P- ISSN 1991-8941 E-ISSN 2706-6703 Journal of University of Anbar for Pure Science (JUAPS) Open Access
2012,(6), (3) :106-108

Further Results on Action of Finite Groups on Commutative Rings

Sind K.Ibrahim

University of Anbar -College of Science.

ARTICLE INFO

Received: 6 / 3 /2012
Accepted: 16 / 7 /2012
Available online: 29/8/2013
DOI: 10.37652/juaps.2012.82140

Keywords:
Action of Finite Groups ,
Commutative Rings.

ABSTRACT

Let R be a commutative ring with identity 1 and G be a finite group of automorphisms of R of order n,Let R^G be the fixed subring of R. In this paper we study the relations between the ideals of R^G and R and we study R^G In case R is e-ring(field).

Introduction

Let R be a commutative ring with identity 1 and let G be a finite group of automorphisms of R of order n. Let:

$R^G = \{r \in R/g(r)=r, \text{ for all } g \in G\}$

The set R^G is a subring of R, it is called the fixed ring of G. A.G.Naoum and the author [1,2] studied the relations between R and R^G, they studied some certain ring theoretic properties of R which satisfied in R^G, for more informations see [3,4].

In this paper we study some further results of the ring R^G. We show that if I is G-invariant and |G| is invertible in R, then I is a maximal ideal in R if and only if $I \cap R^G$ is a maximal ideal in R^G, and if I is a prime ideal in R, then $I \cap R^G$ is a prime ideal in R^G.

Also, we show that if (a,b) is a projective ideal of R, then (a^n,b^n) is a projective ideal in R^G, where |G| =n.

Finally, we show that if R is e-ring (field) and |G| is invertible in R, then R^G is e-ring (field).

Ideals of R and RG:

In this section we study the relation between the elements, and the ideals of R and those of R^G. We start with the following definition, remarks and proposition.

We recall that an ideal M of the ring R is said to be G-invariant if $g(M) \subseteq M$, for all $g \in G$,

Where $g(M)=\{g(m);m \in M\}$.

Remarks:

l- If I is an ideals of R, then $I \cap R^G$ is an ideal of R^G.

2- If an element a in R^G is invertible in R, then a is invertible in R^G.

The converse is clear [5].

* Corresponding author at: University of Anbar -College of Science.E-mail address:

Proposition:

Let G be a finite group of automorphisrns of R of order n. Let $b \in R$,and let $x=\sum_{i=1}^{n} g_i(b)$, $y = \prod_{i=1}^{n} g_i(b)$,

$z = \sum_{i=1}^{n} g_i(b) g_{\delta(i)}(b)$, where δ is a 2-Cycle and $g_i \in G$.Then each of x, y and z belong to R^G, in general, $\sum_{i=1}^{n} g_i(b) g_{\delta(i)}(b) \cdots g(b) \in R^G$, where δ is an m-cycle and $1 \leq m \leq n - 2$.

Proof: [5].

Theorem:

Let R be a commutative ring with identity 1 and G be a finite group of automorphisms of R of order n and |G| is invertible in R, Let M be G-invariant ideal of R, then M is a maximal ideal in R if and only if $M \cap R^G$ is a maximal ideal in R^G.

Proof:

Let $a \in R^G$ and $a \notin M \cap R^G$, then $a \notin M$. But M is maximal in R, so M+Ra=R. Since $1 \in R$, then $1 \in M + Ra$, Thus 1=m+ra, where $m \in M$, $r \in R$. Thus:

$$1= n^{-1}\left(\sum_{i=1}^{n} g_i(m)+a \sum_{i=1}^{n} g_i(r) \right)$$

But M is G-invariant, implies $g_i(M) \subseteq M$, so $g_i(m) \in M$, for all $g_i \in G$ and then,

$n^{-1} \sum_{i=1}^{n} g_i(m) \in M$. By Proposition (1-2) and Remark (l-l) (2),$n^{-1} \sum_{i=1}^{n} g_i(m) \in R^G$, implies $n^{-1}\sum_{i=1}^{n} g_i(m)\in M \cap R^G$, and by Proposition (l-2) ,$\sum_{i=1}^{n} g_i(r) \in R^G$, so $a\sum_{i=1}^{n} g_i(r) \in R^G a$.

Then $1\in (M \cap R^G) + R^G a$. Therefore $M \cap R^G$ is a maximal ideal in R^G.

Conversely, let J be an ideal of R such that $M\subset J \subseteq R$ then,

$M\cap R^G \subset J\cap R^G \subseteq R^G$. But $M\cap R^G$ is a maximal ideal in R^G, so $J\cap R^G = R^G$. Since $1\in R^G$, then $1\in J\cap R^G$ which means $1\in J$. Hence J=R. Therefore M is a maximal ideal in R.

Theorem:

Let R be a commutative ring with identity 1 ring and G be a finite group of automorphisms of R of order n. If I is a prime ideal in R, Then $I\cap R^G$ is a prime ideal in R^G.

Proof:

Let x, y$\in R^G$ such that $x.y\in I \cap R^G$, then $x.y\in I$ and $x.y\in R^G$. Since I is a prime ideal in R, then either $x\in I$ or $y \in I$. Hence either $x\in I \cap R^G$ or $y\in I\cap R^G$, then $I\cap R^G$ is a prime ideal in R^G.

Before we start the next result, we recall that a finitely generated ideal A which is generated by $\{a_l, a_2..., a_n\}$ in R is projective if and only if there exists an n ×n matrix $M=(r_{ij})$ with elements in R such that:

i) UM = U, and

ii) $U^{\perp}$ = ann (M)

where U= $(a_l, a_2..., a_n) \in R^n$ is a vector and $U^{\perp}= \{X\in R^n; UX'=0\}$ is the orthogonal complement of U, X' is the column vector which is the transpose of X [6].

Theorem:

If the ideal (a,b) is a projective in R, then (a^n,b^n) is a projective in R^G.

Proof:

Let (a,b) be an ideal in R^G, implies that $a,b\in R$ and (a,b) is a projective in R,

then there exists a matrix M =$\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$ where $r_{ij}\in R$, i, j =1,2 such that:

l) (a,b) M=(a,b) and

2) ann (a,b)=ann(M), thus:

$$(a,b).\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} =(a,b)$$

Hence $ar_{11}+br_{21}=a$

$ar_{12}+br_{22}=b$. Thus:

$a(1-r_{11})=br_{21}$ and $b(1-r_{22})=ar_{12}$. Therefore:

$$a^n\prod_{i=1}^{n}(1-g_i(r_{11})) = b^n\prod_{i=1}^{n}g_i(r_{21}) \quad , \quad \text{and}$$

$$b^n\prod_{i=1}^{n}(1-g_i(r_{22})) = a^n\prod_{i=1}^{n}g_i(r_{12}) \ , g_i\in G$$

Put $\prod_{i=1}^{n}(1-g_i(r_{11})) = 1-s_{11}, \prod_{i=1}^{n}g_i(r_{21}) = s_{21}$

$\prod_{i=1}^{n}(1-g_i(r_{22})) = 1-s_{22}, \prod_{i=1}^{n}g_i(r_{12}) = s_{12}$

Then $1-s_{11}$, s_{21}, $1- s_{22}$; and s_{12} are in R^G (by Proposition 1-2).

Thus $a^n=a^n s_{11}+ b^n s_{21}$ and $b^n=b^n s_{22}+ a^n s_{12}$

Put $M'=\begin{pmatrix} s_{11}\, s_{12} \\ s_{21}\, s_{22} \end{pmatrix}$.Hence $(a^n,b^n)M'=(a^n,b^n)$.

Now to prove that $ann(a^n,b^n)$= ann (M')

Let $(x,y)\in ann(a^n,b^n)$, To prove $(x,y)\in ann(M')=\{X\in R^n; M' X' =0'\}$

i.e. $M'\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,equivalently

$s_{11}x+s_{12}y =0$ and $s_{21}x+s_{22}y =0$. We use the induction on the order G: If n=1, $(x,y)\in ann(a,b)$ implies $(x,y)\in ann(M)$.So:

$M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.Hence $r_{11}x +r_{12}y =0$ and $r_{21}x +r_{22}y=0$

But |G|=1, so $R=R^G$ and M =M'. Thus $(x,y)\in ann(M')$.

Suppose it is true for n-1 that is $ann(a^{n-1},b^{n-1})\subseteq ann(M')$.

Let $(x,y)\in ann(a^n,b^n)$,then $(xa^{n-1},yb^{n-1})\in ann(a,b)$.So $(xa^{n-1},yb^{n-1})\in ann(M')$.

Thus $s_{11}xa^{n-1} + s_{21}yb^{n-1}=0$, $s_{12}xa^{n-1} + s_{22}yb^{n-1}=0$.

Therefore $(s_{11}x,s_{21}y)\in ann(a^{n-1},b^{n-1})\subseteq ann(M')$ and $(s_{12}x,s_{22}y)\in ann(a^{n-1},b^{n-1})\subseteq ann(M')$.

Hence:

$s_{11}(s_{11}x) + s_{21}(s_{21}y)=0$

$s_{12}(s_{12}x) + s_{22}(s_{22}y)=0$.

Thus, $s_{11}=1-(1-\sum_{k=1}^{n-1}(g_1(r_{11})\ldots g_k(r_{11}))$

$$= \sum_{k=1}^{n-1}(g_1(r_{11})\ldots g_k(r_{11}))$$

Hence

$$g(r_{11})(\sum_{k=1}^{n-1} g_1(r_{11})\ldots g_k(r_{11})) = \sum_{k=1}^{n} g_1(r_{11})\ldots g_k(r_{11}) = s_{11}$$

Similarly for s_{12} s_{21} and s_{22}. Hence:

$s_{11}x + s_{12}y=0$, and $s_{21}x + s_{22}y=0$

Thus (x,y)∈ann(M') and ann(M')⊆ann(a^n,b^n).

Therefore ann(a^n,b^n)=ann(M).So (a^n,b^n) is a projective ideal in R^G.

e-ring and fields:

We start this section by the following:

We recall that a ring R is said to be e-ring if for all x∈R, there exists y∈R such that xy=x [7].

Theorem:

Let R be a commutative ring with identity 1 and G be a finite group of automorphisms of R of order n, and |G| is invertible in R. If R is an e-ring, then R^G is an e-ring.

Proof:

Let x∈R^G, then x ∈ R, but R is an e-ring, then there exist y∈R, such that xy=x, Thus $x\sum_{i=1}^{n} g_i(y) = nx$, g_i∈G.

So $x(n^{-1}\sum_{i=1}^{n} g_i(y)) = x$. This means $n^{-1}\sum_{k=1}^{n} g_i(y) \in R^G$ [by Proposition 1-2 and Remark l-l(2)]. Therefore R^G is an e-ring. Finally, we have the following result.

Theorem:

Let R be a commutative ring with identity 1 and G be a finite group of automoiphisms of R of order n ,such that |G| is invertible in R.If R is a field, then R^G is a field.

Proof:

R is a commutative ring with l, implies that R^G is a commutative ring with l.

Let r∈R^G, r≠0 then r∈R. Hence, there exists s∈R such that r.s=l, then $r\sum_{k=1}^{n} g_i(s)=n$, Since |G| is invertible in R,then $r(n^{-1}\sum_{i=1}^{n} g_i(s)) = 1$.By Remark 1-1(2) and Proposition 1-2,

$n^{-1}\sum_{i=1}^{n} g_i(s)=r^{-1} \in R^G$,then r has a multiplicative inverse in R^G.Therefore R^G is a field.

References:

[1] A.G.Naoum.(1987). The ohm type properties for finitely generated multiplication ideals. *Periodica Mathematica Hungarica* 18 , 287-293.

[2] S. Montgomery.(1980). Fixed rings of finite automorphism groups of associative rings .LNM:818,Springer – Verlag, Berlin Heidelberg New York.

[3] F. Dums.(2011). Non- commutative invariant. Lauret Adrover Republican Argentina Press.

[4] J. Has.(2010). The zero-divisor group under a group action in a commutative ring. *J. Korean Math.*, Soc., 47, No. 5, pp. 1097-1106.

[5] S. K. Ibrahim.(1996)."Action of finite groups on commutative rings". M.Sc. thesis, Univ. of Baghdad.

[6] A.G.Naoum.(1981). On finitely generated projective ideals in commutative rings. *Periodica Mathematics Hangarica*, l2, 287-292.

[7] R.Gilmer.(1972). Multiplicative ideal theory. M. Dekker, New York, MR55:323.

نتائج أخرى حول فعل الزمر المنتهية على الحلقات الأبدالية

سند خليل ابراهيم

الخلاصة:

لتكن R حلقة ابدالية ذات عنصر محايد 1,ولتكن G زمرة منتهية للتشاكلات المتقابلة الذاتية من R الى R ذات رتبة n.لتكن RG حلقة العناصر الصامدة في R.في هذا البحث سندرس العلاقات بين المثاليات في الحلقة RG والحلقة R, كذلك سندرس الحلقة RG عندما تكون الحلقة R حلقة−e وايضا عندما تكون R حقل.