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FULLY BOUNDED MODULES.

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A R T I C L E I N F O

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A B S T R A C T

This paper contains some results a bout fully bounded modules. Various conditions where given to ensure that bounded modules are fully bounded modules.

Introduction:

Let R be a commutative ring with identity and M a unitary (left) R-module. M is called fully bounded R-module if M is bounded and every proper submodule of M is bounded (where M is called bounded R-module provided that there exists an element $x \in M$ such that $ann_R M = ann_R (x)$.[1,p.70])

In the first section of this paper , the relation between fully bounded modules and bounded modules are studied .In fact every fully bounded R-module is bounded , but the converse is not true in general see(1.3) . However, we have shown that the converse holds in the class of faithful fully stable modules, see(1.7). Next we study some classes of modules that also related to fully bounded modules, such as torsion-free, projective, cyclic faithful and free modules.

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In section two, we study some properties of fully bounded, for example if M_1 and M_2 be two fully bounded R-modules, then $M_1 \oplus M_2$ is fully bounded R-module. Next the behavior of fully bounded modules under localization is also considered in this section, see (2.6).

We finally remark that R in this work stands for a commutative ring with identity and all modules are unitary (left) modules.

$1:.Fully Bounded Module

Definition(1.1): An R- module M is said to be fully bounded if M is bounded and every proper submodule is bounded as an R-module. M is bounded if there exists $x \in M$ such that $ann_R M = ann_R (x)$. [1, p.70]

Corollary (1.2): Every fully bounded R-module is bounded.

But the converse is not true in general for example.

Example(1.3):

Let $R=\{f/f:\Re\to\Re$ is map $\}$,we define + and $\bullet$ on R as follows:

$(f+g)(x)=f(x)+g(x)$ and

$(f\bullet g)(x)=f(x)\bullet g(x)$ $\forall f,g\in R$, $\forall x\in\Re$.

$(R,+,\bullet)$ is commutative ring with identity where $I:\Re\to\Re$ such that $I(x)=1$ $\forall x\in\Re$ is the identity element of R.

Let $M=R$ as an R-module,then M is bounded R-module [2,1.1.13(2)].

Let $N=\{f\in R; f(x)=0 \quad \forall x\notin[-n,n]\}$ where $n\geq 0$ is an integer depending on f.

To prove N is a submodule of M.

$N\subseteq M$ and $N=\phi$ since the zero map is in N.

Let $f,g\in N$, then there exists n,m non-negative integers such that $f(x)=0,\forall x\notin[-n,n]$, $g(x)=0$, $\forall x\notin[-m,m]$

If $n>m$,then

$(f-g)(x)=f(x)-g(x)=0$

$\forall x\notin[-n,n]$.Thus $f-g\in N.$

Let $h\in R$ and $f\in N,$ then there exists an integer $n\geq 0$ such that $f(x)=0 \quad \forall x\notin[-n,n]$,

$(h\bullet g)(x)=f(x)\bullet g(x)=0 \quad \forall x\notin[-n,n]$,

then $h\bullet f\in N$,thus N is a submodule of M.

We claim that $ann_R N=\{\ \text{O}\ \}$, let $h\in R$ and $h\neq 0$, then $h(a)\neq 0$ for some $a\in\Re.$ Define $f:\Re\to\Re$ such that:

$$f(x)=\begin{cases}0\cdots if\cdots x\neq a\\ b\ldots if\ldots x=a\end{cases}$$ where $b\neq 0$

Hence $f\in R$ and $f(x)=0\cdots\forall x\notin[-n,n]$, where $n>a$.

Therefore $f\in N$ and $(h\bullet f)(a)=h(a)\bullet f(a)\neq 0$ .Hence $ann_R N=\{\ \text{O}\ \}$.

While ,for each $f\in N, ann_R N\neq\{\ \text{O}\ \}$.For if $f\in N$ then $f(x)=0\cdots\forall x\notin[-n,n]$, n is non-negative integer. Define $h:\Re\to\Re$ such that:

$$h(x)=\begin{cases}0\cdots\forall x\in[-n,n]\\ x\cdots if\cdots x\notin[-n,n]\end{cases}$$ $,x\in\Re$ and $x\neq 0$

then $h\in R$ and

$(h\bullet f)(x)=h(x)\bullet f(x)=0$, implies that $h\in ann_R f$. Therefore N is not bounded R-module.#

However we shell give in anther place the conditions under which the converse of (coro.1.2) is true.

Let R be an integral domain and M be an R-module .An element $x\in M$ is called a torsion element of M if $ann_R M \neq 0$. The set of torsion elements denoted by $T(M)$ is a submodule of M .If $T(M)=0$ the R-module M is said to be torsion-free.[3,p.45]

Proposition (1.4):Every torsion-free R-module (where R is an integral domain) is fully bounded.

Proof : Since every torsion-free R-module is bounded [2,propo. 1.1.6] and every submodule of torsion-free is torsion-free .Which completes the proof .#

So we have the following results.

Corollary (1.5): Let R be an integral domain, then

(a)A projective R-module is fully bounded.

(b)A multiplication faithful R-module is fully bounded.

(c)A cyclic faithful R-module is fully bounded.

(d)A divisible multiplication R-module is fully bounded.

(e)A free R-module is fully bounded .

Corollary (1.6): Z_p (where P is prime) is fully bounded Z -module.

Recall that an R-module M is said to be fully stable if $ann_M(ann_R(x))=(x)$ for each $x \in M$. [5,coro.3.5]

The following proposition gives a partial converse of corollary (1.2).

Proposition (1.7): Let M be bounded faithful fully stable R-module, then M is fully bounded.

Proof : Since M is bounded fully stable ,then M is cyclic [2,propo.1.1.4]. Thus M is fully bounded [coro.1.5(c)].#

In the class of faithful fully stable modules , we have the following characterization .

Proposition (1.8): Let M be faithful fully stable R-module, then M is bounded R-module if and only if M is fully bounded.

An R-module M is said to be uniform module if every non-zero submodule of M is essential [4]. A submodule N of an R-module M is called essential provided that $N \cap K \neq 0$ for every non-zero submodule K of M .[4]

Now, we have the following proposition .

Proposition(1.9): Let M be an R-module and $0 \neq x \in M$ such that :

(1) Rx is an essential submodule of M .

(2) $ann_R(x)$ is a prime ideal of R , and

(3) $ann_R M = ann_R(x)$.

Then M is fully bounded .

Proof : By (1),(2) and (3) every submodule of M is bounded [2,propo.1.2.2]. Thus this completes the proof.#

The following results are consequence of proposition (1.9) .

Corollary (1.10): If M is bounded uniform R-module such that $ann_R M$ is prime ideal of R , then M is fully bounded.

Corollary (1.11): If M is bounded uniform faithful R-module , then M is fully Bounded .

An R-module M is said to be a prime module if $ann_R M = ann_R N$ for every non-zero submodule N of M .[6]

Proposition (1.12) : Let M be a uniform R-module and $ann_R M$ is prime ideal of R. Then the following are equivalent:

(1) M is bounded R-module .

(2) M is prime R-module .

(3) M is fully bounded R-module .

Proof : (1)$\Rightarrow$(2) by [2,propo.1.3.4] .

(2)$\Rightarrow$(3) Since M is prime ,then $ann_R M = ann_R N$ for every non-zero submodule N of M and M is bounded [2,p.24] (i.e. there exists $x \in M$ such that $ann_R M = ann_R(x)$). Thus $ann_R N = ann_R(x)$,therefore N is bounded R- module.

(3)$\Rightarrow$(1) by [coro. 1.2] .#

Proposition(1.13): If R is an integral domain and M is faithful uniform R-module ,then the following are equivalent:

(1) M is bounded R-module .

(2) M is torsion-free R-module .

(3) M is fully bounded R-module .

Proof : (1)$\Rightarrow$(2) by [2,propo.1.3.6] .

(2)$\Rightarrow$(3) by [propo.1.4] .

(3)$\Rightarrow$(1) by [coro. 1.2] .#

$2.Some Properties Of Fully Bounded Modules:

Proposition (2.1): Let M_1 and M_2 be two fully bounded R-modules, then $M_1 \oplus M_2$ is fully bounded R-module.

Proof : Since M_1 is fully bounded , then M_1 is bounded and every proper submodule N_1 is bounded ,that is there exists $x \in N$ such that $ann_R N_1 = ann_R(x)$. Also M_2 is bounded and every proper submodule N_2 is bounded , that is there exists $y \in N_2$ such that $ann_R N_2 = ann_R(y).$ We claim $ann_R(N_1 \oplus N_2) = ann_R((x,y)).$

Let $r(x,y) = (0,0)$, so $(rx,ry) = (0,0)$. It follows that $rx = 0$ and $ry = 0$,that is $r \in ann_R(x)$ and $r \in ann_R(y)$, therefore $r \in ann_R N_1$ and $r \in ann_R N_2$ .Now if $(n_1,n_2) \in N_1 \oplus N_2$,then $r(n_1,n_2) = (rn_1,rn_2) = (0,0)$, implies that $r \in ann_R(N_1 \oplus N_2)$ .Therefore $ann_R(N_1 \oplus N_2) = ann_R((x,y))$ .Since $M_1 \oplus M_2$ is bounded R-module [2,propo.1.1.14], which completes the proof. #

Note that a direct summand of a fully bounded module need not be fully bounded in general. For example:

Let $M = Z \oplus Z_{p^\infty}$ as a Z-module M is fully bounded ,because M is bounded ($ann_Z M = 0 = ann_Z((1,0)))$ and every proper submodule $N = nZ \oplus (\frac{1}{p^n} + Z)$ [2,coro.1.1.3and propo.1.1.14] is bounded , but Z_{p^∞} is not fully bounded since it is not bounded Z-module.

By proposition (2.1) and by mathematical induction we have the following:

Corollary (2.2): A finite direct sum of fully bounded R-modules is fully bounded.

However, an infinite direct sum of fully bounded R-modules need not be fully bounded ,For example:

Z_p as a Z-module is fully bounded for all prime P [coro.1.6] , but $\underset{p-is.prime}{\oplus} Z_P$ is not fully bounded Z-module, because it is not bounded Z-module[2,exa.1.1.16].

Proposition(2.3): Let M be an R-module and let I be an ideal of R , which is contained in $ann_R M$. Then M is fully bounded R-module if and only if M is a fully bounded R/I-module .

Proo f: If M is fully bounded R-module , then M is bounded and every proper submodule N is bounded , that is there exists $x \in N \ni ann_R N = ann_R(x)$.

We claim that $ann_{R/I} N = ann_{R/I}(x)$.

Let $r+I \in ann_{R/I}(x)$,so $(r+I)x = 0$,but $(r+I)x = rx = 0$,that is $r \in ann_R(x)$,therefore $r \in ann_R N$,implies that $rn = 0 \forall n \in N$.

Then $(r+I)n = 0 \forall n \in N$,therefore $r+I \in ann_{R/I} N$, thus N is bounded R/I-module and since M is bounded R/I-module [2,propo.1.1.17].Then M is fully bounded R/I-module.

Next, if M is a fully bounded R/I-module ,then M is bounded and every proper submodule K is bounded , that is there exists $y \in K \ni ann_{R/I} K = ann_{R/I}(y)$.

We claim that $ann_R K = ann_R(y)$.

Let $r \in ann_R(y)$,so $ry = 0$,but $ry = (r+I)y = 0$,that is $r+I \in ann_{R/I}(y)$,

therefore $r+I \in ann_{R/I} K$, implies that $(r+I)k = 0 \forall k \in K$,then $rk = 0 \forall k \in K$,therefore $r \in ann_R K$,so K is bounded M-module and since M is bounded R-module [2,propo.1.1.17] .Then M is fully bounded R-module.#

Corollary(2.4): Let M be an R-module ,then M is fully bounded R-module if and only if M is fully bounded $R/ann_R M-$ module.

Corollary(2.5): For each positive integer $n>1, Z_n$ is a fully bounded Z_n- module.

Proposition(2.6): Let R be a PID and M be a finitely generated fully bounded R-module, S be a multiplicatively closed subset of R, then M_S is fully bounded R_S- module.

Proof : Since M is fully bounded R-module, then M is bounded and every proper submodule N is bounded ,that is there exists $x \in N \ni ann_R N = ann_R(x)$,

so $(ann_R N)_S = (ann_R(x))_S$ .But N is finitely generated [7,coro.4.5.2,p.203] , thus $ann_{R_S} N_S = ann_{R_S}((x)_S)$

[3,propo.3.14,p.43], hence N_S is a bounded R_S- module and since M_S is a bounded R_S- module [2,propo.1.1.28].Then M_S is fully bounded R_S- module.#

Corollary(2.7): If P is a prime ideal of R (where R is PID) and M is finitely generated fully bounded R-module, then M_P is fully bounded R_P- module.

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موديولات المقيدة التامة

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الخلاصة

يحتوي هذا البحث على عدد من النتائج حول الموديولات المقيدة التامه، وقد اعطيت شروط متنوعه تجعل كل موديول مقيد يحققها موديولا مقيدا تاما

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