Nearly Finitely Pseudo-Injective Modules

Nagham Ali Hussen
Department of Mathematics , College of Education, University of Tikrit , Tikrit , Iraq
naghham_hdeel@yahoo.com

(**Received:** 13 / 4 / 2011 ---- **Accepted:** 26 / 10 / 2011)

Abstract

In this work, the concepts of finitely pseudo-injective modules and finitely pseudo-injective modules are generalized to nearly finitely pseudo-injective modules. Many basic properties of nearly finitely pseudo-injective modules are obtained, characterizations of nearly finitely pseudo-injective modules are obtained, and relationships between nearly finitely pseudo-injective modules and other classes of modules are studied. New characterizations of semi-simple Artinian ring in terms of nearly finitely pseudo-injective modules are introduced. And Endomorphisms ring of nearly Finitely-pseudo-injective modules are studies.

Introduction

Throught this paper, R will denote an associative, commutative ring with identity, and all R-modules are unitary. For an R-module M, $J(M)$, E(M) and $S = End_R(M)$ will respectively stand for the Jacobson radical of M, the injective envelope of M and the endomorphism ring of M. An R-module M is called Pseudo-N-injective if for any R-submodule A of N and every R-monomorphism from A into M can be extended to an R-homomorphism from N into M [9]. An R-module M is called Pseudo-injective if M is Pseudo-M-injective [7]. An R-module M is called nearly quasi-injective if for any R-submodule N of M and every R-homomorphism $f: N \to M$ there exists an R homomorphism $g: M \to M$ *such that* $g(n) - f(n) \in J(M)$, for all $n \in N$ where $J(M)$ is the Jacobson radical of M [3]. An R-module M is called nearly Pseudo-injective if for any R- submodule N of M and for each R-monomorphism $f: N \to M$, there exists $g: M \to M$ *such that* $goi(n) - f(n) \in J(M)$, i.e $g(n) - f(n) \in J(M)$ [1]. An R-module M is called nearly P-injective if for any principle ideal I of R and each R-homomorphism $f: I \to M$, there exists an element $m \in M$ such that $f(x) - xm \in J(M), for\ all\ x \in I$ [3]. An R-module M is called nearly injective if for each R-monomorphism $f: A \to B$(where A and B are two R-modules), each R-homomrphism $g: A \to M$, there exists an R-homomorphism $h: B \to M$such that $(hof)(a) - g(a) \in J(M)$, for all $a \in A$ [3]. An R-monomorphism $f: N \to M$ is called nearly split, if there exists an R-homomorphism $g: M \to N$ such that $(gof)(n) - n \in J(N) for\ all\ n \in N$ [3]. An R-monomorphism $f: M \to N$ is called nearly pseudo-split, if there exists an R-homomorphism $g: N \to M$ *such that* $(gof)(m) - I_M \in J(M)$ for all $m \in M$, where I_M is the identity R-homomorphism on M, i.e $(gof)(m) - m \in J(M)$ for all $m \in M$ [1]. A ring R is self-injective, if and only if for every ideal I in R and every R-homomorphism $f: I \to R$, there exists an element $r \in R$ such that $f(x) = rx$ for each $x \in I$ [2].

§1:Basic properties of nearly Finitely-pseudo-N-injective modules

In this section we introduce the concept of nearly Finitely-pseudo-N-injective module as a generalization of Finitely pseudo-injective modules.

Definition (1.1)

Let M and N be two R-modules. M is said to be nearly Finitely-pseudo-N-injective module, if for any finitely generated R-submodule A of N, and any R-monomorphism $f: A \to M$, there exists an R-homomorphism $g: N \to M$ such that $g(a) - f(a) \in J(M)$ for all a in A. i.e $(g \circ i_A)(a) - f(a) \in J(M)$ for all a in A and i_A is the inclusion mapping from A into N. An R-module M is called nearly Finitely-pseudo-injective module if M is nearly Finitely-pseudo-M-injective. A ring R is called nearly Finitely-pseudo-injective, if R is nearly Finitely-pseudo-R-injective as R-module.

Examples and Remarks (1.2)

1-All finitely Pseudo-injective modules are nearly Finitely Pseudo-injective module.

2- All Finitely Pseudo-injective modules are trivial examples of nearly Finitely Pseudo-injective modules.

3-Every nearly injective R-module is nearly Finitely Pseudo-N-injective, for all R-module N.

4- Every nearly quasi- injective R-module (so nearly injective R-module) is nearly Finitely Pseudo-injective.

5-Every nearly Finitely quasi- injective R-module is nearly Finitely Pseudo-injective R-module.

6- Z_2 as a Z-modules is nearly Finitely Pseudo- Z_6 -injective. Since Z_2 is finitely-Z_6-injective

7-Z_4 as a Z-modules is Finitely Pseudo -injective.

8-It is not necessary that every submodule of nearly finitely pseudo-injective module is nearly Finitely-pseudo-injective. For example Q as a Z-module is nearly Finitely- pseudo-injective but Z as a Z-submodule of Q is not nearly Finitely- pseudo-injective.

Proposition (1.3)

Let M and N be two R-modules. If M is nearly Finitely-pseudo-N-injective, then M is nearly Finitely-pseudo-A-injective for each finitely generated R- submodule A of N.

Proof

Suppose that M is nearly finitely-pseudo-N-injective R-module. Let H be a finitely generated R-submodule of A and $f: H \to M$ be an R-monomorphism, let i_H be the inclusion mapping from H into A, and i_A be the inclusion mapping from A into N. Consider the following diagram:

$$\begin{tikzcd} H \arrow[r, "i_H"] \arrow[dr, "f"'] & A \arrow[r, "i_A"] \arrow[d, "g_1"] & N \arrow[dl, "g"] \\ & M & \end{tikzcd}$$

Since M is nearly finitely-pseudo-N-injective, then there exists an R-homomorphism $g: N \to M$ such that

$(g \circ i_A \circ i_H)(x) - f(x) \in J(M)$ *for all* $x \in H$.

Now put $g_1 = g \circ i_A$. To prove that $(g_1 \circ i_H)(x) - f(x) \in J(M)$ *for all* $x \in H$.

$(g_1 \circ i_H)(x) - f(x) = ((g \circ i_A)) \circ i_H)(x) - f(x) \in J(M)$.

Hence M is nearly-pseudo-A-injective.

As an immediate consequence of proposition (1.3), we have the following corollary.

Corollary (1.4)

Let N be any finitely generated submodule of an R-module M. If N is nearly Finitely-pseudo-M-injective, then N is nearly Finitely-pseudo-injective.

The next proposition shows that nearly-pseudo-N-injective is inherited by a direct summand.

Proposition (1.5)

Any direct summand of nearly Finitely-pseudo-N-injective R-module is nearly Finitely-pseudo-N-injective.

Proof

Let M be a nearly Finitely-pseudo-N-injective R-module, and let A be any direct summand R-submodule of M. Thus, there exists an R-submodule A' of M such that $M = A \bigoplus A'$. Let B be any finitely generated R-submodule of N, and $f: B \to A$ be an R-monomorphism. Let $g = J_A \circ f$, where $J_A: A \to M = A \bigoplus A'$ is the injection mapping. Then g is an R-monomorphism. Now, consider the following diagram:

$$\begin{tikzcd} B \arrow[r, "i_B"] \arrow[d, "f"] \arrow[dd, bend right, "g"'] & N \arrow[dl, "\alpha"'] \arrow[ddl, "\lambda"] \\ A \arrow[d, shift right, "J_A"'] & \\ M \arrow[u, shift right, "\pi_A"'] & \end{tikzcd}$$

Since M is nearly Finitely-pseudo-N-injective, then there exists an R-homomorphism $\lambda: N \to M$ such that $(\lambda \circ i_B)(b) - g(b) \in J(M)$.Let $\pi_A: M \to A$ be the natural projection R-homomorphism from $M = A \bigoplus A'$ into A. Then $\alpha = \pi_A \circ \lambda: N \to A$, it is clear that α is an R-homomorphism. To prove that $(\alpha \circ i_B)(b) - f(b) \in J(A)$. Where $i_B: B \to N$ is the inclusion mapping. Since $\pi_A: M \to A$ is an R-homomorphism, then by [4,P.214], $\pi_A(J(M)) \subseteq J(A)$. Thus $\pi_A((\lambda \circ i_B)(b) - g(b)) \in J(A)$. That is $((\pi_A \circ \lambda) \circ i_B)(b) - (\pi_A \circ (j_A \circ f))(b) \in J(A)$.

Hence,

$(\alpha \circ i_B)(b) - f(b) \in J(A)$. Therefore A is nearly Finitely-pseudo-N-injective. #

From proposition (1.5)and corollary (1.4) we have the following corollary.

Corollary (1.6)

Any direct summand of nearly Finitely-pseudo-injective R-module is also nearly Finitely-pseudo-injective.

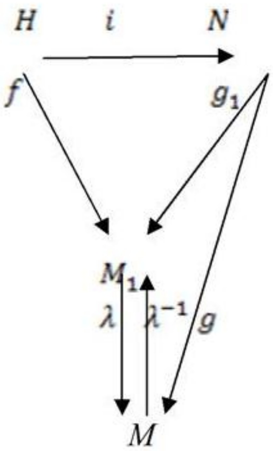
The following proposition shows that nearly Finitely-pseudo-N-injectivity is an algebraic property.

Proposition (1.7)

Isomorphic R-module to nearly Finitely-pseudo-N-injective R-module is nearly Finitely-pseudo-N-injective, for any R-module N.

Proof

Let M be a nearly Finitely-pseudo-N-injective R-module and let $M_1 \cong M$ where M_1 is an R-module. To prove that M_1 is a nearly Finitely-pseudo-N-injective R-module, let H be a finitely generated R-submodule of N, and $f: H \to M_1$ be an R-monomorphism. Since $M_1 \cong M$, then there exists an isomorphism $\lambda: M_1 \to M$. It is clear that $\lambda \circ f: H \to M$ is an R-monomorphism. Now, consider the following diagram, where $i: H \to N$ is the inclusion mapping. Since M is nearly Finitely-pseudo-N-injective R-module, then there exists an R-homomorphism $g: N \to M$ such that $g \circ i = \lambda \circ f$ and $(g \circ i)(x) - (\lambda \circ f)(x) \in J(M)$ for all x in H. Now since λ is an isomorphism, then there exists an isomorphism $\lambda^{-1}: M \to M_1$ such that $\lambda \circ \lambda^{-1} = I_M$ is an identity R-homomorphism.



Put $\lambda^{-1} \circ g = g_1: N \to M_1$, it is clear that g_1 is an R-homomorphism. To prove that $(g_1 \circ i)(x) - f(x) \in J(M_1)$. Since $\lambda^{-1}: M \to M_1$ is an R-monomorphism, then by [4,P.214],$\lambda^{-1}(J(M) \subseteq J(M_1)$.Since $(g \circ i)(x) - (\lambda \circ f)(x) \in J(M)$ *for all x in H*, therefore $\lambda^{-1}((g \circ i)(x) - (\lambda \circ f)(x)) \in J(M_1)$ That is $((\lambda^{-1} \circ g) \circ i)(x) - ((\lambda^{-1} \circ \lambda) \circ f)(x) \in J(M_1)$. Thus $(g_1 \circ i)(x) - f(x) \in J(M_1)$ Hence M_1 is a nearly Finitely- pseudo-N-injective R-module.

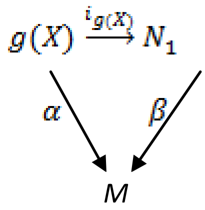
Proposition (1.8)

Let $N_1 and N_2$ be two R-modules, such that $N_1 \cong N_2$. If M is a nearly Finitely-pseudo-N_1-injective R-module, then M is a nearly finitely-pseudo-N_2-injective R-module.

Proof

Suppose that M is a nearly Finitely-pseudo-N_1-injective R-module. To prove that M a is nearly Finitely-pseudo-N_2-injective R-module, let X be any finitely generated R- submodule of N_2, and $f: X \to M$be an R-monomorphism. Since $N_1 \cong N_2$, so there exists an isomorphism $g: N_2 \to N_1$. Since X is a finitely generated R-submodule, then $g(X)$ is a finitely generated R-submodule of N_1. Define $\alpha: g(X) \to M$ *by* $\alpha(g(x)) = f(x)$ *for all x in X*.

Then α is well-define. For if, let $g(x) = g(y)$, *where* $x, y \in X$, since g is an isomorphism, thus $x = y$ *and hence* $f(x) = f(y)$. Now, we have to prove that α is an R-monomorphism. Suppose that $\alpha(g(x)) = \alpha(g(y))$, *where* $x, y \in X$. Thus $f(x) = f(y)$. But f is an R-monomorphism, so $x = y$ which mean that $g(x) = g(y)$. Therefore α is an R-monomorphism. Consider the following diagram:



Since M is a nearly Finitely-pseudo-N_1-injective R-module, there exists an R-homomorphism $\beta: N_1 \to M$ such that $(\beta \circ i_{g(X)})(g(x)) - \alpha(g(x)) \in J(M)$ for all $g(x) \in g(X)$ *and* x *in* X, where $i_{g(X)}: g(X) \to N_1$ is the inclsion mapping. Define, $\gamma: N_2 \to M$ by $\gamma(y) = \beta(g(y))$ *for all* $y \in N_2$. It is clear that γ is an R-homomorphism. In fact, if $y_1, y_2 \in N_2$, then $\gamma(y_1 + y_2) = \beta(g(y_1, +y_2)) = \beta(g(y_1) + g(y_2)) = \beta(g(y_1)) + \beta(g(y_2)) = \gamma(y_1) + \gamma(y_2)$. And if r in R and y in N_2, then $\gamma(ry) = \beta(g(ry)) = \beta(rg(y) = r\beta(g(y)) = r\gamma(y)$. Now, we have to prove that $(\gamma)(x) - f(x) \in J(M)$ where $i_X: X \to N_2$ is the inclusion mapping. Consider $\gamma(x) - f(x) = \beta(g(x)) - \alpha(g(x)) = (\beta \circ i_{g(x)})(g(x)) - \alpha(g(x)) \in J(M)$, which implies that $(\gamma)(x) - f(x) \in J(M)$. Hence M is a nearly Finitely-pseudo-N_2-injective R-module.

The following theorem is a characterization of nearly Finitely-pseudo-N-injective module.

Theorem (1.9)

Let M and N be two R-modules. Then M is nearly Finitely-pseudo-N-injective if and only if M is nearly Finitely-pseudo-N-injective for every free R-module N.

Proof

$(\Rightarrow)$ Trivial.

$(\Leftarrow)$ let A be a finitely generated submodule of R-module N and $g: A \to M$ be an R-monomorphism.

Since N is a set, thus by [10,Th.3.2,P.58], there exists a free R-module, say F, having N as a basis. Now consider the following diagram: where i_A is the inclusion mapping from A into N and i_N is the inclusion mapping from N into F.

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[dr, "g"'] & N \arrow[r, "i_N"] \arrow[d, "h_1"] & F \arrow[dl, "h"] \\ & M & \end{tikzcd}$$

By hypothesis, there exists an R-homomorphism $h: F \to M$ such that $(h \circ (i_N \circ i_A))(a) - g(a) \in J(M) for\ all\ a \in A$.

Put $h_1 = h \circ i_N: N \to M$. It is clear that h_1 is an R-homomorphism. Now, for every $a \in A$, we have $(h_1 \circ i_A)(a) - g(a) = (h \circ i_N) \circ i_A)(a) - g(a) \in J(M)$.

Therefore M is nearly-pseudo-N-injective R-module.

Before we give the next proposition, we introduce the following concept.

Definition (1.10)

Let M and N be two R-modules. An R-monomorphism $f: M \to N$ is said to be nearly Finitely-pseudo-split, if for each finite set $B = \{m_1, m, \dots, m_i\}$ is contained in N, where $i \in Z$ there exists an R-homomorphism $g_B: N \to M$(g_B may depend on B) such that $(g_B \circ f)(m_s) - I_M(m_s) \in J(M) for\ all\ m_s \in B$ where I_M is the identity R-homomorphism on M. That is $(g_B \circ f)(m_s) - m_s \in J(M) for\ all\ m_s \in B$.

Proposition (1.11)

Let M and N be two R-modules. If M is nearly Finitely-pseudo-N-injective, then each R-monomorphism $\alpha: M \to N$ is nearly Finitely-pseudo-split.

Proof

Suppose that M is nearly Finitely-pseudo-N-injective R-module and let $\alpha: M \to N$ be any R-monomorphism, and $a_1, a_2, \dots, a_s \in M$.. Define $\beta: \alpha(M) \to M$ by

$\beta(\alpha(m)) = m\ for\ all\ m \in M$. First, we prove that β is well-defined. Let $\alpha(m_1) = \alpha(m_2)$ where $m_1, m_2 \in M$. Thus $\alpha(m_1) - \alpha(m_2) = \alpha(m_1 - m_2) = 0$ and consequently $m_1 - m_2 \in ker\alpha = \{0\}$, this means that $m_1 - m_2 = 0$, and hence $m_1 = m_2$, which indicates that $\beta(\alpha(m_1)) = \beta(\alpha(m_2))$. Therefore β is well-defined.

Now, we have to prove that β is an R-monomorphism, that is $ker\beta = \{0\}$. Suppose that $\beta(\alpha(m)) = 0, then\ m = 0$, and hence $\alpha(m) = 0$ (since α is an R-monomorphism). There fore $ker\beta = \{0\}$, this proves that β is an R-monomorphism. Now consider the following diagram:

$$\begin{tikzcd} \alpha(M) \arrow[r, "i"] \arrow[dr, "\beta"'] & N \arrow[d, "h"] \\ & M \end{tikzcd}$$

Where $i: \alpha(M) \to N$ is the inclusion mapping. Since M is nearly Finitely-pseudo-N-injective, then there exists an R-homomorphism $h: N \to M$ such that $(h \circ i)(m) - \beta(\alpha(m)) \in J(M)$. That is $(h \circ \alpha)(m) - m \in J(M)$. Hence α is nearly Finitely-pseudo-split R-monomorphism.

As an immediate consequence of Proposition(1.11) we have the following corollary.

Corollary (1.12)

If M is nearly Finitely-pseudo-injective R-module, then each R-monomorphism $f: M \to M$ is nearly Finitely-pseudo-split.

$(h \circ i)(a) - g(a) = ((\beta \circ h_1) \circ i)(a) - g(a) = (\beta \circ (h_1 \circ i))(a) - g(a) = (\beta \circ (\alpha \circ g))(a) - g(a) = (\beta \circ \alpha)(g(a)) - g(a) \in J(M)$.

The following proposition gives a characterization of nearly finitely-pseudo-E(M)-injectivity.

Proposition (1.13)

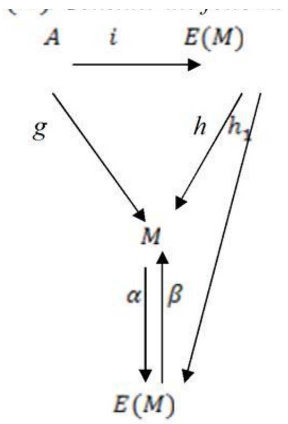
Let M be an R-module. Then M is nearly Finitely-pseudo-E(M)-injective if and only if each R-monomorphism $\alpha: M \to E(M)$ is nearly Finitely-pseudo-split .

Proof

$(\Rightarrow)$ by Proposition (1.11) and Corollary (1.12)

$(\Leftarrow)$ Suppose that each R-monomorphism $\alpha: M \to E(M)$ isnearly Finitely-pseudo-split. To prove that M is nearly Finitely-pseudo-E(M)-injective, let A be a finitely generated R-submodule of E(M) and $g: A \to M$ be an R-monomorphism.

Since E(M) is an extension of M, then by [11,P.43] there exists an R-monomorphism say $\alpha: M \to E(M)$. Consider the following diagram.



Where $i: A \to E(M)$ is the inclusion mapping. Now $\alpha \circ g: A \to E(M)$ is an R-monomorphism. Since E(M) is injective(hence E(M) is a pseudo-injective), there exists an R-homomorphism $h_1: E(M) \to E(M)$ such that $(h_1 \circ i)(a) = (\alpha \circ g)(a)$ for all a in A. Since $\alpha: M \to E(M)$ is nearly Finitely -pseudo-split, therefore, there exists an R-homomorphism $\beta: E(M) \to M$ such that $(\beta \circ \alpha)(a) - a \in J(M)$ for finitely number of a in M. Put $h = \beta \circ h_1$, it is clear that h is an R-homomorphism. For each a in A we have

Thus $(h \circ i)(a) - g(a) \in J(M) for\ all\ a \in A$.

Hence M is nearly Finitely-pseudo-E(M)-injective.

Theorem (1.14)

If $M_1 \oplus M_2$ is nearly Finitely Pseudo-injective R-module, then M_i is nearly Finitely-M_j-injective for each i,j=1,2 ,i$\neq j$

Proof

Let $M_1 \oplus M_2$ be an nearly Finitely Pseudo-injective R-module, we show that $M_1$is nearly Finitely-M_2-injective.Let A be any finitely generated R-submodule of M_2, and f:A→ M_1be any R-monomorphism. Define $g: A \to M_1 \oplus M_2$ by $g(a) = (f(a), a) for\ all\ a \in A, then\ g\ is$ an R-monomorphism. Since $M_1 \oplus M_2$ is nearly Finitely Pseudo-$M_1 \oplus M_2$-injective R-module, and $(0) \oplus M_2$ is an R-submodule of $M_1 \oplus M_2$,thus by Proposition(1.3) $M_1 \oplus M_2$ is nearly Finitely Pseudo-$(0) \oplus M_2$- injective R-module. Since M_2 isomorphic to $(0) \oplus M_2$,thus by Proposition(1.8) $M_1 \oplus M_2$ is nearly Finitely Pseudo-M_2-injective R-module. Thus there exists an R-homomorphism $h: M_2 \to M_1 \oplus M_2$ such that $h(a) - g(a) \in J(M_1 \oplus M_2) for all\ a \in A$. Let $\pi_1: M_1 \oplus M_2 \to M_1$ be the natural projective R-homomorphism of $M_1 \oplus M_2$ to M_1.put $h_1 = \pi_1 \circ h: M_2 \to M_1$ .Thus for each $a \in A$ we have that

$h_1(a) - f(a) = (\pi_1 \circ h)(a) - \pi_1(f(a), a) = \pi_1(h(a)) - \pi_1(g(a))$ $= \pi_1(h(a) - g(a)) \in J(M_1)$ [8]. Therefore M_1 is nearly Finitely-M_2-injective R-module. Consequently, M_2 is nearly Finitely-M_1-injective.

The following corollary is immediately from theorem(1.14).

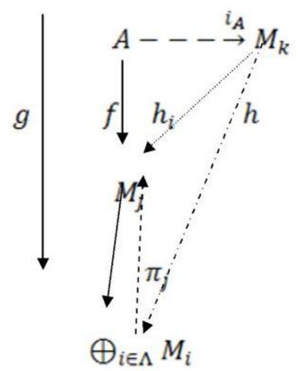
Corollary (1.15)

If $\bigoplus_{i \in \Lambda} M_i$ is nearly Finitely-pseudo-injective R-module, then M_j is finitely- M_k-injective for all $j, k \in \Lambda\ and\ j \neq k$.

Proof:-

Let $\bigoplus_{i \in \Lambda} M_i$ be a nearly finitely-pseudo-injective R-module, we show that M_j is nearly finitely- M_k-injective. Let A be any finitely generated R-submodule of M_k, and let $f: A \to M_j$ be any R-monomorphism. Define $g: A \to \bigoplus_{i \in \Lambda} M_i$ by $g(a) = (0,0, \dots, f(a), 0, \dots, 0), \forall a \in A$, we have to prove that g is monomorphism. Let $a_1, a_2 \in A$, and $g(a_1) = g(a_2)$.Then

$(0,0, \dots, f(a_1), 0, \dots, 0) = (0,0, \dots, f(a_2), 0, \dots, 0)$ and hence $f(a_1) = f(a_2)$. But f is an R-monomorphism, so $a_1 = a_2$. Therefore g is an R-monomorphism. Since $\bigoplus_{i \in \Lambda} M_i$ is a nearly finitely-pseudo- $\bigoplus_{i \in \Lambda} M_i$ –injective R-module, and $0 \oplus 0 \oplus \dots \oplus M_k \oplus 0 \oplus \dots \oplus 0 \oplus 0$ is an R-submodule of $\bigoplus_{i \in \Lambda} M_i$, thus by Proposition (1.3) $\bigoplus_{i \in \Lambda} M_i$ is a nearly finitely-pseudo- $0 \oplus 0 \oplus \dots \oplus M_k \oplus 0 \oplus \dots \oplus 0 \oplus 0$injective R-module. Since M_k isomorphic to $0 \oplus 0 \oplus \dots \oplus M_k \oplus 0 \oplus \dots \oplus 0 \oplus 0$, thus by Proposition (1.8) $\bigoplus_{i \in \Lambda} M_i$ is a nearly finitely-pseudo-M_k-injective R-module, then there exists an R-homomorphism $h: M_k \to \bigoplus_{i \in \Lambda} M_i$ such that $h(a) - g(a) \in J(M_i), \forall a \in A$. Consider the following diagram.



Let $\pi_j:\bigoplus_{i\in\Lambda} M_i \to M_j$ be the canonical projection of $\bigoplus_{i\in\Lambda} M_i$ to M_j, put $h_j = \pi_j \circ h: M_k \to M_j$. Thus $\forall a \in A$. We have that $h_j(a) - f(a) = (\pi_j \circ h)(a) - \pi_j(0,0,\dots,f(a),\dots,0,0) = \pi_j(h(a)) - \pi_j(g(a)) = \pi_j(h(a) - g(a)) \in J(M_j)$.

Therefore M_j *is* a nearly finitely-M_k-injective R-module.

Corollary (1.16)

For any integer $n \geq 2$, M^n is nearly Finitely-pseudo-injective R-module if and only if M is nearly finitely-quasi-injective.

§2: Endomorphisms ring of nearly Finitely-pseudo-injective modules

In this section, we study some properties of endomorphisms rings of nearly Finitely-pseudo-injective modules.

Before we give the main theorem of this section we need to recall the following lemma.

Lemma (2.1) [3]

Let M be an R-module, $S = End_R(M)$ and $\Delta(S) = \{f \in S: N\ ker(f)\ is\ an\ essential\ in\ M\}$, then $\Delta(S)$ is two sided ideal of S.

Theorem (2.2)

Let M be a nearly Finitely-pseudo-injective R-module, and $S = End_R(M)$. Then $S/\Delta(S)$ is a regular ring.

Proof

Let $\lambda + \Delta(S) \in S/\Delta(S)$, where $\lambda \in S$. Put $K = \ker(\lambda)$ and let L be a finitely generated relative complement of K in M. Define $\theta: \lambda(L) \to M, by\ \theta(\lambda(x)) = x, \forall x \in L$. We have to prove that θ is well-defined. Let $\lambda(x_1) = \lambda(x_2), where\ x_1, x_2 \in L.$ Thus $\lambda(x_1) - \lambda(x_2) = 0,$ that is $\lambda(x_1 - x_2) = 0. Hence\ x_1 - x_2 \in \ker(\lambda) = K.$

Since $x_1, x_2 \in L, then\ x_1 - x_2 \in L, where\ x_1 - x_2 \in L \cap K = (0), thus\ x_1 - x_2 = 0$, which implies that $x_1 = x_2$. It follows that $\theta(\lambda(x_1)) = \theta(\lambda(x_2))$. Therefore θ is well-defined. Now, we have to prove that θ is an R-monomorphism. That is $\ker(\theta) = (0)$. Suppose $\theta(\lambda(x)) = 0, then\ x = 0. Thus\ \lambda(x) = \lambda(0) = 0$. Since λ is well-defined monomorphism. Therefore θ is a monomorphism. Consider the following diagram:

$$\begin{tikzcd} \lambda(L) \arrow[r, "i"] \arrow[dr, "\theta"'] & M \arrow[d, "\emptyset"] \\ & M \end{tikzcd}$$

Where $i: \lambda(L) \to M$ is the inclusion mapping. Since M is nearly Finitely-pseudo-injective, then there exists an R-homomorphism $\emptyset: M \to M, such\ that\ \emptyset(\lambda(x)) - \theta(\lambda(x)) \in J(M), \forall x \in L.$ that is for all $x \in L$, we have $\emptyset(\lambda(x)) = \theta(\lambda(x)) + j, for\ some\ j \in J(M)$. Let $u \in K \oplus L, hence\ u = x + y, where\ x \in K, y \in L$. Then $(\lambda - \lambda\emptyset\lambda)(u) = \lambda(u) - (\lambda\emptyset\lambda)(u) = \lambda(x+y) - (\lambda\emptyset\lambda)(x+y) = \lambda(x) + \lambda(y) - (\lambda\emptyset\lambda)(x) - (\lambda\emptyset\lambda)(y)$.

Since $K = \ker(\lambda)$, and $x \in K$, then $\lambda(x) = 0$, whence $(\lambda - \lambda\emptyset\lambda)(u) = \lambda(y) - (\lambda\emptyset\lambda)(y) = \lambda(y) - \lambda\left(\emptyset(\lambda(y))\right)$, but $\emptyset(\lambda(x)) = \theta(\lambda(x)) + j$, then, $(\lambda - \lambda\emptyset\lambda)(u) = \lambda(y) - \lambda(\theta(\lambda(y)) + j) = \lambda(y) - \theta(\lambda(y)) - \lambda(j) = \lambda(y) - \lambda(y) - \lambda(j) \in J(M)$, which implies that $u \in Nker(\lambda - \lambda\emptyset\lambda)$. Hence for each $u \in K \oplus L$, then $u \in Nker(\lambda - \lambda\emptyset\lambda)$. Since $K \oplus L$ is essential in M, then by [8,lemma(5.1.5)a,P.109], $Nker(\lambda - \lambda\emptyset\lambda)$ is essential in M. hence $(\lambda - \lambda\emptyset\lambda) \in \Delta(S)$. Thus $\lambda + \Delta(S) = (\lambda\emptyset\lambda) + \Delta(S)$. That is $\lambda + \Delta(S) = (\lambda + \Delta(S))(\emptyset + \Delta(S))(\lambda + \Delta(S))$. Hence $S/\Delta(S)$ is regular ring.

Proposition (2.3)

If M is nearly Finitely –pseudo-injective R-module, and $S = End_R(M)$, then $J(S) \subseteq \Delta(S)$.

Proof

Let $\alpha \in J(S)$. Since M is nearly Finitely-pseudo-injective, then by Theorem (2.2) $S/\Delta(S)$ is regular ring. That is there exists $\lambda \in S$ such that $\alpha + \Delta(S) = (\alpha + \Delta(S))(\lambda + \Delta(S))(\alpha + \Delta(S))$. Thus $\alpha - \alpha\lambda\alpha \in \Delta(S)$. Put $\beta = \alpha - \alpha\lambda\alpha$. Since $J(S)$ is a two sided ideal of S, then $\lambda\alpha \in J(S)$. Since $J(S)$ is a quasi-regular, then $(I_M - \alpha\lambda)^{-1}$ exists, where $I_M: M \to M$ is the identity R-homomorphism. Hence $(I_M - \alpha\lambda)(I_M - \alpha\lambda)^{-1} = I_M$, since $(I_M - \alpha\lambda)^{-1}(\alpha - \alpha\lambda\alpha) = (I_M - \alpha\lambda)^{-1}(I_M - \alpha\lambda)\alpha = I_M\alpha = \alpha$. Thus $(I_M - \alpha\lambda)^{-1}\beta = \alpha$. Also since $\beta \in \Delta(S), (I_M - \alpha\lambda)^{-1} \in S$ and $\Delta(S)$ is two sided ideal, then Lemma(2.1) $\alpha \in \Delta(S)$. There fore $J(S) \subseteq \Delta(S)$.

The following corollary is immediate from Theorem(2.2) and Proposition(2.3).

Corollary (2.4)

Let M be a nearly Finitely-pseudo-injective R-module, and $S = End_R(M)$. Then $I \cap K = IK + \Delta(S) \cap (I \cap K)$ for each two sided ideals I and K of S.

Proof

Since M is nearly Finitely-pseudo-injective R-module, thus by Theorem (2.2) $S/\Delta(S)$is regular ring. Let I and K be any two sided ideals of S, and let $\lambda \in I \cap K$, thus $\lambda + \Delta(S) \in S/\Delta(S)$. Since $S/\Delta(S)$ is regular ring, then there exists an element $\alpha + \Delta(S) \in S/\Delta(S)$ such that $\lambda + \Delta(S) = (\lambda + \Delta(S))(\alpha + \Delta(S))(\lambda + \Delta(S))$.

Therefore $\lambda - \lambda\alpha\lambda \in \Delta(S) \cap (I \cap K)$.

Put $\lambda_1 = \lambda - \lambda\alpha\lambda$, then $\lambda = \lambda\alpha\lambda + \lambda_1 \in IK \cap \Delta(S) \cap (I \cap K)$.

Thus $I \cap K \subseteq IK + \Delta(S) \cap (I \cap K)$.Since $IK \subseteq I$ and $IK \subseteq K$, Then $IK \subseteq I \cap K$. And since $\Delta(S) \cap (I \cap K) \subseteq I \cap K$, thus $IK + \Delta(S) \cap (I \cap K) \subseteq I \cap K$. Therefore $I \cap K = IK + \Delta(S) \cap (I \cap K)$.

The following corollary is an immediately from Corollary (2.4).

Corollary (2.5)

Let M be a nearly Finitely-pseudo-injective R-module, and $S = End_R(M)$. Then $K = K^2 + \Delta(S) \cap K$ for each ideal K of S.

Proposition (2.6)

Let M be a nearly Finitely-pseudo-injective R-module, and

$S = End_R(M)$. If $\Delta(S) = (0)$, then S is self Finitely-pseudo-injective ring.

Proof

Suppose that M is nearly Finitely-pseudo-injective R-module, and $\Delta(S) = (0)$, then by Theorem (2.2) $S/\Delta(S)$ is regular ring, that is S is a regular ring. To prove that S is self Finitely-pseudo-injective ring, let I be an ideal of S and $f: I \to S$ be an R-monomorphism. Define a finitely generated submodule IM to be generated by $\{\lambda m: \lambda \in I, m \in M\}$, it follows that if $x \in IM$, then there exists $\lambda_1, \lambda_2, \dots, \lambda_n \in I$ and $m_1, m_2, \dots, m_n \in M$, where $n \in Z^+$ such that $x = \sum_{i=1}^{n} \lambda_i m_i$. Define $\theta: IM \to M$ as follows for each $x = \sum_{i=1}^{n} \lambda_i m_i \in IM, \theta(x) = \theta(\sum_{i=1}^{n} \lambda_i m_i) = \sum_{i=1}^{n} f(\lambda_i) m_i$. For each $x, y \in IM$, we have $x = \sum_{i=1}^{n} \lambda_i m_i \text{ and } y = \sum_{j=1}^{n} \alpha_j m'_j$, where $\lambda_i, \alpha_j \in I \text{ and } m_i, m'_j \in M$, for all $i = 1,2, \dots, n, j = 1,2, \dots t \text{ and } t, n \in Z^+$.

Since S is regular ring, thus each finitely generated ideal of S is generated by an idempotent. Thus the ideal of S which is generated by $\lambda_1, \lambda_2, \dots, \lambda_n, \alpha_1, \alpha_2, \dots, \alpha_t$ has the form eS where $e = e^2 \in S$. Since λ_i, α_jbelong to the ideal of S, which is generated by$\lambda_1, \lambda_2, \dots, \lambda_n, \alpha_1, \alpha_2, \dots, \alpha_t$ for all $i = 1,2, \dots, n, j = 1,2, \dots t$. Thus $\lambda_i, \alpha_j \in eS \text{ for all } i = 1,2, \dots, n, j = 1,2, \dots t$, and this implies that $\lambda_i = es_i \text{and } \alpha_j = es'_j$ for some $s_i, s'_j \in S \text{for all } i = 1,2, \dots, n, j = 1,2, \dots t \text{ and } t, n \in Z^+$. Hence $e\lambda_i = e(es_i) = e^2 s_i = es_i = \lambda_i$ for all $i = 1,2, \dots, n$ and $e\alpha_i = e(es'_j) = e^2 s'_j = es'_j = \alpha_j$ for all $j = 1,2, \dots, t$. Therefore $f(\lambda_i) = f(e)\lambda_i \text{and } f(\alpha_j) = f(e)\alpha_j \text{for all } i = 1,2, \dots, n, j = 1,2, \dots t$. Thus $\theta(x) = \theta(\sum_{i=1}^{n} \lambda_i m_i) = \sum_{i=1}^{n} f(\lambda_i) m_i = \sum_{i=1}^{n} f(e)\lambda_i m_i = f(e)\sum_{i=1}^{n} \lambda_i m_i = f(e)x$, and similarly we have $\theta(y) = f(e)y$. θ is well-defined, since for all $x, y \in IM$, if x=y, we have just proved that $\theta(x) = f(e)x$ and $\theta(y) = f(e)y$, Hence $\theta(x) = \theta(y)$. It is clear that θ is an R-monomorphism. Thus we have the following diagram:

$$\begin{tikzcd} IM \arrow[r, "i"] \arrow[dr, "\theta"'] & M \arrow[d, "\emptyset"] \\ & M \end{tikzcd}$$

Where $i: IM \to M$ is the inclusion mapping.

Since M is nearly Finitely-pseudo-injective R-module, thus there exists an R-homomorphism $\emptyset: M \to M$ such that $\emptyset(x) - \theta(x) \in J(M)$ for all $x \in IM$. For each $m \in M$, if $\lambda \in I$, then $(\emptyset\lambda)(m) = \emptyset(\lambda m) = \theta(\lambda m) = \theta(\lambda m) + j = f(\lambda)m + j$ for some $j \in J(M)$. Hence $(\emptyset\lambda - f(\lambda))(m) \in J(M), \forall \lambda \in I$. Therefore $N\ ker(\emptyset\lambda - f(\lambda))$is essential in M for all$\lambda \in I$. Since $\Delta(S) = (0)$, implies that $\emptyset\lambda - f(\lambda) = 0$ that is $\emptyset\lambda = f(\lambda)$ for all $\lambda \in I$. Thus S is self Finitely-pseudo-injective ring.

§3: Characterization of Rings by means of nearly Finitely -pseudo-N-injective R-modules

In this section, we introduce some new characterizations of semi-simple Artinian ring by means of nearly Finitely pseudo- N-injective R-modules.

Theorem (3.1)

The following statements are equivalent for a ring R.

1. R is a semi-simple Artinian.
2. Every R-module is nearly Finitely-pseudo-N-injective for every R-module N.
3. Every finitely generated R-module is nearly Finitely-pseudo-N-injective for every R-module N.

Proof

$(1) \Rightarrow (2)$ Suppose that R is a semi-simple Artinian ring. By [6,Th.3.7, P.439] Every module over semi-simple Artinian ring is injective. That is M is N-injective for every R-module N, then by[1, example and remark (1.2.2)(3),P.19] every R-module is nearly-pseudo-N-injective and hence by example and remark (1.2) every R-module is nearly Finitely-pseudo-N-injective .

$(2) \Rightarrow (3)$ trivial .

$(3) \Rightarrow (1)$ Suppose that every finitely generated R-module is nearly Finitely-pseudo-N-injective for each R-module N. Let M be any simple R-module, Then M is finitely generated. Hence by our assumption M is nearly Finitely-pseudo-N-injective for any R-module N. Now, since M is simple then $J(M) = (0)$[8, p. 218].and Hence by [5, Prop.2.1] M is injective. Now, every simple R-module is injective. Thus by [8,Ex.18,P.272] R is regular ring. And hence R is Jacobson radical. Therefore every finitely generated R-module is injective. Hence R is semi-simple Artinian.

Theorem (3.2)

The following statements are equivalent for a ring R.

1. R is semi-simple Artinian ring.
2. For each R-module M, if M_1 and M_2 are nearly Finitely-pseudo-N-injective R-submodules of M, then $M_1 \cap M_2$ is nearly Finitely-pseudo-N-injective R-submodule of M, for each R-module N.
3. For each R-module M , if M_1 and M_2 are N-injective R-submodules of M, then $M_1 \cap M_2$ is nearly Finitely-pseudo-N-injective R-submodule of M, for each R-module N.
4. For each R-module M, if M_1 and M_2 are pseudo-N-injective submodules of M, then $M_1 \cap M_2$ is nearly Finitely-pseudo-N-injective R-submodule of M, for each R-module N.

Proof

$(1) \Rightarrow (2)$ Follows from Theorem (3.1).

$(2) \Rightarrow (3)$ trivial.

$(3) \Rightarrow (4)$ trivial.

$(4) \Rightarrow (1)$ Let M be any R-module, and $E(M) = E$ is the injective hill of M. Let $Q = E \oplus E$, $K = \{(x,x) \in Q : x \in M\}$ and $\bar{Q} = Q/K$. Put $M_1 = \{y + K \in \bar{Q} : y \in E \oplus (0)\}$ and $M_2 = \{y + K \in \bar{Q} : y \in (0) \oplus E\}$, then $\bar{Q} = M_1 + M_2$, for if $\bar{h} = h + K$ *where* $h \in Q = E \oplus E$. Thus $h = (h_1, h_2) = (h_1, 0) + (0, h_2)$ where $h_1, h_2 \in E$, then $\bar{h} = (h_1, 0) + (0, h_2) + K = ((h_1, 0) + K) + ((0, h_2) + K)$. Since $(h_1, 0) + K \in M_1$ and $(0, h_2) + K \in M_2$, thus $\bar{h} \in M_1 + M_2$. Hence $\bar{Q} \subseteq M_1 + M_2$. It is easily to prove that M_1 and M_2 are R-submodules of $\bar{Q}$. Thus $M_1 + M_2$ is R-submodule of $\bar{Q}$. So $M_1 + M_2 \subseteq \bar{Q}$. From preceding argument, we have $\bar{Q} = M_1 + M_2$. Define $\alpha_1 : E \to M_1$ by $\alpha_1(y) = (y, 0) + K, \forall y \in E$. We have to prove that α_1 is an isomorphism. Assume that $y_1, y_2 \in E$ *and* $r \in R$, then $\alpha_1(y_1 + y_2) = (y_1 + y_2, 0) + K = (y_1, 0) + K + (y_2, 0) + K = \alpha_1(y_1) + \alpha_1(y_2)$ .And $\alpha_1(ry) = (ry, 0) + K = r((y, 0) + K) = r\alpha_1(y)$. Therefore α_1 is an R-homomorphism. Now, let $\alpha_1(y_1) = \alpha_1(y_2)$, then $(y_1, 0) + K = (y_2, 0) + K$, and hence $(y_1 - y_2, 0) \in K$. Thus $y_1 - y_2 = 0$ and so $y_1 = y_2$. Therefore α_1 is R-monomorphism. To prove that α_1 is an epimorphism, let $a + K \in M_1$, whence $a = (d, 0) \in E \oplus 0$, then $\alpha_1(d) = (d, 0) + K = a + K$, which mean that α_1 is an epimorphism. Now, define $\alpha_2 : E \to M_2$ by $\alpha_2(y) = (0, y) + K, \forall y \in E$. Similarly, we can prove that α_2 is an R-isomorphism. Since $(E \oplus (0)) \cap K = (0)$ *and* $((0) \oplus E) \cap K = (0)$ because , if $x \in (E \oplus (0)) \cap K$, then $x \in E \oplus (0)$ *and* $x \in K$. Hence $x = (x_1, 0), x_1 \in E$ and $x = (y_1, y_1), y_1 \in M$. Then $x = (y_1, y_1)$ and consequently $y_1 = 0$. Thus $x = (0,0)$. Since E is an injective R-module, hence E is N-injective for each R-module N. Therefore by [5,Prop. 2.1] M_i is N-injective R-submodule of $\bar{Q}$ for i=1,2. Thus by hypothesis $M_1 \cap M_2$ is nearly Finitely-pseudo-N-injective R-module. Define $f : M \to M_1 \cap M_2$ by $f(m) = (m, 0) + K$ for all $m \in M$. We claim that

$M_1 \cap M_2 = \{y+K \in \bar{Q} : y \in M \oplus (0)\}$. Let $x \in M_1 \cap M_2$, then $x \in M_1$ and $x \in M_2$. Therefore $x = y + K, y \in E \oplus (0)$ and $x = y' + K, y' \in (0) \oplus E$. Then $y = (a,0) + K = (0,a') + K$ which implies that $(a-0, 0-a') \in K$, that is $(a, -a') \in K$. Therefore $a = -a'$, $a \in M$. And consequently $y = (a,0) \in M \oplus (0)$. Then $x = y + K \in \bar{Q}$ and $y \in M \oplus (0)$. Therefore $x \in \{y+K \in \bar{Q} : y \in M \oplus (0)\}$. This shows that $M_1 \cap M_2 \subseteq \{y+K \in \bar{Q} : y \in M \oplus (0)\}$.

Now, let $x \in \{y+K \in \bar{Q} : y \in M \oplus (0)\}$, then $x = y + K \in \bar{Q}$ where $y \in M \oplus (0)$. Thus $y = (a,0) \in M \oplus (0)$ where $a \in M$. It follows that $(a,a) \in K$. Then $k = (a,a) + K = ((a,0)+K) + ((0,a)+K)$. Therefore $x = (a,0) + K = (0,-a) + K$. But $(a,0) + K \in M_1$ and $(0,-a) + K \in M_2$. So $x \in M_1 \cap M_2$. Therefore $\{y+K \in \bar{Q} : y \in M \oplus (0)\} \subseteq M_1 \cap M_2$. From above, we have that $M_1 \cap M_2 = \{y+K \in \bar{Q} : y \in M \oplus (0)\}$. We have to prove now, f is an isomorphism. Let $m_1, m_2 \in M$ and $r \in R$, then $f(m_1 + m_2) = (m_1 + m_2, 0) + K = ((m_1,0)+K) + ((m_2,0)+K) = f(m_1) + f(m_2)$.

$f(rm) = (rm, 0) + K = r(m,0) + K = rf(m)$. This proves that f is an R-homomorphism. Now, let $f(m_1) = f(m_2)$, then $(m_1,0) + K = (m_2,0) + K$, whence $(m_1 - m_2, 0) \in K$, and thus $m_1 - m_2 = 0$, which implies that $m_1 = m_2$. Therefore f is a monomorphism. It is easily to prove that f is an epimorphism, so f is an isomorphism. That is $M \cong M_1 \cap M_2$. Thus M is nearly Finitely-pseudo-N-injective by Proposition (1.7) . Hence every R-module is nearly Finitely-pseudo-N-injective. Therefore by Theorem (3.1) R is a semi-simple Artinian ring.

References

[1]-Al-obaidy,N.J. "On Nearly-Pseudo-injective modules " M.Sc., thesis, Univ. Of Tikrit (2008).

[2]- Al-Tabtabai, M. S. “ On commutative self-injective rings” MSc. Thesis University of Baghdad (1973).

[3]-Al-Yassri,A. R. “Nearly injective modules” M.Sc. Thesis, Al-Mustansiriyah University, (2ooo).

[4]-Burton, D. M. "Abstract and linear algebra" Addison-wesley publish Company, (1972).

[5]-Dinh, H. Q. “ A note on pseudo-injective modules” Comm. Algebra, 33 (2005), p 361-369.

[6]-Hungerford,T. W. "Algebra" Springer-Verlage, New-York, pp.(1974).

[7]-Jain, S. K. and Singh, S. “ On pseudo-injective modules and self pseudo- injective rings” J.Math. Soc. 2 (1967), p. 23-31.

[8]-Kash, F. "Modules and Rings" Academic Press, London, New-York,(1982).

[9]-Pandeya, B. M. and Koirala, S. P." pseudo- M-injective modules" Algebra and its application, Narosa publishing (2001), pp.201-207.

[10]-Rotman, Josef, J. “ An introduction to homological Algebra” Academic press, New York (1979).

[11]-Sharp, D. W. and Vamos, P. "Injective modules" Cambridge, Univ.Press, London,(1972).

المقاسات الاغمارية الكاذبة المنتهية تقريباً

نغم علي حسين

قسم الرياضيات ، كلية التربية ، جامعة تكريت ، تكريت ، العراق

naghham_hdeel@yahoo.com

(**تاريخ الاستلام:** 13 / 4 / 2011 ---- **تاريخ القبول:** 26 / 10 / 2011)

الملخص

في هذا البحث قدمنا مفهوم المقاسات الاغماريه الكاذبه المنتهيه تقريباً بصفته اعماماً الى مفهوم المقاسات الاغماريه الكاذبه المنتهيه. حيث أعطيت العديد من الأمثلة والخواص والتشخيصات للمقاسات الاغمارية الكاذبة المنتهية تقريباً من جانب أخر بحثت عن حلقة التشاكلات الذاتية للمقاسات الاغمارية الكاذبة المنتهية تقريباً وأخيراً أعطيت تشخيصات عن الحلقة الارتينية شبه البسيطة بدلالة المقاسات الاغمارية الكاذبة المنتهية تقريباً.