Degree of Approximation by Taylor Operator of Functions in $I_p(U)$ Space for $p < 1$

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Abstract

In this paper we prove direct theorems for approximation of functions in $I_{p(U)}$ spaces for p < 1 using Taylor operator .

Keywords: Ip spaces, Taylor operator, degree of best approximation.

الخلاصة

برهنا في هذا البحث مبرهنه مباشره من نوع جاكسون في التقريب المعقد للدوال في الفضاءات باستخدام منحني تايلر p <
$I_{p(U)}$ عندما 1

الكلمات المفتاحية: فضاءات Ip، مؤثر تايلر ، درجة افضل تقريب

1.Introductin And Preliminaries

The origin point of the complex interpolating approximation is the Taylor series because it is interpolating polynomials. Many properties of Tylor sections are introduced in(Dienes, 1957). In (Dienes, 1957) Dienes also proved that every point of the circle convergence $|z| = p$ is a limit point of the set of zeros of the sequence $\{S_n\}_{n=1}^{\infty}$ defined below.

Let us now introduce the following notations:

Let $Л_n$ the space of all polynomials of degree $\leq n$ with complex coefficient

And $U = \{[Z \in C: |Z| \leq 1]\}$.

Also we need the following definitions:

Definition 1.1:

If $f: U \to \mathbb{R}$ then

$\|f\|_p \coloneqq \|f\|_{I_p(U)} =(\int_U |f(z)|^p dz)^{\frac{1}{p}}$,$0 < p < \infty$, and if $p = \infty$, we have

$$\|f\|_\infty \coloneqq \|f\|_{I_\infty(U)} = sup_{z\in U}|f(z)|$$

So let

$$I_p(U) = \{f: U \to \mathbb{R}: \|f\|_p < \infty\}.$$

Definition 1.2:

A projection $p: I_p(U) \to P_n$ is bounded linear operator satisfy $p = p^2$and If $I_p(U) = P_n$,then $p = I$ *is* the identity operator.

Definition 1.3:(Carothers,1960)

T is linear on the vector space X over the field F if and only if $T(ax + by) = aT(x) + bT(y)$ $\forall x, y \in X$ and a and b scalars in F ,T is called bounded if and only if there exists $M > 0$, such that $\|T(x)\| \leq M\|x\|$, where $\|x\|$ is the norm of x.

Definition 1.4: (SAFF, 1998)

The Taylor projection is

$$S_n(f,z) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} (z)^n,$$

where $S_n(j)\,(0) = f(j)\,(0)$,$j = 0,1,2,3, \dots \dots, n$

Lemma 1.5: (SAFF, 1998)
For any projection p

$$S_n(p)(z) = \frac{1}{2\pi i}\int_{|z|=1}(A_{\bar{t}}(p)A_t(p))(z) \quad \frac{dt}{t},$$

where A_t is the shift operator defined by

$$A_t(p)(z) := p(tz).$$

2. The Main Results

In (Boricchev, 2011) Alexander study the L^p integrability of polyharmonic functions and he shows:
Given $p \in (0,1)$ and $\alpha \in R$,we study polyharmonic functions u on the unit disc D such that $\int_D |u(z)|^p (1-|z|^2)^{-\alpha}\, dm_2(z) < \infty$.
In this article we improve the works above by studying the approximation of functions in $I_p(U)$ using Taylor project.

Theorem 2.1:
Let p be any projection of the space $I_p(U)$ to the $P_n$then, for the operator norm induced by our norm in Definition1.1 over the unit disk U , we have $\|S_n(p)\|_p \leq \|p\|_p$

Proof:

$$\|S_n(p)\|_p = \left\|\frac{1}{2\pi i}\int_{|t|=1}(A_{\bar{t}}(p)A_t(p))(z)\frac{dt}{t}\right\|_p$$
$$= \left\|\frac{1}{2\pi i}\int_{|t|=1} t\bar{t}\, p^2(z)\frac{dt}{t}\right\|_p$$
$$= \left|\frac{1}{2\pi i}\int_{|t|=1}\frac{dt}{t}\right|_p \|p\|_p$$
$$\leq \left|\frac{1}{2\pi i}\right|\int_{|t|=1}\frac{dt}{|t|}{}_p \|p\|_p$$
$$= \frac{1}{2\pi}\int_{|t|=1} dt \;\; \|p\|_p = \|p\|_p$$

What can be said about the rate of convergence of the Taylor sections? The answer is intimately related to the familiar Cauchy-Hadamard formula for the radius of $\sum_{k=0}^{n} C_k Z^k$ · That is

$$1/p = \lim_{k\to\infty} sup|C_k|^{\frac{1}{k}}.$$

Then we need the following Lemma

Lemma2.2: (Geddes & Mason, 1975)
The Taylor operator defined in Lemma 1.5 satisfy

$$\lim_{n\to\infty} sup|f - S_n|^{\frac{1}{n}} = \frac{1}{p} < 1.$$

As a direct consequence of the above Lemma we have:

Theorem 2.3:
If $f \in I_p(U)$ for the I_p norm $\|f\|_p = (\int_U |f(z)|^p\, dz)^{\frac{1}{p}}$,
then Taylor operators S_n satisfy $\lim_{n\to\infty} sup\|f - S_n\|_p = \frac{1}{p} < 1$, where p is the radius of the largest open disk centered at the origin throughout which f has a single–valued analytic continuation moreover, the sequence S_n converges to f for $|z| < p$.

Proof:
By the well know result $\|f - S_n\|^{\frac{1}{n}}_p < \|f - S_n\|^{\frac{1}{n}}_\infty$, and using Lemma 2.2 we obtain

$$\lim_{n\to\infty}\|f - S_n\|^{\frac{1}{n}}_p \quad < \lim_{n\to\infty} sup\|f - S_n\|^{\frac{1}{n}} = \frac{1}{p} < 1$$

Therefore $\lim_{n\to\infty}\|f-S_n\|_p^{\frac{1}{p}} < 1$ ♦

For interpolating polynomials. We have the following lemma

Lemma 2.4: (SAFF, 1998)

Suppose f is analytic inside and on the simple close contour Γ that the $n+1$ points $z_0, z_1, \dots, z_n$. If P_n is the unique polynomial in $Л_n$ that interpolates f in these points,then

$$f(z) - P_n(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{\pi(z)f(t)}{\pi(t)-(t-z)}\, dt, z \; inside \; \Gamma,$$

where $\pi(z) := \prod_{k=0}^{n}(z-z_k)$.

Since the Taylor series S_n interpolates in the origin of multiplicity $n+1$ by Lemma 2.2 we have

$$f(z) - S_n(z) = \frac{1}{2\pi i}\frac{z^{n+1}f(t)}{t^{n+1}\,(t-z)}\, dt, \quad |z| < r, (2.5)$$

Theorem 2.6:

If $f \in \mathrm{I}_p(\mathrm{U})$, where $U_{r=}\langle z \epsilon \mathbb{C}: |z| \le r\rangle$.Then we have

$$\lim_{n\to\infty} sup\|f-S_n\|_p \le \frac{1}{p}$$

Proof:

The Proof is clear by using Thoerem2.3 and (2.5) ♦

Corollary 2.7:

The Taylor series for $f \in I_p(U_r)$ converges to analytic function on $|z| < p$ for $R > p$.

Proof:

Using Theorem2. 5 we have

$$\left\|\frac{f^n(0)}{n}z^n\right\|_p = \left\|\sum_{k=0}^{n}\frac{f^n(0)}{k}z^k - \sum_{k=0}^{n-1}\frac{f^n(0)}{k}z^k\right\|_p = \|S_{n-}\,S_{n-1}\|_p$$

Using Theorem2.3 to obtain

$$\lim_{n\to\infty} sup\left\|\frac{f^n(0)}{n}z^n\right\|_p = \lim_{n\to\infty} sup\,\|S_n - S_{n-1}\|_p$$ ♦

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