SOME TYPES OF CONTINUITIES ON APPROXIMATION SPACE

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ABSTRACT.

IN THIS PAPER WE DEFINE SOME TYPES OF STRONG CONTINUITIES ON THE APPROXIMATION SPACE IN TERMS OF ROUGH SETS. SOME RESULTS ON THESE TYPES OF CONTINUITIES ARE GIVEN.
Key Words. Rough sets, strong continuity, approximation space

الخلاصة

قمنا في هذا البحث باستخدام المجموعات المضطربة لتعريف بعض انواع الاستمرارية القوية على الفضاءات التقريبية. كما قمنا ببرهان بعض الحقائق حول تلك الانواع من الاستمرارية.

الكلمات المفتاحية: المجموعات المضطربة، الاستمرارية القوية، لفضاءات التقريبية.

1. INTRODUCTION

Rough set theory introduced by (Zdzislaw , 2002) in . It is a formal approximation of crisp set in terms of a pair of sets which give the lower and upper approximation of the original set. Many works appeared recently for example in structural analysis (Lorgerron, S. Bonneray, 2002), in chemistry(M. Stadler, Stadler, 2002), physics(Galton, 2002) and in computer sciences (Davva, 2004 , Lee, Pork ,*at al* 2006)

The interest for rough set theory has come from the need to represent of a universe in terms of equivalence classes of portion of that universe. The portion characterize a topological space called *approximation space* $A = (X, R)$, where X is a set called the *universe* and $R \subset X \times X$ is an *equivalence relation*. Equivalence classes of the relation R and the empty set will be called *elementary* sets (*atoms*) in A (Zdzislaw Pawlak, 1984)

Every union of elementary sets in A will be called a *composed* set in A(Zdzislaw,1984).

The family of all composed sets in A is denoted by $Comp(A)$. Obviously $Comp(A)$ is a Boolean algebra, i. e. the family of all composed sets is closed under intersection, union, and complement of set.

Let U be a subset of X. The least composed set in A containing U called the *upper approximation* of U in A, and write $\bar{A}(U)$. The greatest composed set in A, contained in U called the *lower approximation* of U in A, and write $\underline{A}(U)$.

In other words

$$\underline{A}(U) = \{x \in X : [x]_R \subset U\}$$

and

$$\overline{A}(U) = \{x \in X : [x]_R \cap U \neq \emptyset\}$$

Where $[x]_R$ denotes the equivalence class of relation R containing x (Abo-Table, 2011). Now let us introduce some properties of approximations. One can easily check that the approximation space $A = (X, R)$ defines uniquely the topological space $T_A = \{X, Comp(A)\}$, and $Comp(A)$ is the family of all open and closed sets in T_A.

From the definition of approximations follows that $\overline{A}(U)$ and $\underline{A}(U)$ are closure and interior of U in the topological space T_A respectively.

Thus for every $U, V \subset X$ and every approximation space $A = (X, R)$, the following properties (Zdzislaw Pawlak, 1984) of approximations are valid:

i. $\underline{A}(U) \subset U \subset \overline{A}(U)$

ii. $\underline{A}(X) = \overline{A}(X) = X$

iii. $\underline{A}(\emptyset) = \overline{A}(\emptyset) = \emptyset$.

iv. $\overline{A}(U \cup V) = \overline{A}(U) \cup \overline{A}(V)$

v. $\underline{A}(U \cup V) \supset \underline{A}(U) \cup \underline{A}(V)$

vi. $\overline{A}(U \cap V) \subset \overline{A}(U) \cap \overline{A}(V)$

vii. $\underline{A}(U \cap V) = \underline{A}(U) \cap \underline{A}(V)$

viii. $\overline{A}(U^c) = \left(\underline{A}(U)\right)^c$

ix. $\underline{A}(U^c) = \left(\overline{A}(U)\right)^c$

Also notice that $\underline{A}(U) = \overline{A}(U)$ if and only if X is composed set in A. If R is a reflexive relation on X, then then the upper approximation satisfies the properties above (Allam Bakeir ,Abo-Table,*at al* 2008)

A subset A of a topological space X is called *preopen* (S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, 1982) or nearly open (V. Ptak, 1958) if $A \subset int\left(cl\,(A)\right)$. A function $f: X \to Y$ is called *precontinuous* (Mashhour, Abd El-Monsef ,El-Deeb,*at al* 1982)if the preimage $f^{-1}(V)$ of each open set V of Y is preopen in X. Precontinuity was called near continuity by Ptak (Ptak. 1958) and also called almost continuity by Frolik (Husain. 1966), and Husain (Lorgerron, Bonneray, 2002). Jankovic (Jankovic, 1985) introduced almost weak continuity as a weak form of precontinuity. Popa and Noiri (Popa and Noiri, 1992) introduced weak precontinuity and showed that almost weak continuity is equivalent to weak precontinuity. The purpose of this paper is to introduce and investigate stronger forms of continuities and precontinuities in terms of upper and lower approximations of sets and relate them to some types of separation axioms.

2. The Main Results

In this article we shall introduce our main results. And let us begin with the following proposition:

Proposition 2.1. Let $f: A_1 \coloneqq (X, R_1) \to A_2 \coloneqq (X, R_2)$. If for any set V in Y,

$$f^{-1}\left(\underline{A}_2(V)\right) \subset \underline{A}_1\left(\overline{A_1}\left(f^{-1}\left(\underline{A}_2(V)\right)\right)\right),$$

then

$$\overline{A}_1\left(\underline{A_1}\left(f^{-1}\left(\overline{A}_2(V)\right)\right)\right) \subset f^{-1}\left(\overline{A}_2(V)\right)$$

Proof. By our hypothesis we have,

$$f^{-1}\left(\overline{A}_2(V)\right)^c \subset \underline{A}_1\left(\overline{A_1}\left(f^{-1}(\overline{A}_2(V)^c)\right)\right),$$

$$\begin{aligned} f^{-1}\left(\underline{A}_2(V^c)\right) &= f^{-1}(Y) - f^{-1}\left(\underline{A}_2(V)\right) \\ &= f^{-1}\left(\overline{A}_2(V)\right)^c \\ &\subset \underline{A}_1\left(\overline{A_1}\left(f^{-1}(\overline{A}_2(V)^c)\right)\right) \\ &= \underline{A}_1\left(\overline{A_1}\left(f^{-1}\left(\underline{A}_2(V)\right)^c\right)\right) \\ &= \underline{A}_1\left(\overline{A_1}\left(X - f^{-1}\left(\underline{A}_2(V)\right)\right)\right) \end{aligned}$$

$$= \underline{A}_1\left(\overline{A_1}\left(f^{-1}\left(\underline{A}_2(V)\right)\right)^c\right).$$

This implies

$$\overline{A}_1\left(\underline{A_1}\left(f^{-1}\left(\overline{A}_2(V)\right)\right)\right) \subset f^{-1}\left(\overline{A}_2(V)\right) \qquad ☺$$

Then for our next result we need the following definitions:

Definition 2.2. A map $f: A_1 \coloneqq (X, R_1) \to A_2 \coloneqq (Y, R_2)$ is *weakly approximately pre-continuous* if for each $x \in X$ and any set V in Y, s.t. $f(x) \in \underline{A}_2(V)$ there exists a set U containing x, and satisfying $U \subset \underline{A_1}\left(\overline{A_1}(U)\right)$, we have $f(U) \subset \overline{A_2}\left(\underline{A_2}(V)\right)$.

Definition 2.3. A map $f: A_1 \coloneqq (X, R_1) \to A_2 \coloneqq (Y, R_2)$ is said to be *approximately pre-continuous* if for each $x \in X$ and any set V in Y, s.t. $f(x) \in \underline{A}_2(V)$ there exists a set U in X, with $x \in U \subset \underline{A_1}\left(\overline{A_1}(U)\right)$, we have $f(U) \in \underline{A_2}(V)$.

Definition 2.4. An approximation space $A \coloneqq (X, R)$ is called T_1 *−approximation space*, if for each pair a and b of distinct points of X, there are two sets U and V in X such that a belongs to $\underline{A}(U)$ but b doesn't and b belongs to $\underline{A}(V)$ a doesn't.

Definition 2.5. A T_3 *−approximation space* is a T_1 −approximation space $A \coloneqq (X, R)$ such that for each subset C of X and each point a not in $\overline{A}(C)$ there exist two sets U and V in X such that $\underline{A}(U) \cap \underline{A}(V) = \emptyset$ with $a \in \underline{A}(U)$ and $\overline{A}(C) \subset \underline{A}(V)$.

Theorem 2.6. Let $A_2 \coloneqq (Y, R_2)$ be a T_3 −approximation space and $f: A_1 \coloneqq (X, R_1) \to A_2 \coloneqq (Y, R_2)$. If f is weakly approximately pre continuous, then it is approximately pre-continuous.

Proof. Let x be a point in X and V be a set such that $f(x) \in \underline{A_2}(V)$. Since Y is a T_3 −approximation space there exists a set W such that $f(x) \in \underline{A_2}(W) \subset \overline{A}_2\left(\underline{A_2}(W)\right) \subset \underline{A_2}(V)$. Since f is wealy approximately pre-continuous map, there exists a set U containing x such that $U \subset \underline{A_1}\left(\overline{A_1}(U)\right)$, and $f(U) \subset \overline{A_2}\left(\underline{A_2}(W)\right)$. Therefore we have$f(U) \subset \underline{A_2}(V)$ ☺

For our next results we need the following definitions.

Definition 2.7. A map $f: A_1 \coloneqq (X, R_1) \to A_2 \coloneqq (Y, R_2)$ is said to be *weakly approximately continuous* if for each $x \in X$ and any set V in Y, s.t. $\underline{A}_2(V)$ containing $f(x)$, there exists a set U such that $\underline{A_1}(U)$ containing x and satisfying $f\left(\underline{A_1}(U)\right) \subset \underline{A_2}(V)$.

Definition 2.8. A map $f: A_1 \coloneqq (X, R_1) \to A_2 \coloneqq (Y, R_2)$ is called *strongly approximately pre-continuous* if for each $x \in X$ and each set V in Y, s.t. $\underline{A}_2(V)$ containing $f(x)$, there exists a set U such that $U \subset \underline{A_1}\left(\overline{A_1}(U)\right)$ containing x and satisfying $f\left(\overline{A_1}(U)\right) \subset \underline{A_2}(V)$.

Definition 2.9. $A \coloneqq (X, R)$ is called *p-regular approximation space*, if for each set F and each point $x \in \overline{A}(F)$, there exist disjoint sets U and V such that $x \in U \subset \underline{A}\left(\overline{A}(U)\right)$, and $\overline{A}(F) \subset V \subset \underline{A}\left(\overline{A}(V)\right)$.

Theorem 2.10. An approximately continuous map $f: A_1 \coloneqq (X, R_1) \to A_2 \coloneqq (Y, R_2)$ approximately pre-continuous if and only if A_1 is p-regular approximation space.

Proof. Let f be the identity map. Then f is approximately continuous and it is strongly approximately pre-continuous. So for any set U in X and any point x in $\underline{A_1}(U)$, we have $f(x) = x \in \underline{A_1}(U)$ and there is a set G containing x such that $G \subset \underline{A_1}\left(\overline{A_1}(G)\right)$ and

$f\left(\overline{A_1}(G)\right) \subset \underline{A_2}(U)$. Thus we have $x \in G \subset \overline{A_1}(G) \subset \underline{A_1}(U)$. This implies A_1 is p-regular approximation space.

Assume f is approximately continuous map and X is p-regular approximation space for any $x \in X$ and any set V containing $f(x)$ in $\underline{A_2}(V)$ and $x \in f^{-1}\left(\underline{A_2}(V)\right)$. Since X is p-regular approximation space there exist a set U such that $U \subset \underline{A_1}\left(\overline{A_1}(U)\right)$ with $x \in \underline{A_1}(U) \subset \overline{A_1}(U) \subset f^{-1}\left(\underline{A_1}(V)\right)$. This implies $f\left(\overline{A_1}\left(\underline{A_1}(U)\right)\right) \subset \underline{A_1}(U)$. Therefore f is approximate pre-continuous ☺

For the next result we need the following definition:

Definition 2.11. An approximation space $A \coloneqq (X, R)$ is called *pre-regular approximation space*, if for each set F in X with $\overline{A}\left(\underline{A}(F)\right) \subset F$ and each $x \in F^c$ there exist two disjoint sets U and V such that $U \subset \underline{A}\left(\overline{A}(U)\right)$ and $V \subset \underline{A}\left(\overline{A}(V)\right)$ satisfying $x \in U$ and $F \subset V$.

Theorem 2.12. Let $A_1 \coloneqq (X, R_1)$ be a pre-regular approximation space and $f: A_1 \coloneqq (X, R_1) \to A_2 \coloneqq (Y, R_2)$ is approximately pre-continuous map, then it is strongly approximately pre-continuous.

A_1 is p-regular approximation space.

Proof. Since f is approximately pre-continuous and let $x \in X$, and V be a subset of Y such that $f(x) \in \underline{A_2}(V)$. Since f is approximately pre-continuous we have $f^{-1}\left(\underline{A_2}(V)\right) \subset \underline{A_1}\left(\overline{A_1}\left(f^{-1}\left(\underline{A_2}(V)\right)\right)\right)$, so there exist a set U with $U \subset \underline{A_1}\left(\overline{A_1}(U)\right)$ such that $\overline{A_1}(U) \subset f^{-1}\left(\underline{A_2}(V)\right)$. Thus we obtain $f\left(\overline{A_1}(U)\right) \subset \underline{A_2}(V)$, therefore f is strongly approximately pre-continuous map ☺

For Theorem 2.14 let us define upper and lower approximations of subsets of $X \times Y$.

Definition 2.13. Let $A_1 \coloneqq (X, R_1)$ and $A_2 \coloneqq (Y, R_2)$, $U \subset X$ and $V \subset Y$. Define

$$\underline{A_3}(U \times V) = \{(x, y) \in X \times Y : [x]_R \times [y]_R \subset U \times V\},$$

and

$$\overline{A_3}(U \times V) = \{(x, y) \in X \times Y : ([x]_R \times [y]_R) \cap (U \times V) \neq \emptyset\},$$

where $[x]_R$ and $[y]_R$ are equivalence classes of relations R_1 containing x and of R_2 containing y respectively.

Theorem 2.14. Let $f: A_1 \coloneqq (X, R_1) \to A_2 \coloneqq (Y, R_2)$ be a map and let $g: A_1 \to A_1 \times A_2$ the graph function of f. Then if g is strongly approximately pre-continuous then f is strongly approximately pre-continuous and X is p-regular approximation space.

Proof. Suppose that g is strongly approximately pre-continuous to show f is strongly approximately pre-continuous. Let $x \in X$ and V be a set such that $f(x) \in \underline{A_2}(V)$. Then $g(x) \in \underline{A_3}(U \times V)$. Since g is strongly approximately pre-continuous, there is a set U such that $U \subset \underline{A_1}\left(\overline{A_1}(U)\right)$ and $g\left(\overline{A_1}(U)\right) \subset \underline{A_3}(X \times V)$. Therefore we obtain $f\left(\overline{A_1}(U)\right) \subset \underline{A_2}(V)$.

Next let us prove A_1 is p-regular approximation space. Let U be a subset of X, with $x \in \underline{A_1}(U)$, $g(x) \in \underline{A_3}(U \times Y)$, there exist a set G such that $G \subset \underline{A_1}\left(\overline{A_1}(G)\right)$ with $g\left(\overline{A_1}(G)\right) \subset \underline{A_3}(U \times Y)$. Therefore we obtain $x \in G \subset \overline{A_1}(G) \subset \underline{A_1}(U)$. This implies X is p-regular approximation space ☺

Theorem 2.15. Let $f: A_1 \coloneqq (X, R_1) \to A_2 \coloneqq (Y, R_2)$ be a strongly approximately pre-continuous map and A_1 is pre-regular approximation space, then $g: A_1 \coloneqq (X, R_1) \to A_3 \coloneqq (X \times Y, R_3)$ is strongly approximately pre-continuous.

Proof. Let $x \in X$ and W be any subset of $X \times Y$, with $g(x) \in \underline{A_3}(W)$, so there are two sets $U_1 \subset X$ and $V \subset Y$ with $g(x) = (x, f(x)) \in \underline{A_1}(U) \times \underline{A_2}(V) \subset W$. Since f is strongly approximately pre-continuous there is a set $U_2 \subset X$ with $x \in U_2 \subset \underline{A_1}\left(\overline{A_1}(U_2)\right)$ and $f\left(\overline{A_1}(U_2)\right) \subset \underline{A_2}(V)$. A_1 is pre-regular approximation space and $U_1 \cap U_2 \subset \underline{A_1}\left(\overline{A_1}(U_1 \cap U_2)\right)$. So that

$$g\left(\overline{A_1}(U)\right) \subset U_1 \times f\left(\overline{A_1}(U_2)\right)$$
$$\subset \underline{A_1}(U_1) \times \underline{A_2}(V)$$
$$\subset \underline{A_3}(W).$$

This implies g is strongly approximately pre-continuous ☺

For the following result we need the following definitions:

Definition 2.16. An approximation space $A \coloneqq (X, R)$ is called *pre-$T_2 -$ approximation space* if for each pair of distinct points x and y in X, there are two disjoint sets U and V with $U \subset \underline{A}\left(\overline{A}(U)\right)$ and $V \subset \underline{A}\left(\overline{A}(V)\right)$.

Definition 2.17. An approximation space $A \coloneqq (X, R)$ is called $T_0 -$ *approximation space* if for any two distinct points x and y of X there is a set V such that $\underline{A}(V)$ either containing x or containing y.

Theorem 2.18. Let $f: A_1 \coloneqq (X, R_1) \to A_2 \coloneqq (Y, R_2)$ be a strongly pre-continuous injective map and A_2 is $T_0 -$ approximation space, then A_1 is pre- $T_2 -$ approximation space.

Proof. Let A_2 be a $T_0 -$ approximation space, x and y be two points in X. Since f is injective, $f(x) \neq f(y)$, and there is either a set V such that $f(x) \in \underline{A_2}(V)$ not containing $f(y)$ or a set W such that $f(y) \in \underline{A_2}(W)$. $f(x) \in \underline{A_2}(V)$ there exists a set U such that $x \in U$ and $U \subset \underline{A_1}\left(\overline{A}_1(U)\right)$ also $f\left(\overline{A}_1(U)\right) \subset \underline{A_2}(V)$. This implies $f(y) \in f\left(\overline{A}_1(U)\right)$. Therefore $\left(\overline{A}_1(U)\right)^c \subset \overline{A}_1\left(\underline{A_1}(U)\right)$. Similarly if $f(y) \in \underline{A_2}(W)$.

For our last results we need the following three definitions:

Definition 2.19. A map $f: A_1 \coloneqq (X, R_1) \to A_2 \coloneqq (Y, R_2)$ is said to be *strongly approximately continuous* if for each x in X and each set V with $\underline{A_2}(V)$ containing $f(x)$ there exists a set U such that $\underline{A_1}(U)$ containing x and satisfying $f\left(\overline{A}_1\left(\underline{A_1}(U)\right)\right) \subset \underline{A_2}(V)$.

Definition 2.20. An approximation space $A \coloneqq (X, R)$ is called $p - closed$ *approximation space* if for every family of sets $\{V_\lambda\}_{\lambda \in \wedge}$, such that $X \subset \cup_{\lambda \in \wedge} \underline{A}(V_\lambda)$ there is a finite subset $\acute{\wedge}$ of $\wedge$ such that $X \subset \cup_{\lambda \in \acute{\wedge}} \overline{A}\left(\underline{A}(V_\lambda)\right)$.

Definition 2.21. An approximation space $A \coloneqq (X, R)$ is called compact approximation space if for every family of sets $\{V_\lambda\}_{\lambda \in \wedge}$ such that $X \subset \cup_{\lambda \in \wedge} \underline{A}(V_\lambda)$ there is a finite subset $\acute{\wedge}$ of $\wedge$ such that $X \subset \cup_{\lambda \in \acute{\wedge}} \underline{A}(V_\lambda)$.

Theorem 2.22. If $f: A_1 \coloneqq (X, R_1) \to A_2 \coloneqq (Y, R_2)$ is strongly continuous map and K is a p-closed approximation space relative to X, then $f(K)$ is compact approximation space of Y.

Proof. Let $\{V_\lambda\}_{\lambda \in \wedge}$ be a family of subset of Y such that $f(K) \subset \cup_{\lambda \in \wedge} \underline{A_2}(V_\lambda)$. For each $x \in K$ there is a $\lambda_x \in \wedge$ such that $f(x) \in \underline{A_2}(V_{\lambda_x})$. Since f is strongly approximately continuous map there is a set U_x such that $f\left(\overline{A}_1(U_x)\right) \subset \underline{A_2}(V_{\lambda_x})$. $K \subset \cup_{x\in K} \underline{A_1}(U_x)$, so there is a finite subset E of K such that $K \subset \cup_{x\in E} \overline{A}_1\left(\underline{A_1}(U_x)\right)$. This implies $f(K) \subset \cup_{x\in E} \underline{A_2}(V_{\lambda_x})$. So $f(K)$ is compact approximation space.

Theorem 2.23. Let $f: A_1 \coloneqq (X, R_1) \to A_2 \coloneqq (Y, R_2)$ be a strongly approximately continuous map, then if A_1 is p-closed approximation space, then A_2 is compact approximation space.

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