P- ISSN 1991-8941 E-ISSN 2706-6703 2007,(1), (1) :71-75 *Journal of University of Anbar for Pure Science (JUAPS)* Open Access

ON THE NUMBER AND EQUIVALENT LATIN SQUARES

MAKARIM A. AL-TURKY

Computer College - Univesity of Al-Anbar

ARTICLE INFO

Received: 25 / 8 /2006
Accepted: 1 / 3 /2007
Available online: 14/06/2012
DOI: 10.37652/juaps.2007.15428

Keywords:
Number .
Equivalent Latin Squares.

ABSTRACT

We determine the number of Latin rectangles with 11 columns and each possible number of rows, In clouding the Latin squares of order11. Also answer some questions of Alter by showing that the number of reduced Latin squares of order n is divisible by Fi where f is a particular integer close to $\frac{1}{2}n$.

Introduction:

A Kxn Latin rectangle is a Kxn matrix with entries from {1,2,……n} such that the entries in each row and the entries in each column are distinct.

A Latin square of order n is an nxn Latin rectangle. This essay describes some mathematical structures "equivalent " to Latin squares.

A cording to the Handbook of combinatorial Design [2] Latin square of order n is equivalent to

1- the multiplication table (clayey table) of a group on n elements.

2- a single error detecting code of word length 3, with n^2 words from an n-symbol alphabet.

A Latin square is said to be reduced (or normalized) if its first row and first column are in natural order. For example the Latin square Latin

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

square L is reduced because both its first row and its first column are 1,2,3. we can make any Latin square reduced by permuting the rows and permuting the columns.

* Corresponding author at: Computer College - Univesity of Al-Anbar, Iraq.E-mail address: mak_alturky@yahoo.com

If we permute the rows, permute the columns and permute the names of the symbols of a Latin square, we obtain a new Latin square said to be isotopic to the first L and the set of all squares isotopic to L is called an isotopy class. In the special case when the same permutation is applied to the rows, columns and symbols we say that the isotopism is an isomorphism. An isotopism that maps L to itself is called an autotopism of L and any autotopism that is an isomorphism is called an automotphism . The number of isomorphism classes, isotopy classes and main classes has been determined by Mchay, Meynert and Myrvold [6]

Producing all possible equivalence Latin squares

Isotopism is an equivalence relation, so the set of all Latin squares is divided into subsets called isotopy classes, such that two squares in the same class are isotopic and two squares in different classes are not isotopic. If each entry of nxn Latin square is written as a triple (r,c,s), where r is row, c is column, and s is the symbol. We obtain aset of n^2 triples.

so, we can replace each triple (r,c,s) by (c,r,s) or (c,s,r). Altogether there are 6 possibilities giving us 6 Latin square are called conjugates of the original square.

two Latin squares are said to be paratopic (main class isotopic) if one of them is isotopic to a conjugate of the other.

Each main class contain up to isotopy classes.

For each n the number of Latin square is n!(n-1)! and each isotopy class contains upto $(n!)^3$ Latin squares.

2-1 Proposition

(The number Ln of Latin squares of order n, is given by the formula:

Ln= n!(n-1)!Ln ,n≥2 where Ln is the number of reduced Latin squares.

Proof: There are Ln reduced Latin squares of order n. A reduced Latin square of order n is one where both the first row and first column are ordered 0,1,…,n-1.

The columns of each reduced Latin square can be inter changed in n! ways, resulting in a Latin square. Each square formed from each permutation is unique.

After the permutation of the columns, the bottom (n-1) rows can be permuted in (n-1)! ways, yielding Latin squares that are distinct from each other and from those obtained by the column permutations.

This is because the first row is not permuted Hence, the number of Latin squares of order n is given by Ln=n! (n-1)! Ln.□

3- Description of the computations.

Let *R* be a k x n Latin rectangle. We can define an associated K-regular bipartite graph G=G (*R*) thus: V (G)= C U S, where C = { $c_1,c_2,\ldots,c_n$} and S = { $s_1,s_2,\ldots,s_n$} and E (G) = {$c_i.s_j$ | column i contains symbol j}. We will call this graph the template of R. Clearly, many Latin rectangles may have the same template; for example, every Latin square of order n has the complete bipartite graph $K_{n,n}$ as its template.

A one – factor of a graph G is a spanning regular subgraph of degree one. A one-factorization of G is a partition of E(G) into one-factors. Clearly, the rows of a Latin rectangle R correspond to the one-factors in a one-factorization of G(R). For any template G, define N(G) to be the number of one-factorizations of G, or equivalently the number of normalized Latin squares with template G. In forming this count, one-factorizations which differ only in the order of the one-factors are not counted separately. The value of N(G) can be found from the following recursion.

$$N(G)=\sum_F N(G-F) \qquad \ldots\ldots (1)$$

Where the sum is over one-factors F of G which contain some fixed edge of G. The feasibility of this computation for n=10 is due to the fact that N (G) is an invariant of the isomorphism class of G. thus we need only apply (1) to one member of each isomorphism class. The challenge with efficiency is that the templates on the right need to be identified according to which isomorphism class they belong.

The two computations differed in the types of isomorphism recognized between two templates. In the first computation, isomorphism's fixing the sets C and S were used, while in the second the exchange of C and S was also permitted. In order to apply recursion (1), it is necessary to be able to identify G- F from amongst the templates for to applay recursion (1), it is necessary to be abe to identify G – F from amongst the templates for which, the value of N( ) is already known. in the first computation, this was achieved by defining a canonical labeling for templates. templates were stored in canonical form, and templates G-F were identified by, converting them to canonical form. in the second computation, a

P- ISSN 1991-8941 E-ISSN 2706-6703 Journal of University of Anbar for Pure Science (JUAPS) Open Access
2007,(1), (1) :71-75

combinatorial invariant was devised such that no two templates had the same invariant. the invariant had tow components. The first component was a quickly-computed number depending on the distribution of the cycles of length 4 in G- F. This proved sufficient to identify the great majority of templates uniquely. for those not uniquely identified, there was a second component formed from a canonical labeling of the template, using the first author's graph isomorphism program natuy [11]. The set of nonismorphic template, was determined in advance using nautty.

The number of distinct templates under the two definitions of equivalence for n= 10 and k=1,……..,5 were 1, 12, 1165, 121790, 601055 for the first computation, and 1,12, 725, 62616, 304496 for the second computation.

When N (G) is known for each template G, the number of normalized Latin rectangles can be determined. in terms of the second computation, we have

$$L(k,n)=2\,n\,k!\,(n-k)!\sum_{G}\frac{N(G)}{|Aut\,(G)|} \quad (2)$$

where the sum is over all templates of degree k, and Aut(G) is the automorphism group of G.

The reason for (2) is that 2n!k! (n-k)! /|Aut(G)| is the number of labellings of G in which the neighbors of c_1 are $\{s_1,\ldots,s_k\}$, and (n-1)! is removed to allow for normalization of the first row. In the case of k=n, equation (2) simplifies to $L(n,n) = N(K_{n,n})/ (n-1)!$.

3-1 Theorem: The number of reduced kxn Latin rectangles is given by

$L_{k,n}$=2nk! (n-k)!∑N(G).|Aut(G)|

Theorem: The number of reduced Latin squares of order n is given by

L_n=2nk!(n-)!∑N(G).N(G).$|Aut(G)|^{-1}$

Where G is the bipartite complement of G and K is any integer in the rang 0≤K≤n

3-3 Theorem: let G Є G(k,n) for K≥1 let e be an arbitrary edge of G Then

$$N(G) = \sum_F N(G-F)$$

Where the sum is over all 1-factors F of G that include e.

4-Some divisibility properties of L_n we have the following simple divisibility properties.

4-1 Theorem: For each integer n≥1

1. L_{2n+1} is divisible by gcd (n!(n-1)! L_n,(n+1)!)
2. L_{2n} is divisible by n!.

Proof: consider L_{2n+1} first. we define an equivalence relation on reduced Latin squares of order 2n+1 such that each equivalence class has size either n!(n-1)!L_n or (n+1)!.

Let A be the leading principal minor of $L=(l_{ij})$ of order n. If A is a (reduced) Latin subsquare, then the squares equivalent to L are those obtainable by possibly replacing A by another reduced subsquare, permuting the n partial rows (li, $_{n+1}$, li, $_{n+2}$,…..,l_{2n+1}, j) for 1 ≤ i ≤ n. permuting the n-1 partial columns (l_{n+1}, j, l_{n+2}, j,…….l_{2n+1},j) for 2 ≤ j ≤ n Then permuting columns n+1, n+2,….2n+1 to put the first row into natural order. these n! (n-1) Ln operations are closed under composition and give different reduced Latin squares, so each equivalence class has size n! (n-1)! Ln.

If A is not a Latin subsquare, the squares equivalent to L are those obtainable by applying one of the (n+1)! isomorphism's in which the underlying permutation fixes each of the points 1,2,….n.

No isomorphism of this form can be an automorphism of a square in which A is not a subsquare.

See [6,theorem 1]. Hence the squares obtained are different and the equivalence class has (n+1)! elements.

The case of L_{2n} is the same except the second argument gives n! instead of (n+1)!.

4-2 Corollary: If n=2p-1 for some prime P, then L_n is divisible by [(n-1)/2]!. otherwise, L_n is divisible by [(n+1)/2]!.

Proof: This follows from table 1 for n≤8.

For n ≥ 9, note that m|(m-2)|! for m>4.

Note that, for n≥ 12, the corollary gives the best divisor that can be inferred from table 1 and theorem (4.1).

except that L_{13} is divisible by 7! and not merely by 6!.

Alter [1] (see also Mullen [2]) asked whether an increasing power of two divides L_n as n increases and whether L_n is divisible by 3 for all n≥6.

Theorem (4.1) answers both these questions in the affirmative.

References.

[1]. R. Alter, How many Latin squares are there? Amer. Math. Monthly, 82(1975) 632-634.

[2]. C. J. Colbourn and J. Dinitz (editors), CRC Handbook of combinatorial Design, CRC Press, Boca Raton, 1996.

[3]. G. L. Mullen, How many i-j reduced Latin squares are there? Amer.Math. Monthly,85 (1978) 751-752.

[4]. B. D. McKay and E.Rogoyski, Latin squares of order 10, Electronic J. combina-torics, 2 (1995)# N3 (4eP).

[5]. Charles F. Laywin and Gary L. Mullen, "ABrief Introdaction to Latin squares" in Discrete Mathematics uing Latin squares, 4-5 (New York: Johnwiley & Sons, Inc, 1998).

[6]. B. D. McKay, A. Meynert and W. Myrvold, small Latin squares. quasi groups and Loops, Submitted.

[7]. S. E. Bammel and J. Rothstein, the number of 9x9 Latin squares, Discrete Math., 11 (1975)93-95.

[8] Brendan D. McKay and I an M. wanless." on the number of Latin squares' Australian National University.

[9]. G. Kolesova, C.W. H. Lam and Thiel, on the number of 8x8 Latin squares, J combinatorial theory , ser .A, 54 (1990)143-148.

[10]. Jia-yashao and wan-di wei, a formula for the number of Latin squares, Discrete Math. 110 (1992) 293-296.

[11]. B. D. McKay, nauty user's guide (version 1-5) Technical report TR-CS-90-20, computer science Dept. Australian National university, 1990.

Table 1: Reduced Latin rectangles

1 1 1 2 1 1 2 1 3 1 1 2 1 3 1 4 1 1 2 2 3 3 4 4	9 1 1 2 16687 3 103443808 4 207624560256 5 112681643083776 6 12952605404381184 7 224382967916691456 8 377597570964258816 9 377597570964258816
5 1 1 2 11 3 46 4 56 5 56	10 1 1 2 148329 3 8154999232 4 147174521059584 5 746988383076286464 6 870735405591003709440 7 177144296983054185922560 8 4292039421591854273003520 9 7580721483160132811489280 10 7580721483160132811489280
6 1 1 2 53 3 1064 4 6552 5 9408 6 9408 7 1 1 2 309 3 35792 4 1293216 5 11270400 6 16942080 7 16942080 8 1 1 2 2119 3 1673792 4 420909504	10 1 2 1468457 3 798030483323 4 143968880078466048 5 7533492323047902093312 6 962995552373292505158778880 7 240123216475173515502173552640 8 6810820435778726678085834375168 0 9 290599031003388269311398902759424 0 10 5363937773277371298119673540771840 11 5363937773277371298119673540771840

5 27206658048 6 335390189568 7 535281401856 8 535281401856	

المربعات أللاتينية المتكافئة وإعدادها

مكارم عبدالواحد عبدالجبار

Email: ***mak_alturky@yahoo.com***

الخلاصة :

نحدد عدد المستطيلات ألاتينية المتكونة من ١١ عمود وعدد الصفوف المحتملة والتي تتضمن المربعات اللاتينية من الرتبة ١١. كذلك قام العالم Alter (١) بالإجابة على بعض الأسئلة عن طريق ملاحظة الإعداد من المربعات اللاتينية القياسية من الرتبة n هي قابلة للقسمة من fi بحيث f هي صحيح خاص مغلق لقيمة $\frac{1}{2}n$.