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On Dimension Theory by Using $N-Open$ Sets

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Abstract

In this paper we discuss new type of dimension theory by using $N-open$ sets. We the concept of $indX, IndX, dimX,$ for a topological space X have been studied. In this work, these concepts will be extended by using $N-open$ sets.
Key words: $indX, IndX, dimX,$

الخلاصة

في هذا البحث نناقش نوع جديد لنظرية البعد بأستخدام المجموعات المفتوحة من النمط N لقد درسنا المفاهيم $indX, IndX, dimX,$ للفضاءات التبولوجية X في هذا العمل سوف نوسع هذه المفاهيم بأستخدام المجموعات المفتوحة من النمط N.
الكلمات المفتاحية: $indX, IndX, dimX$

1. Introduction

Dimension theory starts with "dimension function" which is a function defined on the class of topological spaces such that $d(X)$ is an integer or∞, with the properties that $d(X) = d(Y)$ if X and Y are homeomorphism and $d(R^n) = n$ for each positive integer n. The dimension functions taking topological spaces to the set$\{-1,0,1,\dots\}$. The dimension functions $ind, Ind, dim,$ were investigated by [Pears ,1975]. Actually the dimension functions, $S-indX, S-IndX, S-dimX$ by using $S-open$ sets were studied in [Raad Aziz Hussain AL-Abdulla,1992], also the dimension functions, $b-indX, b-IndX, b-dimX,$ by using $b-open$ sets were studied in [Sama Kadhim Gabar,2010], and the dimension functions, $f-indX, f-IndX, f-dimX,$ by using $f-open$ sets were studied in [Nedaa Hasan Hajee ,2011]. In this paper we recall the definitions of ind, Ind, dim, from [Pears ,1975], then the dimension functions, $N-ind, N-Ind, N-dim$ are introduced by using $N-open$ sets. Finally some relations between them are studied and some results relating these concepts are proved.

2. Preliminaries

In this section, we recall some of the basic definitions.

Definition 2.1[Omari, and Noorani,2009]: A sub set A of a space X is said to be an $N-open$ if for every $p \in A$ there exist an open sub set U_p in X such that $U_p - A$ is a finite set. The complement of an $N-open$ set is said to be $N-closed$.

Remark 2.2: 1. Every open set is an $N-open$ set.

2. Every closed set is an $N-closed$ set.

The converse of (1) and (2) is not true in general as the following example shows:

Let Z be the set of integer numbers and T be indiscrete topology on Z, then $Z-\{2,3\}$ is an $N-open$ set, but its not an open set and $B=\{2,3\}$ is an $N-closed$ set, but not a closed set.

Remark 2.3: The family of all $N-open$ sub set of a space (X,T) is denoted by T^N.

Theorem 2.4[Omari, and Noorani,2009]: Let X be a topological space, then X with the set of all $N-open$ sub set of X is a topological space.

Corollary 2.5[Omari, and Noorani,2009]: Let X be a topological space, then the intersection of an open set and an $N-open$ set is an $N-open$ set.

Remark 2.6[Hamza and Majhool,2011]: Let X be a space and Y be a sub space of X such that $A \subseteq Y$, if A is $N-open$ sub set in X then A is $N-open$ in Y.
Proposition 2.7[Hamza and Majhool,2011]: 1. Let X be a space and Y be an $N-open$ of X, if A is $N-open$ in Y then A is an $N-open$ in X.
2. Let X be a space and Y be a sub set of X if B is an $N-open$ in X then $B \cap Y$ is $N-open$ in Y.
Definition 2.8[Hashmiya Ibrahim Nasser,2012]: A space X is called $NT_1-space$ if and only if for each $x \neq y$ in X, there exists disjoint $N-open$ sets U and V such that $x \in U, y \notin U$ and $y \in V, x \notin V$.
Remark 2.9: It is clear that every $T_1-space$ is $NT_1-space$ but the converse is not true in general, as the following example shows: Let $X=\{1,2,3\}, T=\{X,\emptyset,\{1\},\{2\},\{1,2\}\}$, the $N-open$ set is $\{X,\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$, It is clear to see that X is $NT_1-space$ but is not $T_1-space$.
Proposition 2.10[Hashmiya Ibrahim Nasser,2012]: Let X be a topological space, and then X is $NT_1-space$ if and only if $\{p\}$ is $N-closed$ set for each $p \in X$.
Definition 2.11[Hashmiya Ibrahim Nasser,2012]: A space X is called $N-Hausdorff$ if and only if any two distinct points of X has disjoint an $N-open$ neighborhoods.
Remark 2.12: Every $Hausdorff$ space is $N-Hausdorff$. But the convers is not true in general.
Definition 2.13: A space X is said to be $N-regular$ space if and only if for each $p \in X$ and C closed sub set such that $p \notin C$, there exist disjoint $N-open$ sets U,V in X such that $p \in U, C \subseteq V$.
Definition 2.14: A space X is said to be $N^*-regular$ space if and only if for each $p \in X$ and C $N-closed$ sub set such that $p \notin C$, there exist disjoint $open$ sets U,V in X such that U open set, V is an $N-open$ set and $p \in U, C \subseteq V$.
Remark 2.15: 1. Every regular space is $N-regular$ space but the convers is not true.
2. Every $N^*-regular$ space is $N-regular$ space but the convers is not true.
As the following example shows: let $X=\{1,2,3\}, T=\{X,\emptyset,\{1\},\{2\},\{1,2\}\}$, the $N-open$ set is $\{X,\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$, It is clear to see that X is $N-regular$ space, but X is not regular since $\{2,3\}$ is closed set, $1 \notin \{2,3\}$ and there exist no disjoint two open set U,V such that $1 \in U, \{2,3\} \subseteq V$. Also X is not $N^*-regular$, since $\{1,2\}$ is $N-closed$ set and $3 \notin \{1,2\}$, but there exist no disjoint open set U and $N-open$ set V such that $3 \in U, \{1,2\} \subseteq V$.
Definition 2.16: A space X is said to be $N-normal$ space if and only if for every disjoint closed sets C_1, C_2 there exist disjoint $N-open$ sets V_1, V_2 such that $C_1 \subset V_1, C_2 \subset V_2$.
Definition 2.17: A space X is said to be $N^*-normal$ space if and only if for every disjoint $N-closed$ sets C_1, C_2 there exist disjoint $open$ sets V_1, V_2 such that $C_1 \subset V_1, C_2 \subset V_2$.
Definition 2.18: A space X is said to be $N^{**}-normal$ space if and only if for every disjoint $N-closed$ sets C_1, C_2 there exist disjoint $N-open$ sets V_1, V_2 such that $C_1 \subset V_1, C_2 \subset V_2$.
Remark 2.19: 1. Every normal space is $N-normal$ but the convers is not true.
2. Every $N^*-normal$ space is normal but the convers is not true.
Definition 2.20[N.Burbaki,1989]: A topological space X is said to be compact if every open cover of X has a finite sub cover.
Definition 2.21[S.H.Hamza and F.M.Majhool,2011]: A topological space X is said to be $N-compact$ if every $N-open$ cover of X has a finite sub cover.

Lemma 2.22[Maheshwari, S.N.and Thakur,S.S., 1985]: 1. Every closed sub set of compact is compact.
2. Every $N-closed$ sub set of $N-compact$ is compact.
Definition 2.23: Let X be a set and A a family of sub sets of X, by the order of the family A we mean the largest integer n such that the family A contains $n+1$ sets with a non-empty intersection, if no such integer exists, we say that the family A has order ∞. The order of a family A is denoted by ord A.
Definition 2.24: Let X be a topological space the family β of $N-open$ sets is called a $N-base$ if and only if for each $N-open$ set a union of members of a family β.
Definition 2.25[S.H.Hamza and F.M.Majhool,2011]: Let X be a space and $A \subseteq X$. The intersection of all $\mathrm{N-closed}$ sets of X contained in A is called the $\mathrm{N-closure}$ of A and is denoted by $\overline{A}^{N}$.

3. On ind by using $N-open$ sets

Definition 3.1: The $N-$small inductive dimension of a space$X$, $N-indX$, is defined inductively as follows. A space X satisfies $N-indX=-1$ if and only if X is empty. If n is a non-negative integer, then $N-indX \le n$ means that for each point p of X and each open set G such that $p \in G$ there exists an $N-open$ set U such that $p \in U \subset G$ and $N-indb(U) \le n-1$. We put $N-indX=n$ if it is true that $N-indX \le n$, but it is not true that $N-indX \le n-1$. If there exists no integer n for which $N-indX \le n$ then we put $N-indX=\infty$.
Definition 3.2: The N^*-small inductive dimension of a space$X$, N^*-indX, is defined inductively as follows. A space X satisfies $N^*-indX=-1$ if and only if X is empty. If n is a non-negative integer, then $N^*-indX \le n$ means that for each point p of X and each $N-open$ set G such that $p \in G$ there exists an $N-open$ set U such that $p \in U \subset G$ and $N^*-indb(U) \le n-1$. We put $N^*-indX=n$ if it is true that $N^*-indX \le n$, but it is not true that $N^*-indX \le n-1$. If there exists no integer n for which $N^*-indX \le n$ then we put $N^*-indX=\infty$.
Proposition 3.3: Let X be a topological space. If $indX$ is exists then $N-indX \le indX$.
Proof: By induction on n. It is clear $n=-1$. Suppose that it is true for $n-1$. Now suppose that $indX \le n$, to prove $N-indX \le n$, let $p \in X$ and G is an open set in X such that $p \in G$ since $indX \le n$, then there exists an open set U in X such that $p \in U \subset G$ and $indb(U) \le n-1$ and since every open set is $N-open$ set then U is an $N-open$ set such that $p \in U \subset G$ and $N-indb(U) \le n-1$. Hence $N-indX \le n$.■
Theorem 3.4: Let X be a topological space, then $indX=0$ if and only if $N-indX=0$.
Proof: By proposition (3.3). If $indX=0$ then $N-indX \le 0$, and since $X \ne \emptyset$ then $N-indX=0$.
Now
Let $N-indX=0$ and Let $p \in X$ and each open set $G \subset X$ of the point p, since $N-indX=0$ then there exists an $N-open$ set $U \subset X$ such that $p \in U \subset G$ and $N-indb(U) \le -1$. Then $b(U)=\emptyset$, therefore U is both open and closed, and thus $indb(U)=-1$. So that $indX \le 0$ and since $X \ne \emptyset$ then $indX=0$. ■
Remark 3.5[A.P.Pears ,1975]: Let X be a topological space with $indX=0$ then X is a regular space.
Corollary 3.6: Let X be a topological space, if $N-indX=0$ then X is a regular space.
Remark 3.7[A.P.Pears ,1975]: A space X satisfies $indX=0$ if and only if it is not empty and has a base for its topology which consists of open and closed sets.

Corollary 3.8: A space X satisfies $N-indX=0$ if and only if it is not empty and has a base for its topology which consists of open and closed sets.
Proposition 3.9[A.P.Pears ,1975]: For every sub space A of a space X, we have $indA \leq indX$.
Theorem 3.10: For every sub space A of a space X and A is open, we have

$$N-indA \leq N-indX$$

Proof: By induction on n. If $n=-1$ then theorem is true. Suppose that the theorem is true for $n-1$.
Now
Suppose that $N-indX \leq n$ to prove $N-indA \leq n$. Let $p \in A$ and G is an open set in A such that $p \in G$. Since G is open set in A then there exists U is an open set in X such that $G=U \cap A$. Since $p \in A$ and $N-indX \leq n$ then there exists an $N-open$ set W in X such that $p \in W \subset U$ and $N-indb(W) \leq n-1$. Let $V=W \cap A$ is $N-open$ in A, by proposition(2.7).
Thus $p \in V=W \cap A \subset U \cap A=G$ to prove $N-indb_A(V) \leq n-1$ *then* $N-indA \leq n$.

$$\begin{aligned} b_A(V) &\subseteq b(V) \cap A=(\bar{V}-V^{\circ}) \cap A \subset (\bar{W}-V^{\circ}) \cap A=\left(\bar{W} \cap V^{\circ^C}\right) \cap A \\ &=\left[\bar{W} \cap \left(W^{\circ^C} \cup A^{\circ^C}\right)\right] \cap A=\left[\left(\bar{W} \cap W^{\circ^C}\right) \cup \left(\bar{W} \cap A^{\circ^C}\right)\right] \cap A \\ &\subset \left(b(W) \cup A^{\circ^C}\right) \cap A=(b(W) \cap A) \cup \left(A^{\circ^C} \cap A\right) \subset b(W). \end{aligned}$$

Thus $N-indb_A(V) \leq N-indb(W)$. Since $N-indb(W) \leq n-1$ then $N-indb_A(V) \leq n-1$ (By induction). Therefore $N-indA \leq n$. ■

4. On Ind by using $N-open$ sets

Definition 4.1: The $N-$large inductive dimension of a space X, $N-IndX$, is defined inductively as follows. A space X satisfies $N-IndX=-1$ if and only if X is empty. If n is a non-negative integer, then $N-IndX \leq n$ means that for each closed set E and each open set G of X such that $E \subset G$ there exists an $N-open$ set U such that $E \subset U \subset G$ and $N-Indb(U) \leq n-1$. We put $N-IndX=n$ if it is true that $N-IndX \leq n$, but it is not true that $N-IndX \leq n-1$. If there exists no integer n for which $N-IndX \leq n$ then we put $N-IndX=\infty$.
Definition 4.2: The N^*-large inductive dimension of a space X, N^*-IndX, is defined inductively as follows. A space X satisfies $N^*-IndX=-1$ if and only if X is empty. If n is a non-negative integer, then $N^*-IndX \leq n$ means that for each $N-closed$ set E and each open set G of X such that $E \subset G$ there exists an $N-open$ set U such that $E \subset U \subset G$ and $N^*-Indb(U) \leq n-1$. We put $N^*-IndX=n$ if it is true that $N^*-IndX \leq n$, but it is not true that $N^*-IndX \leq n-1$. If there exists no integer n for which $N^*-IndX \leq n$ then we put $N^*-IndX=\infty$.
Definition 4.3: The $N^{**}-$large inductive dimension of a space X, $N^{**}-IndX$, is defined inductively as follows. A space X satisfies $N^{**}-IndX=-1$ if and only if X is empty. If n is a non-negative integer, then $N^{**}-IndX \leq n$ means that for each $N-closed$ set E and each $N-open$ set G of X such that $E \subset G$ there exists an $N-open$ set U such that $E \subset U \subset G$ and $N^{**}-Indb(U) \leq n-1$. We put $N^{**}-IndX=n$ if it is true that $N^{**}-IndX \leq n$, but it is not true that $N^{**}-IndX \leq n-1$. If there exists no integer n for which $N^{**}-IndX \leq n$ then we put $N^{**}-IndX=\infty$.
Proposition 4.4: Let X be a topological space, if $IndX$ is exist then $N-IndX \leq IndX$.
Proof: By induction on n. It is clear $n=-1$. Suppose that it is true for $n-1$. Now suppose that $IndX \leq n$, to prove $N-IndX \leq n$, let C be a $closed$ set in X and G is an open set in X such that $C \subset G$ since $IndX \leq n$, then there exists an open set U in X such that $C \subset U \subset G$ and $Indb(U) \leq n-1$ and since every open set is $N-open$ set then U is an $N-open$ set such that $C \subset U \subset G$ and $N-Indb(U) \leq n-1$. Hence $N-IndX \leq n$.■

Proposition 4.5: Let X be a topological space then:
1. If $N^{**}-IndX$ is exist then $N^{*}-IndX \leq N^{**}-IndX$.
2. If $N^{*}-IndX$ is exist then $N-IndX \leq N^{*}-IndX$.

Theorem 4.6: Let X be a topological space, then $IndX=0$ if and only if $N-IndX=0$.

Proof: By proposition (4.5). If $IndX=0$ then $N-IndX \leq 0$, and since $X \neq \emptyset$ then $N-IndX=0$.

Now

Let $N-IndX=0$ and Let F is closed set in X and each open set G in X such that $F \subset U$. Since $N-IndX=0$ then there exists an $N-open$ set U in X such that $F \subset U \subset G$ and $N-Indb(U)=-1$. Then $b(U)=\emptyset$, therefore U is both open and closed, and thus $Indb(U)=-1$. So that $IndX \leq 0$ and since $X \neq \emptyset$ then $IndX=0$. ■

Remark 4.7[A.P.Pears ,1975]: Let X be a topological space with $IndX=0$ then X is a normal space.

Corollary 4.8: Let X be a topological space with $N-IndX=0$ then X is a normal space.

Proof: It follows from theorem (4.6).

Remark 4.9[A.P.Pears ,1975]: A space X satisfies $IndX=0$ if and only if it is non-empty and has a base for its topology which consist of open and closed sets.

Corollary 4.10: A space X satisfies $N-IndX=0$ if and only if it is non-empty and has a base for its topology which consist of open and closed sets.

Proposition 4.11[A.P.Pears ,1975]: Let X be a topological space, if A is a closed sub set of a space X we have $IndA \leq IndX$.

Theorem 4.12: Let X be a topological space, if A is a closed and open sub set of a space X, we have

$$N-IndA \leq N-IndX.$$

Proof: By induction on n. If $n=-1$ then theorem is true. Suppose that the theorem is true for $n-1$.

Now

Suppose that $N-IndX \leq n$ to prove $N-IndA \leq n$. Let F is a closed set in A and G is an open set in A such that $F \subset G$. Since F is a closed set in A and A is a closed set in X then F is a closed set in X. Since G is open set in A then there exists U is an open set in X such that $G=U \cap A$. Since $F \subset U$ and $N-IndX \leq n$ then there exists an $N-open$ set W in X such that $F \subset W \subset U$ and $N-Indb(W) \leq n-1$. Let $V=W \cap A$ is $N-open$ in A, by proposition(2.7). Thus $F \subset V = W \cap A \subset U \cap A = G$

$$\begin{aligned} b_A(V) &\subseteq b(V) \cap A = (\bar{V}-V^{\circ}) \cap A \subset (\bar{W}-V^{\circ}) \cap A = \left(\bar{W} \cap V^{\circ^C}\right) \cap A \\ &= \left[\bar{W} \cap \left(W^{\circ^C} \cup A^{\circ^C}\right)\right] \cap A = \left[\left(\bar{W} \cap W^{\circ^C}\right) \cup \left(\bar{W} \cap A^{\circ^C}\right)\right] \cap A \\ &\subset \left(b(W) \cup A^{\circ^C}\right) \cap A = (b(W) \cap A) \cup \left(A^{\circ^C} \cap A\right) \subset b(W). \end{aligned}$$

$\overline{b_A(V)}^{b(W)} = \overline{b_A(V)} \cap b(W) = b_A(V) \cap b(W) = b_A(V)$. $b_A(V)$ is closed set in A, since A is a closed set in X then $b_A(V)$ is a closed set in $b(W)$. Thus $N-Indb_A(V) \leq N-Indb(W)$.

Since $N-Indb(W) \leq n-1$ then $N-Indb_A(V) \leq n-1$ (By induction). Therefore $N-indA \leq n$. ■

5. On dim by using $N-open$ sets

Definition 5.1: The $N-$covering dimension $N-dimX$ of a topological space X is the least integer n such that every finite open covering of X has an $N-open$ refinement of order not exceeding n or is ∞ if there is no such integer. Thus $N-dimX=-1$ if and only if X is empty, and $N-dimX\leq n$ if each finite open covering of X has $N-open$ refinement of order not exceeding n. We have $N-dimX=n$ if it is true that$N-\ dimX\leq n$ but it is not true that $N-dimX\leq n-1$. Finally $N-dimX=\infty$ if for every integer n it is false that $N-dimX\leq n$.

Definition 5.2: The N^*-covering dimension N^*-dimX of a topological space X is the least integer n such that every finite $N-open$ covering of X has an $N-open$ refinement of order not exceeding n or is ∞ if there is no such integer. Thus $N^*-dimX=-1$ if and only if X is empty, and $N^*-dimX\leq n$ if each finite $N-open$ covering of X has $N-open$ refinement of order not exceeding n. We have $N^*-dimX=n$ if it is true that$N^*-\ dimX\leq n$ but it is not true that $N^*-dimX\leq n-1$. Finally $N^*-dimX=\infty$ if for every integer n it is false that $N^*-dimX\leq n$.

Proposition 5.3: Let X be a topological space, if $dimX$ is exists then

$$N-dimX\leq dimX$$

Proof: By induction on n. If $n=-1$ then $dimX=-1$ and $X=\emptyset$, so that $N-dimX=-1$.
Suppose statement is true for $n-1$, now let $dimX\leq n$ to prove $N-dimX\leq n$.
Let $\mathcal{U}=\{U_1,U_2,\dots,U_k\}$ be a finite open cover of X. Since $dimX\leq n$ then $\mathcal{U}$ has $N-open$ refinement $\mathcal{V}$ of $order\leq n$. Hence $N-dimX\leq n$.

Proposition 5.4: Let X be a topological space, if N^*-dimX is exists then

$$N-dimX\leq N^*-dimX$$

Theorem 5.5: Let X be a topological space, if X has a base of sets which are both $N-open$ and $N-closed$ then $N-dimX=0$, for a $T_1-space$ the converse is true.

Proof: Suppose X has a base of sets which are both $N-open$ and $N-closed$. Let $\{U_i\}_{i=1}^{K}$ be a finite open cover of X, it has an $N-open$ refinement W, if $w\in W$ then $W\subset U_i$ for some i.
Let each $w\ in\ W$ be associated with one of the sets U_i containing it and let V_i be the union of those members of W thus associated with U_i. Thus V_i is $N-open$ set and hence $\{V_i\}_{i=1}^{K}$ forms disjoint $N-open$ refinement of $\{U_i\}_{i=1}^{K}$ then $N-dimX=0$.
Conversely
Suppose X is $T_1-space$such that $N-dimX=0$. Let $p\in X$ and G be an open set in X such that $p\in G$. Then $\{p\}$ is closed set and $\{G,X-\{p\}\}$ is finite open cover of X so it has an $N-open$ refinement of order 0. Let C_1 be the union of $N-open$ sets in G and C_2 be the union of the $N-open$ sets in $X-\{p\}$. Then $C_1\cap C_2=\emptyset$, $C_1\cup C_2=X$ and C_1,C_2 are $N-open$ sets and $N-closed$ sets in X. Thus $N-closed$ set in X and hence X has a base of sets which are both $N-open$ and $N-closed$ sets. ■

Theorem 5.6: Let X be a topological space, if X has $N-base$ of sets which are both $N-open$ and $N-closed$ then $N^*-dimX=0$, for a $NT_1-space$ the converse is true.

Proof: Suppose X has $N-base$ of sets which are both $N-open$ and $N-closed$. Let $\{U_i\}_{i=1}^{K}$ be a finite $N-open$ cover of X, it has an $N-open$ refinement W, if $w\in W$ then $W\subset U_i$ for some i. Let each $w\ in\ W$ be associated with one of the sets U_i containing it and let V_i be the union of those members of W thus associated with U_i. Thus V_i is $N-open$ set and hence $\{V_i\}_{i=1}^{K}$ forms disjoint $N-open$ refinement of $\{U_i\}_{i=1}^{K}$ then $N^*-dimX=0$.
Conversely
Suppose X is $NT_1-space$ such that $N^*-dimX=0$. Let $p\in X$ and G be an $N-open$ set in X such that $p\in G$. Then $\{p\}$ is closed set and $\{G,X-\{p\}\}$ is finite $N-open$ cover of X.

Since $N^* - dimX = 0$ then there exists an $N - open$ refinement $\{V, W\}$ of order 0. Such that $V \cap W = \emptyset$, $\cup W = X$, $V \subset G$ and $W \subset X - \{p\}$. Then V is $N - open$ and $N - closed$ set in X such that $p \in W^C \in V \subset G$ and hence X has $N - base$ of $N - open$ and $N - closed$ sets. ■

Remark 5.7[A.P.Pears ,1975]: Let X be a topological space with $dimX = 0$ then X is a normal space.

Theorem 5.8: Let X be a topological space with $N - dimX = 0$ then X is $N - normal$ space.

Proof: Let $F_1$and F_2 be disjoint closed sets of X, then $\{X \setminus F_2, X \setminus F_1\}$ is an open covering of X. Since $N - dimX = 0$ then it has $N - open$ refinement of order 0, hence there exist $N - open$ sets H and G such that $H \cap G = \emptyset$, $\cup G = X$, $H \subset X \setminus F_1$ and $G \subset X \setminus F_2$. Thus $F_1 \subseteq H^C = G$, $F_2 \subseteq G^C = H$ and since $H \cap G = \emptyset$ then X is $N - normal\ space$. ■

Theorem 5.9: Let X be a topological space with $N^* - dimX = 0$ then X is $N^{**} - normal$ space.

Proof: Let $F_1$and F_2 be disjoint $N - closed$ sets of X, then $\{X \setminus F_2, X \setminus F_1\}$ is an $N - open$ covering of X. Since $N^* - dimX = 0$ then it has $N - open$ refinement of order 0, hence there exist $N - open$ sets H and G such that $H \cap G = \emptyset$, $\cup G = X$, $H \subset X \setminus F_1$ and $G \subset X \setminus F_2$. Thus $F_1 \subseteq H^C = G$, $F_2 \subseteq G^C = H$ and since $H \cap G = \emptyset$ then X is $N^{**} - normal\ space$. ■

Remark 5.10: Let $X = \{a, b, c\}$, $T = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$.In example show that $dimX = N - dimX = 0$. Since X is the open cover of X and its only refinement of it. Then $dimX = 0$ and since $N - dimX \leq dimX$, $X \neq \emptyset$ then $dimX = N - dimX = 0$.

Proposition 5.11[A.P.Pears ,1975]: Let X be a topological space, if A is a closed sub set of a space X we have $dimA \leq dimX$.

Theorem 5.12: Let X be a topological space, if A is a closed and open sub set of a space X, we have

$$N - dimA \leq N - dimX$$

Proof: Suppose that $N - dimX \leq n$ to prove $N - dimA \leq n$. Let $\{U_1, U_2, \dots, U_k\}$ be an open covering of A. Then for each i, $U_i = A \cap V_i$, where V_i is open set in X.

The finite open covering $\{V_1, V_2, \dots, V_k, X \setminus A\}$ of X has an $N - open$ refinement W of order $\leq n$. Let $V = \{w \cap A : w \in W\}$ where $w \cap A$ is an $N - open$ in A by proposition(2.7).

Then V is an $N - open$ refinement of $\{U_1, U_2, \dots, U_k\}$ of order$\leq n$. Thus $N - dimA \leq n$. ■

Theorem 5.13: Let X be a topological space, if A is a closed and open sub set of a space X, we have

$$N^* - dimA \leq N^* - dimX$$

Proof: Suppose that $N^* - dimX \leq n$ to prove $N^* - dimA \leq n$. Let $\{U_1, U_2, \dots, U_k\}$ be an $N - open$ covering of A. Then for each i, $U_i = A \cap V_i$, where V_i is $N - open$ set in X.

The finite $N - open$ covering $\{V_1, V_2, \dots, V_k, X \setminus A\}$ of X has an $N - open$ refinement W of order $\leq n$. Let $V = \{w \cap A : w \in W\}$ where $w \cap A$ is an $N - open$ in A. Then V is an $N - open$ refinement of $\{U_1, U_2, \dots, U_k\}$ of order$\leq n$. Thus $N^* - dimA \leq n$. ■

6. Relation between the dimensions ind and Ind by using $N - open$ sets

Proposition 6.1: Let X be a topological space, if X is $T_1 - space$ then

$$N - indX \leq N - IndX$$

Proof: By induction on n. If $n = -1$ then the statement is true. Suppose that the statement is true for $n - 1$.

Now

Suppose that $N - IndX \leq n$, to prove $N - indX \leq n$. Let $p \in X$ and each open set $G \subset X$ of the point p, since X is $T_1 - space$ then $\{p\} \subseteq G$ such that $\{p\}$ is closed set. Since $N - IndX \leq n$

then there exists an $N-open$ set V in X such that $p \subset V \subset G$ and $N-Indb(V) \leq n-1$. Hence $N-indb(V) \leq n-1$ and $p \in V \subset G$. Then $N-indX \leq n$. ■

Proposition 6.2: Let X be a topological space, if X is $NT_1-space$ then

$$N-indX \leq N^*-IndX$$

Proof: By induction on n. If $n=-1$ then the statement is true. Suppose that the statement is true for $n-1$.

Now

Suppose that $N^*-IndX \leq n$, to prove $N-indX \leq n$. Let $p \in X$ and each open set $G \subset X$ of the point p, since X is $NT_1-space$ then $\{p\} \subseteq G$ such that $\{p\}$ is $N-closed$ set. Since $N^*-IndX \leq n$ then there exists an $N-open$ set V in X such that $p \subset V \subset G$ and $N^*-Indb(V) \leq n-1$. Hence $N-indb(V) \leq n-1$ and $p \in V \subset G$. Then $N-indX \leq n$. ■

Proposition 6.3: Let X be a topological space, if X is $NT_1-space$ then

$$N^*-indX \leq N^{**}-IndX$$

Proof: By induction on n. If $n=-1$ then the statement is true. Suppose that the statement is true for $n-1$.

Now

Suppose that $N^{**}-IndX \leq n$, to prove $N^*-indX \leq n$. Let $p \in X$ and each $N-open$ set $G \subset X$ of the point p, since X is $NT_1-space$ then $\{p\} \subseteq G$ such that $\{p\}$ is $N-closed$ set.

Since $N^{**}-IndX \leq n$ then there exists an $N-open$ set V in X such that $p \subset V \subset G$ and $N^{**}-Indb(V) \leq n-1$. Hence $N^*-indb(V) \leq n-1$ and $p \in V \subset G$. Then $N^*-indX \leq n$. ■

Proposition 6.4: Let X be a topological space, if X is a regular space then

$$N-indX \leq N-IndX$$

Proof: By induction on n. If $n=-1$ then the statement is true. Suppose that the statement is true for $n-1$.

Now

Suppose that $N-IndX \leq n$, to prove $N-indX \leq n$. Let $N-IndX \leq n$ and let $p \in X$ and each open set $G \subset X$ of the point p, since X is a regular space then there exists an $open$ set V in X such that$p \in V \subset \bar{V} \subset G$.

Also.. Since $N-IndX \leq n$ and $\bar{V}$ is closed, $\bar{V} \subset G$ then there exists an $N-open$ set U in X such that $\bar{V} \subset U \subset G$ and$N-Indb(U) \leq n-1$. Hence $N-indb(U) \leq n-1$ and $p \in U \subset G$ (by induction). Therefore $N-indX \leq n$. ■

Proposition 6.5: Let X be a topological space, if X is a $N-regular$ space then

$$N-indX \leq N^*-IndX$$

Proof: By induction on n. If $n=-1$ then the statement is true. Suppose that the statement is true for $n-1$.

Now

Suppose that $N^*-IndX \leq n$, to prove $N-indX \leq n$. Let $N^*-IndX \leq n$ and let $p \in X$ and each open set $G \subset X$ of the point p, since X is $N-regular$ space then there exists an $N-open$ set V in X such that$p \in V \subset \bar{V}^N \subset G$.

Also.. Since $N^*-IndX \leq n$ and $\bar{V}^N$ is $N-closed$, $\bar{V}^N \subset G$ then there exists an $N-open$ set U in X such that $\bar{V}^N \subset U \subset G$ and $N^*-Indb(U) \leq n-1$. Hence $N-indb(U) \leq n-1$ and $p \in U \subset G$ (by induction). Therefore $N-indX \leq n$. ■

Proposition 6.6: Let X be a topological space, if X is a $N^*-regular$ space then

$$N^*-indX \leq N^{**}-IndX$$

Proof: By induction on n. If $n = -1$ then the statement is true. Suppose that the statement is true for $n - 1$.
Now
Suppose that $N^{**} - IndX \leq n$, to prove $N^{*} - indX \leq n$. Let $N^{**} - IndX \leq n$ and let $p \in X$ and each $N - open$ set $G \subset X$ of the point p, since X is $N^{*} - regular$ space then there exists an $open$ set V in X such that $p \in V \subset \bar{V}^{N} \subset G$.
Also.. Since $N^{**} - IndX \leq n$ and $\bar{V}^{N}$ is $-closed$, $\bar{V}^{N} \subset G$ then there exists an $N - open$ set U in X such that $\bar{V}^{N} \subset U \subset G$ and $N^{**} - Indb(U) \leq n - 1$. Hence $N^{*} - indb(U) \leq n - 1$ and $p \in U \subset G$ (by induction). Therefore $N^{*} - indX \leq n$. ■
Theorem 6.7: Let X be a topological space, if X is a compact space and $N - indX = 0$ then $N - IndX = 0$.
Proof: Let X is compact space such that$N - indX = 0$.
Let F be a closed set of X and G be an open set of X such that $F \subset G$.
Since$N - indX = 0$, for each $p \in F$ there exist an $N - open$ and $N - closed$ sets U_p such that $p \in U_p \subset G$ hence$F \subset \bigcup_{p \in F} U_p \subset G$.
Since F be a closed set in the compact space$X$, F is compact , hence there exist points $p_1, p_2, \ldots\ldots, p_k$ of F such that$F \subset \bigcup_{i=1}^{k} U_{p_i} \subset G$. Since $\bigcup_{i=1}^{k} U_{p_i}$ is $N - open$ and $N - closed$ of X. It following that $N - IndX = 0$. ■
Theorem 6.8: Let X be a topological space, if X is a $N - compact$ space and $N - indX = 0$ then $N^{*} - IndX = 0$.
Proof: Let X be $N - compact$ space such that$N - indX = 0$.
Let F be an $N - closed$ set of X and G be an open set of X such that $F \subset G$.
Since$N - indX = 0$, for each $p \in F$ there exist an $N - open$ and $N - closed$ sets U_p such that $p \in U_p \subset G$ hence$F \subset \bigcup_{p \in F} U_p \subset G$.
Since F be an $N - closed$ set in the $N - compact$ space$X$, F is $N - compact$ space, hence there exist points $p_1, p_2, \ldots\ldots, p_k$ of F such that$F \subset \bigcup_{i=1}^{k} U_{p_i} \subset G$. Since $\bigcup_{i=1}^{k} U_{p_i}$ is $N - open$ and $N - closed$ of X. It following that $N^{*} - IndX = 0$. ■
Theorem 6.9: Let X be a topological space, if X is a $N - compact$ space and $N^{*} - indX = 0$ then $N^{**} - IndX = 0$.
Proof: Let X be $N - compact$ space such that $N - indX = 0$.
Let F be an $N - closed$ set of X and G be an $N - open$ set of X such that $F \subset G$.
Since$N - indX = 0$, for each $p \in F$ there exist an $N - open$ and $N - closed$ sets U_p such that $p \in U_p \subset G$ hence$F \subset \bigcup_{p \in F} U_p \subset G$.
Since F be an $N - closed$ set in the $N - compact$ space$X$, F is $N - compact$ space, hence there exist points $p_1, p_2, \ldots\ldots, p_k$ of F such that$F \subset \bigcup_{i=1}^{k} U_{p_i} \subset G$. Since $\bigcup_{i=1}^{k} U_{p_i}$ is $N - open$ and $N - closed$ of X. It following that $N^{*} - IndX = 0$. ■

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