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DIMENSIONAL REDUCTION OF SYMMETRIES

Maruf J. Rbiea **Saad N. Abood**

College of Science - University of Anbar.

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ABSTRACT

It has been shown that ordinary (d-2)-dimensional quantum field theories are equivalent to corresponding quantum field theories defined on a (d+2) dimensional superspace with two anticommuting variables. This dimensional reduction is a consequence of super rotation invariance in the superspace. In this paper we study the generate conformal transformations in the properties of field theories which are invariant under such transformations. We show that the symmetries are dimensionally reduced.

Introduction

Quantum field theories defined on a graded manifold $\sum = \{ y^{\alpha} \} = \{ x^{\mu}, \theta, \theta' \}$ …. (1)

Where $\{ x^{\mu} \}$ is a d-dimensional Euclidean space and θ, θ' are anticommuting scalars, have been investigated [1-4]. Such theories are applicable to a variety of problems including the theories of ferromagnetic in a quenched random magnetic field [1,3,4], the problem of quark confinement [5], study of dilute branched polymers [6] and the unification of weak electromagnetic and strong interactions [7].

These super space quantum field theories were originally introduced by Parisi and Sourlas [1] to explain the equivalence between classical stochastic field theories in d-dimensions and their quantum counterparts in d-2 dimensions [8]: both of these theories are related to the corresponding superspace quantum field theories in the (d+2) dimensional super space, and the dimensional reduction $d \to d+2$ is attributed to the negative dimensionality of the two ant commuting variables θ, θ' [1, 2].

In this paper we address the question how symmetries are dimensionally reduced: We study general conformal symmetries in the super and show that if there is an invariance of the (d-2) dimensional Lagrangian, then the corresponding (d+2)-dimensional superspace Lagrangian invariant under the pertinent super space transformation.

As a consequence we find that the full symmetry structure of the superspace field theories is dimensionally reduced.

As in ref. [2] we define the inner product of two supervectors by introducing the metric tensor $g_{\alpha\beta}$:

$$g_{\alpha\beta}\, y^{\alpha} y^{\beta} = y^{\alpha} y_{\alpha} = x^2 + (4/\Delta)\theta'\theta \quad (2)$$

Note that $g_{\alpha\beta}$ is not symmetric so that care must be taken with the order of indices. We define

$$\partial_{\alpha} = (\partial/\partial x^{\mu}, \partial/\partial\theta, \partial/\partial\theta') \quad (3)$$

So that

$\partial_{\alpha} y^{\alpha} = d+2$ ………… (4) but $\partial^{\alpha} y_{\alpha} = d-2$ ..…………… (5)

In the super space we define the symplectic conformal group Osp(1, d+2) [9] by a generalization of the ordinary conformal group to the superspace of rotations translations, dilatation and special conformal transformations are taken to be:

$$\delta^{\alpha\beta} y^{\gamma} = g^{\alpha\gamma} y^{\beta} - (-)^{\alpha\beta} g^{\alpha\gamma} y^{\gamma} \quad (6a)$$

$$\delta_T^{\alpha}\, y^{\beta} = g^{\alpha\beta} \quad …………….. (6b)$$

$$\delta_D\, y^{\alpha} = -y^{\alpha} \quad ……… (6c)$$

* Corresponding author at: College of Science - University of Anbar, Iraq.E-mail address: scicolanb@yahoo.com

$\delta_C^\alpha\ y^\beta = 2y^\alpha y^\beta - g^{\alpha\beta} y^2$ (6d)

Here $(-)^{\alpha\beta}$ denotes the grading of the indices and in exhibiting these formulas we have not included the infinitesimal parameters that specify the transformations. By considering the combined actions of these infinitesimal transformations in different orders the graded Lie algebra of the symplectic conformal group can be abstracted: upon defining $R^{\alpha\beta}$, T^α, D, C^α to be the generators of superrotations, super translations, super dilatations and special super conformal transformation respectively, we find the following graded Lie algebra:

$[R^{\alpha\beta}, R^{\gamma\delta}\} = g^{\beta\gamma} R^{\alpha\delta} + (-)^{\alpha\beta}(-)^{\gamma\delta} g^{\alpha\delta} R^{\beta\gamma} - (-)^{\gamma\delta} g^{\beta\delta} R^{\alpha\gamma} - (-)^{\alpha\beta} g^{\alpha\gamma} R^{\beta\delta}$ … (7a)

$[R^{\alpha\beta}, T^\gamma\} = (-)^{\alpha\beta}(-)^{\alpha\gamma} g^{\beta\gamma} + T^\alpha - (-)^{\beta\gamma} g^{\alpha\gamma} T^\beta$ …… (7b)

$[T^\alpha, T^\beta\} = 0$ … (7c)

$[D, T^\alpha\} = -T^\alpha$ … (7d)

$[D, C^\alpha\} = C^\alpha$ …… (7e)

$[R^{\alpha\beta}, C^\gamma\} = g^{\beta\gamma} C^\alpha - (-)^{\alpha\beta} g^{\alpha\gamma} + T^\alpha C^\beta$ ……. ……… (7f)

$[T^\alpha, C^\beta\} = 2g^{\alpha\beta} D - 2R^{\alpha\beta}$ …..…………… (7g)

$[D, D\} = [D, R^{\alpha\beta}\} = [C^\alpha, C^\beta] = 0$ ………… (7h)

Where [, } denotes the graded commutator [2]. (For a related analysis see refs. [9, 10].) Before, we establish the conditions under which a superspace Lagrangian which depends on a super field ϕ is invariant under sympletic conformal transformation we must decide how the field ϕ transform. For simplicity we take ϕ to be a superscalar [2] so that

$\delta_R^{\alpha\beta}\ \phi = [y^\alpha \partial^\beta - (-)^{\alpha\beta} y^\beta \partial^\alpha]\phi$ (8a)

$\delta_T^\alpha\ \phi = -\partial^\alpha \phi$ ……… (8b)

$\partial_D\ \phi = (y^\alpha \partial_\alpha + D)\phi$ … (8c)

$\delta_C^\alpha\ \phi = (2y^\alpha y^\beta \partial_\beta - y^\beta y_\beta \partial^\alpha + 2Dy^\alpha)\phi$ … (8d)

Which is consistent with (eq.7). Here D is in general some matrix but in the case of superspace D is a commuting number that we shall call the scale dimension of ϕ. {for super vectors and suppressions [2] the pertinent super rotation terms must be added in eq.8 }.

Results and Discussion

We first discuss superspace field theories invariant under superrotations and super translations [1-4, 11]: Obviously the super laplacian

$$\Box_{ss} = \partial^\alpha \partial_\alpha\ \Box_{ss} + \Delta\partial_\theta - \partial_\theta \dots (9)$$

is a superscalar: consequently the Lagrangian

$$\delta_{ss}[\phi] = \frac{1}{2}\phi(-\Box_{ss})\phi + V[\phi]$$

……… (10)

Where $V[\phi]$ is an arbitrary local polynomial in ϕ is invariant under super rotations and super translations [1-4, 11]. The super field $\phi(x\theta\theta')$ can be expanded in terms of its component fields,

$\phi(x\theta\theta') = \phi(x) + \varphi(x)\theta' + \theta\varphi'(x) + w(x)\theta\theta'$ …. ………… (11)

If we substitute this expansion to eq.10 and integrate over θ and θ' we find the following component field representation of the superspace Lagrangian:

$$\int d\theta' d\theta \delta_{ss}[\phi] = \frac{1}{2}\Delta w^2 + w[- \quad \phi + V'(\phi)]$$

… (12)

The effect of the super rotation (eq.8a) on the component fields (eq.11) is given by a super symmetry transformation which is parameterized by an infinitesimal tensor $\sigma_{\alpha\beta}$ that characterize the pertinent super rotation: As an example a super rotation parameterized by non-vanishing $\sigma_{\mu\theta}$ and $\sigma_{\mu\theta'}$ leads to the super symmetry transformation

$$\left.\begin{aligned} \delta_R\,\phi &= -\frac{1}{2}\Delta x^{\mu}\,\sigma_{\mu\theta}\,\varphi - \frac{1}{2}\Delta x^{\mu}\,\sigma_{\mu\theta'}\,\varphi' \\ \delta_R\,\varphi &= -\frac{1}{2}\Delta x^{\mu}\,\sigma_{\mu\theta'}\,w - \partial^{\mu}\,\phi\,\sigma_{\mu\theta'} \\ \delta_R\,\varphi' &= \frac{1}{2}\Delta x^{\mu}\,\sigma_{\mu\theta}\,w + \partial^{\mu}\,\phi\,\sigma_{\mu\theta} \end{aligned}\right\} \quad \ldots\ldots(13)$$

$$\delta_R\,w = -\partial^{\mu}\,\varphi\,\sigma_{\mu\theta} - \partial^{\mu}\,\varphi'\,\sigma_{\mu\theta}$$

The invariance of the super space Lagrangian (eq.10) under this super rotation is then exposed by the invariance of the component field Lagrangian (eq.12) under the corresponding super symmetry transformation (eq.13) [1, 2].

The super translation invariance, if expressed in terms of the component fields (eq.11), also leads to a super symmetry that leaves the component field Lagrangian (eq.12) invariant [4, 11]: e.g.,

$$\delta_T\,\phi = \varepsilon_{\theta}\,\varphi' - \varepsilon_{\theta'}\,\varphi$$

$$\delta_T\,\varphi = \varepsilon_{\theta}\,w \quad , \qquad \delta_T\,\varphi' = \varepsilon_{\theta'}\,w \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

under the super dilatation the free Lagrangian

$$\delta_{ss}[\phi] = \frac{1}{2}\,\partial^{\alpha}\phi\;\partial_{\alpha}\phi \ldots\ldots (15)$$

transforms in the following way:

$$\delta_D\,\delta_{ss} = y^{\beta}\,\partial_{\beta}\,\delta_{ss} + 2(D+1)\,\delta_{ss} \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

As a consequence of (eq.4) we find that (eq.16) is a total divergence in the super space

$$\delta_D\,\delta_{ss} = \partial^{\beta}\,(y_{\beta}\,\delta_{ss}) \quad \ldots (17)$$

if we choose the scale dimension *D* to be

$$D = \frac{1}{2}(d-4) \ldots\ldots\ldots (18)$$

Consequently, eq.15 is invariant under super dilatations. Recall, that in the case of conventional field theories the scale dimension of a scalar field is $\frac{1}{2}(d-4)$. Hence the scale dimension has been dimensionally reduced, $d+2 \rightarrow d-2$, As a consequence of this dimensional reduction of the scale dimension we conclude that if an ordinary interacting (*d*-2) dimensional field theory is dilatation invariant, the corresponding (*d*+2) dimensional superspace field theory is invariant under super dilatations a direct consequence of the relation (eq.4). for example, the Lagrangian

$$\delta_{ss}[\phi] = \frac{1}{2}\,\partial^{\alpha}\,\phi\,\partial_{\alpha}\,\phi + (\lambda/4)\phi^4 \quad (19)$$

is super dilatation invariant in the $(\sigma+2)$-dimensional super space.

In the case of ordinary field theories the scale dimension of a field agrees with its canonical dimension of a field agrees with its canonical dimension. However, due to the negative dimensions of the anticommuting coordinates θ and θ' this is not true in the case of superspace field theories: for example in the $(\sigma+2)$-dimensional super space the canonical dimension of the component fields (eq.11) is two. But if we express (eq.8C) in terms of the component fields, we find

$$\left.\begin{aligned} \delta_D\,\phi &= (x^{\mu}\,\partial_{\mu} + 1)\,\phi \\ \delta_D\,\varphi &= (x^{\mu}\,\partial_{\mu} + 2)\,\varphi \\ \delta_D\,\varphi' &= (x^{\mu}\,\partial_{\mu} + 2)\,\varphi' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (20)$$

$$\delta_D\,w = (x^{\mu}\,\partial_{\mu} + 3)\,w$$

Hence only for the anticommuting fields φ, φ' are the canonical and scale dimension equal.

We now consider field theories invariant under the superspace special conformal transformation (eq.8d). In this case a straight forward computation that uses the relations (eq.4 and eq.5) reveals that if an interacting (d-2)-dimensional ordinary space Lagrangian is information, then the corresponding superspace Lagrangian (eq.10) is invariant under the special super conformal transformation (eq.8d). As an example the Lagrangian (eq.19) is invariant under the transformation (eq.8d) in $(\sigma+2)$ dimensional superspace.

The differential representation (eq.8d) of the spatial super conformal transformation mixes commuting and anticommuting variables. Consequently, if we substitute the component field expansion (eq.11) into (eq.8d) we find that there is super symmetry transformation of the component fields akin the super rotation (eq.13) and super translation (eq.14). The invariance of the superspace Lagrangian under special super conformal transformations is equivalent to the invariance of the corresponding component field Lagrangian under the pertinent super symmetry transformation. For example, if we parameterize the transformation (eq.8d) with the super vector

$$v_\alpha = (0, \ldots, a, 0) \ldots\ldots (21)$$

In a $(\sigma+2)$-dimensional super space, with a an anticommuting constant we find the super symmetry

$$\left.\begin{aligned} \delta_C\,\phi &= -\frac{1}{2}\,\Delta x^2\, a\varphi \\ \delta_C\,\varphi' &= 2\,a x^\mu\,\partial_\mu \phi + 2a\phi + \frac{1}{2}\Delta x^2\, a\, w \end{aligned}\right\rbrace$$

$$\ldots\ldots\ldots\ldots (22)$$

$$\delta_C\, w = 2\,a x^\mu\,\partial_\mu \varphi + 6a\varphi$$

Which leaves the component field representation of (eq.19)

$$\delta = -\frac{1}{2}\Delta w^2 + w[- \quad \phi + (\lambda/3!)\phi^3] + \varphi'[- \quad + (\lambda/2!)\phi^2]\,\varphi$$

…… (23)

invariant.

Finally, we discuss shortly the effects of finite sympletic conformal transformations: The finite special super conformal is

$$y^\alpha \to (y^\alpha + c^\alpha\, y^2)/(1+2c\,y+c^2\,y^2) \ldots\ldots (24)$$

For large c^2 we have

$$y^\alpha \to c^\alpha/c^2 + (1+c^2 y^2)(g^{\alpha\beta} - 2c^\alpha c^\beta/c^2)\,y_3 + O(1/c^3)$$

……… (25)

Hence we conclude that invariance under superspace inversions

$$y^\alpha \to (1/y^2)\,y^\alpha \ldots\ldots\ldots (26)$$

is a consequence of invariance under superrotations, super translations and special super conformal transformations. Notice that we indeed can define the inverse of y^2 in the superspace

$$y^2\,(1/y^2) = [x^2 + (4/\Delta)\theta'\theta][1/x^2 - (4/\Delta)(1/x^4)\theta'\theta] = 1$$

… (27)

even though the inverse of the coordinates θ and θ' does not exist.

Consequently we have shown that the inversion invariance is dimensionally reduced. This can also be verified by a direct calculation. Let us first consider a free field theory $\delta = \frac{1}{2}\,\phi\,(- \quad _{ss})\,\phi$ … (28)

The exact two point function is [2]

$$\langle \phi(y)\,\phi(0)\rangle = \frac{1}{(2\pi)^d}\int d^d\,k\,d\alpha'\,d\alpha\,\frac{\exp(i\,y^\beta P_\beta)}{k^2 + \Delta\alpha'\alpha} \ldots\ldots\ldots$$

………… (29)

Where we have used the super translation invariance of (eq.28). Under an inversion (eq.26) the R.H.S of (eq.29) transforms into

$$\frac{1}{(2\pi)^d}\,(y^2)^{d-2}\int d^d\,k\,d\alpha'\,d\alpha\,\frac{\exp(i\,y^\beta P_\beta)}{k^2 + \Delta\alpha'\alpha} \ldots\ldots$$

(30)

The dimensional reduction $d+2 \to d-2$ is thus a consequence of the properties of the Jacobian determinant in the superspace: The two-point function (eq.29) and hence also the Lagrangian (eq.28) is inversion invariant is the superspace if we define the super field $\phi(y)$ to transform in the following way

$$\phi(y^{\alpha}) \to (1/y^2)^{(d/2)-1}\, \phi(1/y^2)\, y^{\alpha} \quad (31)$$

This result also follows from fact that under an inversion the superspace integration measure transforms as

$$d^d x\, d\theta'\, d\theta = d^{d+2} y \to (1/y^2)^{d-2} d^{d+2} y \ldots\ldots\ldots$$

(32)

Using (eq.31) and (eq.32) we can characterize inversion invariant interactions in the superspace. Combining these results we conclude that if an interacting Lagrangian in (d-2)-dimensional ordinary space is inversion invariant, then the superspace inversion. As an example, the Lagrangian (eq.19) is indeed inversion invariant in the $(\sigma+2)$ dimensional superspace.

Conclusion

We have shown that for any invariance of the (d-2) dimensional field theory. there is a corresponding invariance of the (d-2) dimensional superspace field theory. Hence there is also an invariance of the corresponding Stochastic differential equation in d-dimensions [2]. As a consequence, the dimensional reduction $d+2 \to d-2$ goes for beyond the simple diagrammatic equivalence. In fact, all nother conserved currents and charges defined for a (d-2)-dimensional field theory have their $d+2$ -dimensional counterparts, hence also, d-dimensional stochastic counterparts exist [2]. Thus the formulation of quantum field theories in terms of stochastic classical differential equations and Nicolai transformations [2,12] is completely equivalent to conventional approaches.

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الاختزال البعدي للتماثلات

معروف جميل ربيع سعد ناجي عبود

E.mail: scicolanb@yahoo.com

الخلاصة:

في هذا البحث تمت دراسة نظريات المجال لكمي الاعتيادية للبعد (d-2) بأنها تكافئ نظريات المجال الكمي المعرفة على البعد (d+2) للفضاء الأمثل على متغيرين غير متبادلين، حيث أن الاختزال البعدي هو نتيجة للدوران الأمثل غير المتغير في الفضاء الأمثل. إضافة إلى دراسة التحويلات المتطابقة في الفضاء المثل وخصائص نظريات المجال الكمي تحت هذه التحويلات.