Irreducible submoduls and strongly irreducible submodules

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(**Received:** 20 / 9 / 2011 ---- **Accepted:** 13 / 3 / 2012)

Abstract

Throughout this work , all rings are considered commutative ring with identity and all modules are unitary. A submodule N of an R-module M is called irreducible if for each submodules N_1 and N_2 of M , such that $N=N_1\cap N_2$, implies that either $N= N_1$ or $N = N_2$. In this work we give generlization for the concepts irreducible submodule . We call a submodule N of M is a strongly irreducible submodule if for each submodules N_1 and N_2 of M , such that $N_1\cap N_2 \subseteq N$, implies that either $N_1 \subseteq N$ or $N_2 \subseteq N$. Furthermore , we generalize some properties of them .

Introduction :

In this work , all rings are considered commutative with identity and all R-modules are unitary. If R is a ring and N is a submodule of an R-module M . The ideal (N:M) ={r $\in$ R : rM $\subseteq$ N} is called the resutial. Then (0:M) is the annihilator of M , denoted by ann M . A ring R is said to be arithmetical if for all ideals I , J and K of R , we have $(I \cap J) + K = (I + K) \cap(J + K)$, [8] . An ideal I of a ring R is said to be irreducible , if I is not the intersection of two ideals of R that properly contain it , [6] . And a submodule N of an R-module M is said to be irreducible , if N is not the intersection of two submodules of M that properly contain it , [8] .This work consists four sections. In section one we consider some properties of irreducible submodules , so we present some characterizations which are relative to our work . as we see in proposition (1-3) that a submodule N of M is an irreducible if and only if L + N is also irreducible submodule of M , for each submodule L of M . Proposition (1-5) and (1-10) show that a submodule N of an R-module M is an irreducible if and only if the intersection (union) respectively , is also irreducible submodule between N and L , where L is any submodule of M . In proposition (1-15) we prove that N is an irreducible submodule of M if and only if N/L is also irreducible submodule of M/L , where $L\subseteq N$. Section two is devoted to study irreducible submodules in multiplication R-modules. Also we study some result of irreducible submodules in multiplication R-modules. In proposition (2-2) , we prove that if R is an integral domain , then every prime element is an irreducible. The main result is a proposition (2-9) that a submodule N of a multiplication R-module M is an irreducible if and only if (N:M) is an irreducible ideal . In section three we study strongly irreducible submodules . Our main concerns in this section is to give a characterization for strongly irreducible submodules and generalize some know properties of irreducible submodules to strongly irreducible submodules . Also we study new properties for it. Moreover, we study the relation between irreducible submodules and strongly irreducible submodules. Finally in section four , we defined a submodule N of an R-module M is a pseudo irreducible if N=LK, with L+K =M for any submodules L and K of M , then L=M or K=M . Also we prove some results which are relative with it, which are appear in [11] that if M is a finitely generated faithful multiplication R-module. Then the submodule N of M is a pseudo irreducible, if and only if (N:M) is pseudo irreducible ideal in aring R. And a submodule N of a finitely generated faithful multiplication R-module is a pseudo irreducible if and only if I is pseudo irreducible ideal in a ring R, where N = IM. Finally , if N is pseudo irreducible submodule of a multiplication R-module M , then N is pseudo prime .

1- Basic properties of irreducible submodules

Definition (1-1)

A submodule N of an R-module M is called irreducible if for each submodules N_1 and N_2 of M , such that $N=N_1\cap N_2$, implies that either $N= N_1$ or $N = N_2$,[8] .

Examples (1-2)

(4) is an irreducible submodule of the Z_{12}-module . But (6) is not irreducible submodule of Z_{12} , since $(2) \cap (3) =(6)$.

The purpose of this section is to introduce interesting and useful properties of irreducible submodules of an R-modules .

Proposition (1-3)

Let R be arithmetical ring , and let M be an R-module. N and L are submodules of M. Then N is an irreducible submodule if and only if L+N is an irreducible submodule .

Proof : Suppos N is an irreducible submodule , thus there exist submodules N_1 and N_2 of M , such that $N_1\cap N_2 = N$, implies that either $N= N_1$ or $N = N_2$.

Thus we get $L+N = L+(N_1\cap N_2) =(L+ N_1) \cap (L+N_2)$, if $N=N_1$, implies $L+N = L+N_1$, and if $N=N_2$, implies $L+N = L+N_2$. Therefore L+N is an irreducible submodule .

Conversely; suppose L+N is an irreducible , and suppose there exist submodules N_1 and N_2 of M , such that $N_1\cap N_2 = N$. Hence $L+N =L +(N_1\cap N_2) = (L+N_1) \cap (L+N_2)$, since L+N is an irreducible , thus either $L+N =L+N_1$, implies $N=N_1$ or $L+N = L+N_2$ implies $N=N_2$. Therefore N is an irreducible submodule .

The following corollary follows directly from the previous proposition.

Corollary (1-4)

Let R be arithmetical ring , and let M be an R-module . N and L are submodules of M , such that N is an irreducible . Then L is an irreducible if and only if and only if L+N is an irreducible submodule .

Proposition (1-5)

Let N and L are submodules of an R-module M , if N∩L is an irreducible . Then N is an irreducible submodule

Proof : Suppose there exist submodules N_1 and N_2 of M , such that $N_1 \cap N_2 = N$. Hence $N \cap L = (N_1 \cap N_2) \cap L = (N_1 \cap L) \cap (N_2 \cap L)$, since $N \cap L$ is an irreducible submodule . Thus either $N_1 \cap L = N \cap L$ or $N_2 \cap L = N \cap L$, implies that either $N_1 \subseteq N$ or $N_2 \subseteq N$. Since $N \subseteq N_1 \cap N_2$, thus $N \subseteq N_1$ and $N \subseteq N_2$, implies $N = N_1$ or $N = N_2$. Therefore N is an irreducible submodule .

The converse is not true in general , as we see in the following example.

Example (1-6)

Let (4) be an irreducible submodule of the Z_{12}-module , and (3) is a submodule of Z_{12}. But $(4) \cap (3) = (12)$ is not an irreducible submodule of Z_{12} .

In the following proposition we give acondition for the converse of proposition (1-5) to hold .

Proposition (1-7)

Let N and L are submodules of an R-module M , such that $N \subseteq L$. Then

N is an irreducible submodule if and only if $N \cap L$ is an irreducible submodule .

Proof : Suppose N is an irreducible submodule . since $N \subseteq L$, thus $N \cap L = N$. Therefore $N \cap L$ is also irreducible .

The converse is similar to proposition 1-5 , hence we omitted.

The following corollaries follows directly from proposition (1-5) .

Corollary (1-8)

Let N and L are submodules of an R-module M , such that L is an irreducible . If $N \cap L$ is an irreducible , then N is an irreducible .

Corollary (1-9)

Let N and L are submodules of an R-module M , such that $N \subseteq L$, and L is an irreducible . then $N \cap L$ is an irreducible if and only if N is an irreducible.

Proposition (1-10)

Let N and L are submodules of an R-module M , if $N \bigcup L$ is an irreducible submodule . Then N is an irreducible submodule .

Proof : Suppos there exist submodules N_1 and N_2 of M , such that $N_1 \cap N_2 = N$. Then we get $N \bigcup L = (N_1 \cap N_2) \bigcup L = (N_1 \bigcup L) \cap (N_2 \bigcup L)$, since $N \bigcup L$ is an irreducible , thus either $N \bigcup L = N_1 \bigcup L$ or $N \bigcup L = N_2 \bigcup L$. Implies that either $N = N_1$ or $N = N_2$. Therefore N is an irreducible submodule .

The converse of proposition (1-10) is not true in general, as we see in the following example.

Example :1-11

Let (4) be an irreducible submodule of the Z_{12}-module , and (2) is a submodule of Z_{12} . But $(4) \bigcup (2) = (2)$ is not an irreducible submodule of Z_{12}-module .

The following remarks we give a condition for the converse of proposition (1-10) to hold .

Remarks (1-12)

Let N and L be submodules of an R-module M , such that $L \subseteq N$, then N is an irreducible submodule if and only if $N \bigcup L$ is an irreducible submodule .

Proof: Suppose N is an irreducible submodule , since $L \subseteq N$, thus N. Therefore $N \bigcup L$ is also irreducible submodule . $= N \bigcup L$

The converse is similar to proposition (1-10) , hence we omitted .

The following corollaries follows directly from proposition (1-10) . Corollary (1-13)

Let N and L are submodules of an R-module M , such that L is an irreducible . If $N \bigcup L$ is an irreducible , then N is an irreducible .

Corollary (1-14)

Let N and L are submodules of an R-module M , such that $L \subseteq N$, and L is an irreducible . Then $N \bigcup L$ is an irreducible if and only if N is an irreducible submodule .

Proposition (1-15)

N Let N and L are submodules of an R-module M , such that $L \subseteq$ Then N is an irreducible submodule of M if and only if N/L is an irreducible submodule of M/L .

Proof : Suppose N is an irreducible submodule , to prove that N/L is irreducible. Let N_1/L and N_2/L are submodules of M/L , such that $N_1/L \cap N_2/L = N/L$. But $N_1/L \cap N_2/L = (N_1 \cap N_2) / L = N / L$, thus $N_1 \cap N_2 = N$. Hence N is an irreducible, thus either $N = N_1$ or $N = N_2$. Implies that either $N/L = N_1/L$ or $N/L = N_2/L$ .Therefore N/L is an irreducible submodule of M/L .

Conversely ; suppose N/L is an irreducible submodule of M/L , to prove that N is irreducible , let $N_1$and N_2 are submodules of M , such that $N = N_1 \cap N_2$. So $N/L = (N_1 \cap N_2) /L = N_1/L \cap N_2 /L$, since N/L is irreducible submodule of M/L.Thus either $N/L = N_1/L$ or $N/L = N_2/L$, hence either $N = N_1$ or $N = N_2$ .Therefore N is an irreducible submodule .

Proposition (1-16)

If R is a Noetherian ring , then every non-zero R-module M contains a non-zero irreducible submodule , [7] .

Recall that a proper submodule N of an R-module M is called a prime if $r m \in N$, for some $r \in R$ and $m \in M$, implies that either $m \in N$ or $r \in (N:M)$, [8] . And a proper submodule N of an R-module M is called a primary if $rm \in N$, for some $r \in R$ and $m \in M$, then either $m \in N$ or $r^n m \in N$ for some positive integer n. [8] , so every prime submodule of M is primary.

Recall that an R-module M is said to be satisfy the ascending chain condition (Acc) if each the ascending chain of submodules of M terminates , [7] .
The following results follows from [8] .

Proposition (1-17)
If M satisfies (Acc) then every irreducible submodule of M is primary .

Proposition (1-18)
If M satisfies (Acc) then every irreducible submodule of M can be written as an intersection of a finite number of irreducible submodules of M . Therefore if M satisfies (Acc) it contains any primary submodules .If we combine propositions (1-17) and (1-18) , we obtain the following result . Which appear in [8] .

Proposition (1-19)
f the R-module M satisfies (Acc) , then every submodule of M can be written as an intersection of finite number of primary submodules .
Recall that a proper submodule N of M is called semi-prime if for every $r \in R$, $x \in M$, $k \in Z^+$, such that $r^k x \in N$. Then $rx \in N$, [4] .
Recall that an R-module M is called a quasi-prime R-module if and only if $ann_R N$ is a prime ideal in a ring R for each non-zero submodule N of M , [14] .

Proposition (1-20)
Let N be an irreducible submodule of an R-module , then the following statements are equivalent :
1/ N is a prime submodule .
2/ N is a quasi-prime submodule .
3/ N is a semi-prime submodule . [14]

2- Irreducible submodules in multiplication R-module

An element $r \in R$ is called irreducible element if it is a non-zero and non-unite and whenever r = a b , where $a , b \in R$, then either a or b is a unite of R , [8] .

Definition (2-1)
An ideal I of a ring R is called irreducible if for each ideals I_1 and I_2 of R such that $I_1 \cap I_2 = I$, implies that either $I=I_1$ or $I=I_2$, [6] .

Proposition (2-2)
Let R be an integral domain . Then every prime element is an irreducible

Proof : Let $p \in R$ is a prime element , since $(\overline{P}) \neq 0$, and since $(\overline{P}) \neq R$.
We conclude that p is not a zero and not a unit .
Suppose p=ab where $a,b \in R$. Then either $a \in (\overline{P})$ or $b \in (\overline{P})$, let $a \in (\overline{P})$, we can write a=pd. Then p=ab =pdb , and since R is an integral domain , and $(\overline{P}) \neq 0$. Thus db =1; that b is a unit of R . Therefore p is an irreducible element.
Recall that an R-module M is called a multiplication if for each submodule N of M , there exists an ideal I of R , such that N = I M , [5] . One can easily show that , if M is a multiplication module , then N(N : M)M for every submodule N of M .

Recall that a multiplication of two elements $x , y \in M$, then the cyclic submodules Rx and Ry we have some ideals I and J of R , such that Rx =IM and Ry =JM, and so xy=(Rx) (Ry) = (IM) (JM) = (I.J)M , [2].
Recall that N and L are submodules of multiplication R-module M is called a multiplication of submodules in a multiplication R-module , if there exist ideals I and J of a ring R , such that N = IM and L=JM .Hence the multiplication of N and L is defined as , NL=(IM) (JM) =(IJ)M ,[12] .

Lemma (2-3)
A submodule N of an R-module M is a prime if and only if for some submodules L and K , such that $LK \subseteq N$, implies that either $L \subseteq N$ or $K \subseteq N$, [2] .

Proposition (2-4)
If M is a multiplication R-module , and R is a Notherian ring . Thus each submodule of M can be written as an intersection of a finite number of irreducible submodules , [13] .

Proposition (2-5)
Let N be a submodule of a multiplication R-module M . Then N is irreducible ideal of aring R , [13] .
It is clear that each prime ideal of aring R is irreducible ideal . Hence we get the following result .

Proposition (2-6)
Let M be a multiplication R-module, then each prime submodule of M is irreducible , [13] .
Recall that aring R is called a regular if and only if for each $x \in R$, there exists $y \in R$ such that x = xyx , [10].

Proposition (2-7)
Each irreducible ideal in a regular ring is a prime,[7] .
From the privuo results, we can get the following result .

Proposition (2-8)
Let R be a regular ring , and let M be a multiplication R-module . If N is submodule of M , then N is irreducible if and only if N is prime submodule, [13] .

Proposition (2-9)
Let M be a multiplication R-module , and let N be a submodule of M. Then N is an irreducible submodule if and only if (N:M) is an irreducible ideal .
Proof : Suppose N is an irreducible submodule , and suppose there exist ideals I_1 and I_2 of R , such that $I_1 \cap I_2 =(N:M)$. Its clear that ann $M \subseteq I_1 \cap I_2$, so $(N:M) \subseteq I_1$ and $(N:M) \subseteq I_2$. By [1] we get $(I_1 + annM) M \cap (I_2 + annM) M = I_1 M \cap I_2 M = (I_1 \cap I_2) M=(N:M)M= N$.
But N is an irreducible submodule , thus either $N=I_1M$ or $N=I_2M$. Its mean that either $I_1 \subseteq (N:M)$ or $I_2 \subseteq (N:M)$, so either $I_1=(N:M)$ or $I_2=(N:M)$. Therefore (N:M) is an irreducible ideal .
Conversely ; Suppose (N:M) is an irreducible ideal , and suppose there exist submodules N_1 and N_2 of M , such that $N = N_1 \cap N_2$. Thus $(N:M) = (N_1 \cap N_2 : M)= (N_1:M) \cap (N_2:M)$, since (N:M) is an irreducible ideal . thus either $(N_1:M)= (N :M)$ or $(N_2:M)= (N :M)$. And since M is a multiplication R-module , thus

either $N_1 = N$ or $N_2 = N$. Therefore N is an irreducible submodule .

3- Basic properties of strongly irreducible submodules

In this section we list some basic properties concerning strongly irreducible submodules .

Definition (3-1)

A submodule N of an R-module M is called strongly irreducible if for each submodules N_1 and N_2 of M , such that $N_1 \cap N_2 \subseteq N$, implies that either $N_1 \subseteq N$ or $N_2 \subseteq N$, [4] .

An ideal I of a ring R is called strongly irreducible if for each ideals I_1 and I_2 of R such that $I_1 \cap I_2 \subseteq I$, implies that either $I_1 \subseteq I$ or $I_2 \subseteq I$,[6]. An ideal of a ring R which is strongly irreducible submodules is called a strongly irreducible ideal , [4] .

A ring R is said to be arithmetical if for all ideals I , J and K of R , we have $(I + J) \cap K = (I \cap K) + (J \cap K)$. This property is equivalent to the condition that for all ideals I , J and K of arithmetical ring R, we have $(I \cap J) + K = (I + K) \cap (J + K)$.

If R is an arithmetical ring , then I is an irreducible ideal if and only if I is a strongly irreducible ideal , [6].

Theorem (3-2)

Let R be arithmetical, and let M be an R-module . N is a submodule of M. Then N is an irreducible submodule if and only if N is a strongly irreducible submodule.

Proof : Assume that N is strongly irreducible submodule, there exist submodules N_1 and N_2 of M , such that $N_1 \cap N_2 = N$. Then $N_1 \cap N_2 \subseteq N$ and $N \subseteq N_1 \cap N_2$. Since N is strongly irreducible , thus $N_1 \subseteq N$ or $N_2 \subseteq N$. And since $N \subseteq N_1 \cap N_2$, implies that $N \subseteq N_1$ and $N \subseteq N_2$, hence either $N = N_1$ or $N = N_2$. Therefore N is an irreducible submodule of M .

Conversely ; Assume that N is an irreducible submodule of M , there exist submodules N_1 and N_2 of M , such that $N_1 \cap N_2 \subseteq N$. Since N is an irreducible submodule of M , thus $N_1 \cap N_2 = N$, implies that either $N = N_1$ or $N = N_2$, So $N_1 \subseteq N$ or $N_2 \subseteq N$ .Therefore N is a strongly irreducible submodule of M .

Proposition (3-3)

If M is a Noetherian R-module , and N is a strongly irreducible submodule of M. Then N is a primary submodule , [4] .

Proposition (3-4)

If N is a strongly irreducible submodule of an R-module M , such that $Rm \cap Rn \subseteq N$, where Rn and Rm are cyclic submodule of M , then either $m \in N$ or $n \in N$.

Proof : Assume N is strongly irreducible submodule , with $Rn \cap Rm \subseteq N$, where Rn and Rm are cyclic submodules of M , and let $Rn \not\subset N$, its mean that $n \notin N$, since N is strongly irreducible submodule , and since $n \notin N$. Hence $Rm \subseteq N$, so $m \in N$.

Propositon (3-5)

Let N and L are submodules of an R-module M , such that L is contained in N . If N is a strongly irreducible submodule , then N/L is a strongly irreducible submodule of M/L , [4] .

Propositon (3-6)

Suppose N and L are submodule of an R-module M .Then N is a strongly irreducible submodule if and only if L + N is a strongly irreducible submodule .

Proof : Suppose L + N is a strongly irreducible submodule , there exist submodules N_1 and N_2 of M , such that $N_1 \cap N_2 \subseteq N$. Then we get $(L+N_1) \cap (L+N_2) = L+(N_1 \cap N_2) \subseteq L+N$, since L+N is a strongly irreducible submodule . Thus either $L+N_1 \subseteq L+N$ or $L+N_2 \subseteq L+N$, implies that either $N_1 \subseteq N$ or $N_2 \subseteq N$. Therefore N is a strongly irreducible submodule .

Conversely ; clear .

Corollary (3-7)

Let N and L are submodules of an R-module M .

(1) If N and L are strongly irreducible submodules then N+L is a strongly irreducible .

(2) If N is strongly irreducible , and N+L is strongly irreducible . Then L is strongly irreducible .

Proof : (1) Clear ,

(2) Suppose N+L is strongly irreducible submodule . Since N is a strongly irreducible submodule , hence there exists submodules N_1 and N_2 , such that $N_1 \cap N_2 \subseteq N$, implies that either $N_1 \subseteq N$ or $N_2 \subseteq N$, to prove L is also strongly irreducible submodule , suppose there exist submodules L_1 and L_2 , such that $L_1 \cap L_2 \subseteq L$.

$(N_1 \cap N_2)+(L_1 \cap L_2) = [N_1+(L_1 \cap L_2)] \cap [N_2+(L_1 \cap L_2)] = [(N_1+L_1) \cap (N_1+L_2)] \cap [(N_2+L_1) \cap (N_2+L_2)] \subseteq N+L$, since N+L is a strongly irreducible submodule , thus either $(N_1+L_1) \cap (N_1+L_2) \subseteq N+L$ or $(N_2+L_1) \cap (N_2+L_2) \subseteq N+L$. So either $(N_1+L_1) \subseteq N+L$ or $(N_1+L_2) \subseteq N+L$. If $N_1 \subseteq N$, implies that either $L_1 \subseteq L$ or $L_2 \subseteq L$. also either $N_2+L_1 \subseteq N+L$ or $(N_2+L_2) \subseteq N+L$. If $N_2 \subseteq N$ implies either $L_1 \subseteq L$ or $L_2 \subseteq L$. Therefore L is strongly irreducible submodule .

The following result follows direct from the prevue propositions. So the prove of them are similar to the prevue proposition , hence we omitted .

Corollary (3-8)

Let N and L are submodules of an R-module M . such that N is strongly irreducible submodule. Then L is strongly irreducible submodule if and only if N + L is strongly irreducible submodule .

Proposition (3-9)

Let N and L are submodule of an R-module M , if $N\cap L$ is a strongly irreducible submodule. Then N is a strongly irreducible submodule .

Proof : Suppose there exist submodules N_1 and N_2 of M , such that $N_1 \cap N_2 \subseteq N$. Since $(N_1 \cap L) \cap (N_2 \cap L) = (N_1 \cap N_2) \cap L \subseteq N \cap L$, and since $N\cap L$ is a strongly irreducible submodule. Thus either $N_1 \cap L \subseteq N\cap L$ or $N_2\cap L \subseteq N\cap L$, implies that either $N_1 \subseteq N$ or $N_2 \subseteq N$. Therefore N is strongly irreducible submodule .

The converse is not true in general, unless we put some condition as we see in the following proposition.

Proposition (3-10)

Let N and L are submodules of an R-module M , such that $N \subseteq L$. Then N is a strongly irreducible submodule if and only if $N \cap L$ is a strongly irreducible submodule .

Proof : Clear .

Corollary (3-11)

Let N and L are submodules of an R-module M , such that L is a strongly irreducible submodule . If $N\cap L$ is a strongly irreducible submodule , then N is a strongly irreducible submodule .

Corollary (3-12)

Let N and L are submodules of an R-module M , such that $N \subseteq L$ and L is a strongly irreducible submodule . Then N is a strongly irreducible submodule if and only if $N\cap L$ is a strongly irreducible submodule.

Proposition (3-13)

Let N and L are submodules of an R-module M , if $N \bigcup L$ is a strongly irreducible submodule. Then N is a strongly irreducible submodule .

Proof : Suppose there exists submodules N_1 and N_2 of M , such that $N_1 \cap N_2 \subseteq N$. Since $(N_1 \cup L) \cap (N_2 \cup L) = (N_1 \cap N_2) \cup L \subseteq N \cup L$, and since $N \cup L$ is a strongly irreducible submodule . Thus either $N_1 \cup L \subseteq N \cup L$ or $N_2 \cup L \subseteq N \cup L$, implies that either $N_1 \subseteq N$ or $N_2 \subseteq N$. Therefore N is strongly irreducible submodule .

The converse of the proposition (3-13) is not true in general , unless we put some condition as we see in the following proposition .

Proposition (3-14)

N and L are submodules of an R-module M , such that $L \subseteq N$. Let

Then N is a strongly irreducible submodule if and only if $N \bigcup L$ is a strongly irreducible submodule .

Proof : Clear .

Corollary (3-15)

Let N and L are submodules of an R-module M , such that L is a strongly irreducible submodule . If $N \bigcup L$ is a strongly irreducible submodule , then N is a strongly irreducible submodule .

Corollary (3-16)

Let N and L are submodules of an R-module M , such that $L \subseteq N$ and L is a strongly irreducible submodule . Then N is a strongly irreducible submodule if and only if $N \bigcup L$ is a strongly irreducible submodule .

The following corollaries follow directly from the previous proposition .

The proof of the following corollaries is similar to the last proposition , hence we omitted .

Corollary (3- 17)

Let N and L are submodules of an R-module M , If N is a strongly irreducible , and $N \cup L$ is irreducible submodule . Then L is strongly irreducible submodule .

the following result is given in [4] .

Proposition (3-18)

Each prime submodule of multiplication R-module is a strongly irreducible .

Proof : Suppose N is a prime submodules of M , and suppose N_1 and N_2 are submodules of M , such that $N_1 \cap N_2 \subseteq N$, but $N_1 \not\subset N$ and $N_2 \not\subset N$. Since M is a multiplication R-module , so we can write $N_1 = I_1 M$ and $N_2 = I_2 M$, for some ideals I_1 and I_2 of a ring R . So there exist $a_1 \in I_1$ and $a_2 \in I_2$, so m_1 , $m_2 \in M$, such that $a_1m_1 \in N_1$ and $a_2m_2 \in N_2$, with $a_1m_1 \notin N$ and $a_2m_2 \notin N$. Hence $a_1a_2m \in N_1 \cap N_2$, implies that $a_2M \subseteq N$, since N is a prime submodule of M . Thus $a_2m_2 \in N$, which is contradiction . Therefore $N_1 \subseteq N$ or $N_2 \subseteq N$, thus N is strongly irreducible submodule .

4- Pseudo irreducible submodules .

In this section , we shall study pseudo irreducible submodules in multiplication R-module .

Definition (4-1)

Let M be a multiplication R-module , a proper submodule N of M is called pseudo irreducible if N = LK , with L+K = M for any submodules L and K of M , then L = M or K = M , [11] .

And an ideal I of a ring R is called pseudo irreducible if one can not write I = JK with J+K = R where $I \neq R$, $J \neq R$ be ideals of R , [11] .

Our starting point is the following result is given in [11] .

Proposition (4-2)

Let M be a finitly generated faithful multiplication R-module. Then the submodule N of M is pseudo irreducible , if and only if (N:M) is pseudo irreducible ideal in aring R .

Proposition (4-3)

Let M be a finitely generated faithful multiplication R-module , and let N be a submodule of M. Then N is a pseudo irreducible if and only if I is a pseudo irreducible ideal in a ring R , where N = IM ,

Proof: suppose N is pseudo irreducible submodule of M and I = JK , where I and J are maximal ideals of R . Then N = IM = (JK)M = (JM) (KM) and we get M = RM = (J+K)M = JM + KM . Since N is pseudo irreducible we have JM=M or KM=M. And since M is faithful we get $J = R$ or $K = R$. Thus I is pseudo irreducible ideal .

Conversely ; suppose the ideal I is pseudo irreducible in R , and suppose there exist submodules L_1 and L_2 of M , such that $L_1 \cap L_2 = N$ and $L_1 + L_2 = M$. Since M is a multiplication R-module , there exist ideals I_1 and I_2 of R , such that $L_1 = I_1 M$ and $L_2 = I_2 M$. Then we get $N = L_1 L_2 = (I_1 M)(I_2 M) = (I_1 I_2) M$. Hence $M = L_1 + L_2 = I_1 M + I_2 M = (I_1 + I_2) M$. Since I is pseudo irreducible ideal , we obtain $I_1 = R$ or $I_2 = R$ that is $L_1 = I_1 M = R M = M$ or $L_2 = I_2 M = R M = M$. Therefore N is pseudo irreducible submodule of M .

Now to study the relation between pseudo irreducible submodules and pseudo prime submodules. First we must defined pseudo prime submodule .

Recall that a proper submodule N of the multiplication R-module M is called pseudo prime submodule if $N_1 + N_2 = M$ and $N_1 N_2 \subseteq N$ for any submodules N_1 and N_2 of M , then either $N_1 \subseteq N$ or $N_2 \subseteq N$, [11] .

Proposition (4-4)

Let N be a submodule of a multiplication R-module M . If N is pseudo irreducible submodule , then N is pseudo prime .

Proposition (4-5)

Let M be a cyclic R-module , and let N be a submodule of M , with the property if $x , y \in M$ with $x y \in N$, and $(x) + (y) = M$, then $x \in N$ or $y \in N$. Then N is pseudo irreducible submodule of M .

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المقاسات الجزئية الغير قابلة للتحليل والمقاسات الجزئية الغير قابلة للتحليل بقوة

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(**تاريخ الاستلام:** 20 / 9 / 2011 ---- **تاريخ القبول:** 13 / 3 / 2012)

الملخص

لتكن R حلقة ابدالية بمحايد ، و لتكن M مقاسا ايمن احادي معرف على الحلقة R . يتناول الباحث موضوع المقاسات الجزئية الغير قابلة للتحليل ، لما لها من دور هام في كثير من المقاسات الجزئية ، منها على سبيل المثال المقاسات الجزئية الاولية والابتدائية وغيرها بالاضافة الى ذلك علاقتها بالمثالي (N:M) في الحلقة R اذا كان N مقاس جزئي من المقاس M .