . Since $\psi(1) = q^2 + q$ and $\sum_{i=1}^{q-1}\tau_i(1) = q^2 - q$ we have $\sum_{j=1}^{t} m_j\phi_j(1) = 2q$. Since H possesses q^2 -1 non-principal linear characters, there exists at least one non-principal linear character φ such that $\langle \psi_H, \varphi\rangle = 0$ .Put $\langle \psi_H, 1\rangle = m.$ Since $(\upsilon_t)_H = \chi_H - \psi_H + 1$ we have $\langle (\upsilon_t)_H, 1\rangle = 2 - m$. Hence

m ≤ 2 and so $\sum_{j=1}^{t} m_j\phi_j(1) \geq 2q - 2$ where $\sum$ runs over $\phi_j \neq 1$.This means there exists some non-principal linear character ϕ of H such that $\langle \psi_H, \phi\rangle \neq 0$. If we suppose $\langle \psi_H, \phi\rangle = n$, then $\langle (\upsilon_t)_H, \phi\rangle = 1 - n$. Since $n \neq 0$ this shows that n = 1.Therefore there exists a non-principal linear character ϕ such that $\langle \psi_H, \phi\rangle = 1$.

Theorem (3.5) :

Let G = SL(3,q) where q > 2 is a power of prime p and d = 1 and let H be a sylow p-subgroup of G. Then for all irreducible characters χ of G, H is a χ-subgroup.

Proof:

By table (3.3) the characters ρ and ψ have degrees q^3 and q^2+q respectively.
Now if we restrict them to H we see that for all nontrivial $x \in H$ we have $\rho_H(x) = 0$ and $\psi_H(x) = q$ or 0. Thus from the values of the other characters of G on H we get:

$$(\zeta_i)_H = \psi_H + 1,\quad (\eta_j)_H = \rho_H + \psi_H,$$

$$(\varepsilon_r)_H = \rho_H + 2\psi_H + 1,\quad (\mu_s)_H = \rho_H - 1, \text{ and}$$

$$(\upsilon_t)_H = \rho_H - \psi_H + 1.$$

Since $\rho(1) = q^3$ is the order of H, Lemma (2.3) shows that ρ_H is the regular character of H and $\rho_H = \sum_{\upsilon\in Irr(H)}\upsilon(1)\upsilon$. On the other hand by the lemma (3.4) there exists a non-principal linear character φ of H such that $\langle \psi_H, \varphi\rangle = 0$ then, since $\langle \rho_H, \varphi\rangle = 1$, we have

$$\langle \rho_H + \psi_H, \varphi\rangle = 1 ,\ \langle \rho_H + 2\psi_H + 1, \varphi\rangle = 1 ,$$

$\langle \rho_H - 1, \varphi\rangle = 1$ and $\langle \rho_H - \psi_H + 1, \varphi\rangle = 1$.

Also by the lemma (2.3) there exists a non-principal linear character ϕ of H such that $\langle \psi_H, \phi\rangle = 1$. Thus $\langle \psi_H + 1, \phi\rangle = 1$,and H is a χ-subgroup.

Now we consider the case that d=3 .The following table in [6] show of the values of irreducible characters of SL(3, q) on the conjugacy classes which contain the elements of H .

Table (3.6) : Values of characters of SL(3,q) on elements of H when d = 3

	$C_1^{(0)}$	$C_2^{(0)}$	$C_3^{(0,0)}$
1	**1**	**1**	**1**
ψ	**$q^2 + q$**	**q**	**0**
ρ	**q^3**	**0**	**0**
ζ_i	**$q^2 + q + 1$**	**q +1**	**1**

η_j	$q^3 + q^2 + q$	q	0
θ_k	$(q^3 + 2q^2 + 2q+1)/3$	$(2q+1)/3$ or $(1-q)/3$	$(2q+1)/3$ or $(1-q)/3$
ε_r	$q^3 + 2q^2 + 2q+1$	$2q+1$	1
μ_s	$q^3 -1$	-1	-1
ν_t	$q^3- q^2 - q +1$	$1-q$	1
ω_m	$(q^3- q^2 - q +1)/3$	$(1-q)/3$ or $(2q +1)/3$	$(1-q)/3$ or $(2q +1)/3$
γ_n	$(q^3- q^2 - q +1)/3$	$(1-q)/3$ or $(2q +1)/3$	$(1-q)/3$ or $(2q +1)/3$

where $1 \le i, j \le q-2$, $1 \le r \le (q^2+5q+4)/6$, $1 \le s \le (q^2 - q)/2$, $1 \le t \le (q^2+q-2)/3$ and $1 \le k, m, n \le 3$.
By the values of characters ω_m and γ_n on the conjugacy classes $C_1^{(0)}, C_2^{(0)}$ and $C_3^{(0,I)}$ we have

$$\{(\omega_1)_H, (\omega_2)_H, (\omega_3)_H\} = \{(\gamma_1)_H, (\gamma_2)_H, (\gamma_3)_H\} .$$

Definition (3.7) :

Let H be a normal subgroup of group G and α and β be characters of H. Then β is called **a conjugate** of α in G if there exists $g \in G$ such that $\beta = \alpha^g$, where

$\alpha^g(h) = \alpha(ghg^{-1}) = \alpha(h^{g^{-1}})$ for $h \in H$.

Theorem (3.8) [6] :

Let G = GL(3,q) then the elements of each set of characters $\{\theta_1, \theta_2, \theta_3\}, \{\omega_1, \omega_2, \omega_3\}$ and $\{\gamma_1, \gamma_2, \gamma_3\}$ of SL(3,q) are conjugate in G.

Lemma (3.9) :

Let H be a subgroup of G, $x \in N_G(H)$ and ϑ be a character of H. Then ϑ^x is a character of H and $\vartheta^x(1) = \vartheta(1)$. Furthermore $\langle \vartheta^x, \vartheta^x \rangle = \langle \vartheta, \vartheta \rangle$ and in particular ϑ^x is irreducible if and only if ϑ is irreducible.

Proof:

Since $x \in N_G(H)$ so $xhx^{-1} \in H$ for $h \in H$. On the other hand by definition (3.7)

$\vartheta^x(h) = \vartheta(xhx^{-1}) = \vartheta(h^{x^{-1}})$ so ϑ^x is a character of H. Also

$\vartheta^x(1) = \vartheta(xx^{-1}) = \vartheta(1)$.

$$\langle \vartheta^x, \vartheta^x \rangle = \frac{1}{|H|}\sum_{h\in H} \vartheta^x(h)\overline{\vartheta^x(h)} = \frac{1}{|H|}\sum_{h\in H} \vartheta^x(h^{x^{-1}})\overline{\vartheta(h^{x^{-1}})}$$

$$\ldots\ldots\ldots\ldots = \frac{1}{|H|}\sum_{z=h^{x^{-1}}\in H} \vartheta(z)\overline{\vartheta(z)} = \langle \vartheta, \vartheta \rangle.$$

Therefore $\vartheta^x \in Irr(H)$ if and only if $\vartheta \in Irr(H)$.

Lemma (3.10) :

Let G be a normal subgroup of group L and H be a Sylow subgroup of G. Let χ and ϑ be irreducible characters of G and H, respectively. Let $I \in L$ then

$\langle \chi_H, \vartheta \rangle = \langle \chi_H^I, \vartheta^x \rangle$ for some $x \in N_L(H)$.

In particular $\langle \chi_H, 1 \rangle = \langle \chi_H^I, 1 \rangle$.

Proof:

By the Frattini argument lemma (1.6) shows $L = GN_L(H)$. If $I \in L$ then $l = gx$ where $g \in G$ and $x \in N_L(H)$ .Thus

$$\langle \chi_H, \vartheta \rangle = \frac{1}{|H|}\sum_{h\in H} \chi_H(h)\vartheta(h) = \frac{1}{|H|}\sum_{h\in H} \chi_H(h^{x^{-1}})\vartheta(h^{x^{-1}})$$

$$\ldots\ldots\ldots\ldots = \frac{1}{|H|}\sum_{h\in H^x=H} \chi_H^x(h)\vartheta^x(h) = \langle \chi_H^x, \vartheta^x \rangle.$$

Since $\chi \in Irr(G)$ so $\chi^g = \chi$ thus $\chi_H^I = \chi_H^{gx} = \chi_H^x$ and this implies $\langle \chi_H, \vartheta \rangle = \langle \chi_H^I, \vartheta^x \rangle$.

In particular if $\vartheta = \mathbf{1}$ then since $\mathbf{1}^x = \mathbf{1}$ so $\langle \chi_H, 1 \rangle = \langle \chi_H^I, 1 \rangle$.

Theorem (3.11) :

Let G = SL(3,q) where q > 2 is a power of the prime p and d = 3.
Let H be a Sylow p-subgroup of G. If χ is an irreducible character of G then H is a χ-subgroup.

Proof:

We use the same method as we used to prove theorem (3.5) .Thus ρ_H is the regular character of H and ψ_H is the character of degree $q^2+ q$ of H. Therefore for characters ζ_i , η_j , ε_r , μ_s and ν_t we have:

$(\zeta_i)_H = \psi_H + 1, \quad (\eta_j)_H = \rho_H + \psi_H,$

$(\varepsilon_r)_H = \rho_H + 2\psi_H + 1, \quad (\mu_s)_H = \rho_H - 1,$ and

$(\upsilon_t)_H = \rho_H - \psi_H + 1.$

Lemma (3.4) shows H has some non-principal linear characters φ and ϕ such that $\langle \psi_H, \varphi \rangle = 0$ and $\langle \psi_H, \phi \rangle = 1$. Since ρ_H is the regular character of H so $\langle \rho_H, \varphi \rangle = \langle \rho_H, \phi \rangle = 1$ .Hence

$\langle (\eta_j)_H, \varphi \rangle = \langle (\varepsilon_r)_H, \varphi \rangle = \langle (\mu_s)_H, \varphi \rangle = \langle (\upsilon_t)_H, \varphi \rangle = 1$

and $\langle (\zeta_i)_H, \varphi \rangle = 0$ .Also $\langle \psi_H, \phi \rangle = 1$ implies $\langle (\zeta_i)_H, \phi \rangle = 1$.Therefore the restriction of characters ζ_i , η_j , ε_r , μ_s and ν_t on H have at least one linear constituent with multiplicity one. So the only

remaining characters to consider are θ_k, ω_m and γ_n for $1 \le k,m,n \le 3$.
Using the Frobenius reciprocity we have,

$\langle \eta_j, \varphi^G \rangle = \langle \varepsilon_r, \varphi^G \rangle = \langle \mu_s, \varphi^G \rangle = \langle \upsilon_t, \varphi^G \rangle = 1$ and $\langle \zeta_i, \varphi^G \rangle = 0$. Also if

$\langle (\theta_k)_H, \varphi \rangle = K_k, \langle (\omega_m)_H, \varphi \rangle = M_m \quad and \langle (\gamma_n)_H, \varphi \rangle = N_n,$

then

$\langle \theta_k, \varphi^G \rangle = K_k, \langle \omega_m, \varphi^G \rangle = M_m \quad and \langle \gamma_n, \varphi^G \rangle = N_n,$ for $1 \le k, m, n \le 3$.

Therefore if we induce φ to G we get

$$\varphi^G = \rho + (q-2)\ \eta_j + ((q^2 - 5q + 4)/6)\varepsilon_r + ((q^2 - q)/2)\mu_s + ((q^2 + q - 2)/3)\upsilon_t + \sum_{k=1}^{3} K_k \theta_k + \sum_{m=1}^{3} M_m \omega_m + \sum_{n=1}^{3} N_n \gamma_n.$$

Using the fact that $\varphi^G(1) = |G : H| \varphi(1)$ we calculate the value at 1 and simplify the above equation we have

$$|G:H| = -q^2 - q^3 + q^5 + \sum_{k=1}^{3} K_k \theta_k(1) + \sum_{m=1}^{3} M_m \omega_m(1) + \sum_{n=1}^{3} N_n \gamma_n(1).$$

Since $|G:H| = q^5 - q^3 - q^2 + 1$ we get

$$\sum_{k=1}^{3} K_k \theta_k(1) + \sum_{m=1}^{3} M_m \omega_m(1) + \sum_{n=1}^{3} N_n \gamma_n(1) = q^3 + 1.$$

Since

$\theta_k(1) = (q^3 + 2q^2 + 2q + 1)/3$ and $\omega_m(1) = \gamma_n(1) = (q^3 - q^2 - q + 1)/3$

so we have

$$(\sum_{k=1}^{3} K_k)((q^3 + 2q^2 + 2q + 1)/3)$$

$$+(\sum_{m=1}^{3} M_m + \sum_{n=1}^{3} N_n)((q^3 - q^2 - q + 1)/3) = q^3 + 1.$$

Hence by considering $K = \sum_{k=1}^{3} K_k, M = \sum_{m=1}^{3} M_m$ and $N = \sum_{n=1}^{3} N_n$

We get

$K((q^3 + 2q^2 + 2q + 1)/3) + (M + N)((q^3 - q^2 - q + 1)/3) = q^3 + 1$

So

$(K+M+N)q3 + (2K-(M+N))q^2 + ((2K-(M+N))q + (K+M+N) = 3(q^3+1)$.

Thus

$$(A - 3)(q^3 + 1) = -B(q^2 + q) \quad \ldots\ldots (*)$$

where A = K+M+N and B = 2K-(M+N). Since K, M and N are non negative integers and are not all equal 0, so A is a positive integer. Since q \ -B(q2 + q) so q \ A-3 and this means that A- 3 = tq for some integer t. Hence simplifying equation (*) implies $-B = t(q^2 - q + 1)$. Thus

$$0 \le 3K = A + B = 3 - t(q-1)^2.$$

Since d = gcd(3, q-1) = 3 so we can consider q > 3, which in this case

$A = 3 + tq > 0$ implies $t \ge 0$ and $A + B = 3 - t(q-1)^2 \ge 0$ shows $t \le 0$.

Thus t = 0, A = 3 and B = 0 which these conclude K = 1 and M + N = 2.

Therefore $\sum_{k=1}^{3} K_k = 1$ and $(\sum_{m=1}^{3} M_m + \sum_{n=1}^{3} N_n) = 2$. So for some k, $K_k = 1$ and

$\langle (\theta_k)_H, \varphi \rangle = 1$. Let $\langle (\theta_1)_H, \varphi \rangle = 1$ then by theorem (3.8) the characters θ_1, θ_2 and θ_3 are conjugate in L = GL(3,q) .Hence by lemma(3.10) we have

$\langle (\theta_1)_H, \varphi \rangle = \langle (\theta_2)_H, \varphi^x \rangle = \langle (\theta_3)_H, \varphi^y \rangle = 1$ for some x , $y \in N_L(H)$.

On the other hand by lemma(3.9) , φ^x and φ^y are linear characters of H so the restriction of characters θ_1, θ_2 and θ_3 to H have at least a constituent of degree one with multiplicity one.

Also by table (3.6),

$$\{(\omega_1)_H, (\omega_2)_H, (\omega_3)_H\} = \{(\gamma_1)_H, (\gamma_2)_H, (\gamma_3)_H\}$$

so $\sum_{m=1}^{3} M_m = 1$ and $\sum_{n=1}^{3} N_n = 1$ .Therefore for some m and n we have $N_n = 1$ and $M_m = 1$ which this means $\langle (\omega_m)_H, \varphi \rangle = \langle (\gamma_n)_H, \varphi \rangle = 1$.Now we can suppose $\langle (\omega_1)_H, \varphi \rangle = \langle (\gamma_1)_H, \varphi \rangle = 1$. Theorem (3.8) shows the elements of each set of characters $\{\omega_1, \omega_2, \omega_3\}$ and $\{\gamma_1, \gamma_2, \gamma_3\}$ are conjugate in L= GL(3, q).

Therefore by lemma (3.9) and lemma (3.10) there exist r, s, t, u $\in N_L(G)$ such that $\varphi^r, \varphi^s, \varphi^t and \varphi^u$ are linear characters of H and

$\langle (\omega_2)_H, \varphi^r \rangle = \langle (\omega_3)_H, \varphi^s \rangle = \langle (\gamma_2)_H, \varphi^t \rangle = \langle (\gamma_3)_H, \varphi^u \rangle = 1$ Hence for $1 \le m, n \le 3$ the characters $(\omega_m)_H$ and $(\gamma_n)_H$ have a linear constituent with multiplicity 1. This completes the proof .

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شخوص الزمرة الجزئية للزمرة الخطية الخاصة SL(n,q)

افراح محمد ابراهيم

الخلاصة :

في هذا البحث قمنا بايجاد χ –للزمر الجزئية لشخوص غير قابلة للتحليل χ للزمرة الخطية الخاصة SL(n,q) عندما n=2, n=3 .