Spiral like and Uniformly Convexity Properties for Hypergeometric Functions

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Abstract

The object of this paper is to investigate some properties related to spiral likeness and uniformly convexity for certain hyper geometric functions. Further , the convolution properties and the operators related are also considered

Interdiction

Let T be the class of functions $f(z)$ defined as :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1)$$

That analytic and univalent in the unit disk :

$u = \{z : |z| < 1\}$ And let :

$$g(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq 0 \qquad (2)$$

Which are analytic and univalent in u

A function $f(z)$ belong to UCV(α) (uniformly convex class) if :

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \alpha \left|\frac{zf''(z)}{f'(z)}\right|, z \in u, 0 \leq \alpha < 1 \qquad (3)$$

. Where $f(z) \in T$

Furthermore a function f in T is said to be $\lambda-$ Spiral-like function $(SP(\lambda))$ if :

$$\text{Re}\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\} > 0, |\lambda| < \frac{\pi}{2}. \qquad (4)$$

Consider the hyper geometric function

$${}_5F_4 = F\begin{bmatrix} a,b,c,d,e; & \\ & z \\ f,g,h,k; & \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n(d)_n(c)_n}{(f)_n(g)_n(h)_n(k)_n(I)_n} z^n. \qquad (5)$$

Many researchers have studied the various choices of the parameters on hypergeometric functions like M. K. Aouf, , H. M. [1], S. Ponnusamy[4], Ghaedi and S. R H. M. Hossen and A Y [1] Srivastava and S. Owa. Lashin Kulkarni [2]

2- Orders of Spiral likeness and uniformly convexity:

Then: $b+c+d-1$, $a >$ 0 and $>$ a,b,c,d

Theorem 2.1 : If

$$z\,{}_5F_4\begin{bmatrix} a, 1+\frac{a}{2}, b, c, d; & \\ & z \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d; & \end{bmatrix}$$

If : $SP(\lambda)$ belongs in

$$\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)} \times$$

$$\left[1 + \frac{bcd(2+a)(a-b-c)(a-b-d)(a-c-d)\sec\lambda}{(1+a-b)(1+a-c)(1+a-d)(1+a)(a-b-c-d)(a-b-c-d-1)}\right] \leq 2.$$

(10)

Proof : Let

Now define : [7]

$$z\,{}_5F_4\begin{bmatrix} a, 1+\frac{a}{2}, b, c, d; & \\ & 1 \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d; & \end{bmatrix}$$
$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}$$

$\text{Re}(b+c+d-a-1)$

Also we define :

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)_{n-1} \qquad (7)$$

Is called

Gamma function . Γ Is called Pochhammer symbol and $(a)_n$ Where In order to make our result we need the following :a

If : $SP(\lambda)$ of the form (1) is in $f(z)$ A function

$$\sum_{n=2}^{\infty}(1+(n-1)\sec\lambda)|a_n| \leq 1, |\lambda| < \frac{\pi}{2}. \qquad (8)$$

Theorem 2 [3]

If and only if : $UCV(\alpha)$ of the form (2) is in $f(z)$ A function

$$\sum_{n=2}^{\infty} n(n(\alpha+1)-\alpha)a_n \leq 1. \qquad (9)$$

$$z\,{}_5F_4\begin{bmatrix} a, 1+\frac{a}{2}, b, c, d; & \\ & z \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d; & \end{bmatrix}$$

$$= z + \sum_{n=1}^{\infty} \frac{(a)_n(1+\frac{a}{2})_n(b)_n(c)_n(d)_n}{(\frac{a}{2})_n(1+a-b)_n(1+a-c)_n(1+a-d)_n(1)_n} z^{n+1}$$

$$= z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(1+\frac{a}{2})_{n-1}(b)_{n-1}(c)_{n-1}(d)_{n-1}}{(\frac{a}{2})_{n-1}(1+a-b)_{n-1}(1+a-c)_{n-1}(1+a-d)_{n-1}(1)_{n-1}} z^n$$

By Theorem 1 we want to show that :

$$\sum_{n=2}^{\infty}(1+(n-1)\sec\lambda)\frac{(a)_{n-1}(1+\frac{a}{2})_{n-1}(b)_{n-1}(c)_{n-1}(d)_{n-1}}{(\frac{a}{2})_{n-1}(1+a-b)_{n-1}(1+a-c)_{n-1}(1+a-d)_{n-1}(1)_{n-1}} \leq 1$$

$$, |\lambda| < \frac{\pi}{2}.$$

The left hand side of the last inequality is equal to :

$$\frac{abcd(1+\frac{a}{2})\sec\lambda}{\frac{a}{2}(1+a-b)(1+a-c)(1+a-d)}\times$$

$$\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(2+\frac{a}{2})_{n-1}(b+1)_{n-1}(c+1)_{n-1}(d+1)_{n-1}}{(\frac{a}{2}+1)_{n-1}(2+a-b)_{n-1}(2+a-c)_{n-1}(2+a-d)_{n-1}(1)_{n-1}}+$$

$$\sum_{n=1}^{\infty}\frac{(a)_n(1+\frac{a}{2})_n(b)_n(c)_n(d)_n}{(\frac{a}{2})_n(1+a-b)_n(1+a-c)_n(1+a-d)_n(1)_n}.$$

$$=\frac{abcd(1+\frac{a}{2})\sec\lambda}{\frac{a}{2}(1+a-b)(1+a-c)(1+a-d)}\times$$

$$\left[\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(a-b-c-d-1)}{\Gamma(2+a)\Gamma(a-b-c)\Gamma(a-b-d)\Gamma(a-c-d)}\right]+$$

$$\left[\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}-1\right]$$

$$=\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}\times$$

$$\left[\frac{bcd(2+a)(a-b-c)(a-b-d)(a-c-d)\sec\lambda}{(1+a-b)(1+a-c)(1+a-d)(1+a)(a-b-c-d)(a-b-c-d-1)}+1\right]-1.$$

Then by (10) we have the last expression bounded above by 1.

Theorem 2.2 : The function :

$\in UCV(\alpha)$

$b,c,d>-1, bcd<0, a>-2,$ If and only if

$$(\alpha+1)\frac{(a+1)(b+1)(c+1)(d+1)(\frac{a}{2}+)}{(1+\frac{a}{2})(2+a-b)(2+a-c)(2+a-d)}+$$

$$(\alpha+3)\frac{(a+2)(a-b-c-d-2)(a-b-c-d-3)}{(a-b-c-1)(a-b-d-1)(a-c-d-1)}+$$

$$[(a-b-c-d-1)(a+1)(1+a-b)(1+a-d)(a-b-c-d-3)\times$$

$$(a-b-c-d)(a-b-c-d-2)]/[bca((a-b-c)(a-b-c-d-1)\times$$

$$(a-b-d)(a-b-d-1)(a-c-d)(a-c-d-1)]\leq 0. \quad (11)$$

Proof :

$$z_5F_4\left[\begin{matrix} a,1+\frac{a}{2},b,c,d; \\ \frac{a}{2},1+a-b,1+a-c,1+a-d; \end{matrix}\; z\right]$$

$$=z-\left|\frac{abcd(1+\frac{a}{2})}{(\frac{a}{2})(1+a-b)(1+a-c)(1+a-d)}\right|\times$$

$$\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}(c+1)_{n-2}(d+1)_{n-2}(2+\frac{a}{2})_{n-2}}{(1+\frac{a}{2})_{n-2}(2+a-b)_{n-2}(2+a-c)_{n-2}(2+a-d)_{n-2}(1)_{n-1}}z^n$$

Our claim by Theorem 2,

$$\sum_{n=2}^{\infty}n[n(\alpha+1)-\alpha]\frac{(a+1)_{n-2}(b+1)_{n-2}(c+1)_{n-2}(d+1)_{n-2}}{(1+\frac{a}{2})_{n-2}(2+a-b)_{n-2}(2+a-c)_{n-2}}\times$$

$$\frac{(2+\frac{a}{2})_{n-2}}{(2+a-d)_{n-2}(1)_{n-1}}\leq\left|\frac{(1+a-b)(1+a-c)(1+a-d)}{(2+a)bcd}\right|.$$

Let :

$(n+2)^2(\alpha+1)-(n+2)\alpha=(n+1)^2(\alpha+1)+(n+1)(\alpha+2)+1.$ Now let :

$$M(m,n)=\frac{(a)_n(b)_n(c)_n(d)_n(1+\frac{a}{2})_n}{(\frac{a}{2})_n(1+a-b)_n(1+a-c)_n(1+a-d)_n}.$$

$$M(m+1,n)=\frac{(a+1)_n(b+1)_n(c+1)_n(d+1)_n(2+\frac{a}{2})_n}{(1+\frac{a}{2})_n(2+a-b)_n(2+a-c)_n(2+a-d)_n}.$$

Then:

$$\sum_{n=0}^{\infty}[(n+2)^2(\alpha+1)-(n+2)\alpha]\frac{M(m+1,n)}{(1)_{n+1}}$$

$$=\sum_{n=0}^{\infty}(n+1)\frac{M(m+1,n)}{(1)_n}(\alpha+1)+(\alpha+2)\times$$

$$\sum_{n=0}^{\infty}\frac{M(m+1,n)}{(1)n}+\sum_{n=0}^{\infty}\frac{M(m+1,n)}{(1)n+1}$$

$$=(\alpha+1)\frac{(a+1)(b+1)(c+1)(d+1)(2+\frac{a}{2})}{(1+\frac{a}{2})(2+a-b)(2+a-c)(2+a-d)}\times$$

$$\sum_{n=0}^{\infty}\frac{M(m+2,n)}{(1)n}+(\alpha+3)\times\sum_{n=0}^{\infty}\frac{M(m+1,n)}{(1)n+1}+$$

$$\cdot\frac{(1+a-b)(1+a-c)(1+a-d)}{(2+a)bcd}\sum_{n=1}^{\infty}\frac{M(m,n)}{(1)_n}$$

$$=\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(a-b-c-d-3)}{\Gamma(3+a)\Gamma(a-b-c-1)\Gamma(a-b-d-1)\Gamma(a-c-d-1)}\times$$

$$\left[\frac{(\alpha+1)(a+1)(b+1)(c+1)(d+1)(2+\frac{a}{2})}{(1+\frac{a}{2})(2+a-b)(2+a-c)(2+a-d)}\right]+$$

$$\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(a-b-c-d-1)}{\Gamma(2+a)\Gamma(a-b-c)\Gamma(a-b-d)\Gamma(a-c-d)}(\alpha+3)+$$

$$\frac{(1+a-b)(1+a-c)(1+a-d)}{(2+a)bcd}\times$$

$$\left[\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(a-b-c-d+1)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}-1\right].$$

$$=\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(a-b-c-d-3)}{\Gamma(3+a)\Gamma(a-b-c-1)\Gamma(a-b-d-1)\Gamma(a-c-d-1)}\times$$

$$[\frac{(\alpha+1)(a+1)(b+1)(c+1)(d+1)(2+\frac{a}{2})}{(1+\frac{a}{2})(2+a-b)(2+a-c)(2+a-d)}+$$

$$\frac{(a-b-c-d-2)(a-b-c-d-3)(a+2)}{(a-b-c-1)(a-b-d-1)(a-c-d-1)}(\alpha+3)+$$

$$\frac{(a-b-c-d-3)(a-b-c-d)(a-b-c-d-2)}{bcd}\times$$

$$\frac{(a-b-c-d-1)(a+1)(1+a-b(1+a-c)}{(a-b-c)(a-b-c-1)(a-b-d)(a-b-d-1)}\times$$

$$\frac{(1+a-d)}{(a-c-d)(a-c-d-1)}]-\frac{(1+a-b)(1+a-c)(1+a-d)}{(2+a)bcd}.$$

Then the above expression is bounded above by :

$\left|\frac{(1+a-b)(1+a-c)(1+a-d)}{(2+a)bcd}\right|$ If and only if (11) holds .

3- Convolution Properties and Integral Operator I :

Let $h(z)=1-\sum_{n=1}^{\infty}a_n z_n$ analytic and univalent in U, and let

$$H[X]=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n(d)_n(1+\frac{a}{2})_n}{(\frac{a}{2})_n(1+a-b)_n(1+a-c)_n(1+a-d)_n}z^n, \quad (12)$$

where :

$$[X]=\begin{bmatrix} a,1+\frac{a}{2},b,c,d; & \\ & z \\ \frac{a}{2},1+a-b,1+a-c,1+a-d; & \end{bmatrix}, z\in u$$

defined as $H[X]*h(z)$ The Hadamard product

$$H[X]*h(Z)=1-\sum_{N=1}^{\infty}\frac{(a)_n(b)_n(c)_n(d)_n(1+\frac{a}{2})_n}{(\frac{a}{2})_n(1+a-b)_n(1+a-c)_n(1+a-d)_n(1)_n}a_n z^n. \quad (13)$$

Theorem 3.1 : The Hadamard product $H[X]*h(z)$ belongs to $UCV(\alpha)$ If and only if :

$$\max\{a_i\}i\in N\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}\times$$
$$\{(\alpha+1)[bcd(b+1)(c+1)(d+1)(\frac{a}{2}+2)(a-b-c)(a-b-c-1)$$
$$(a-b-d)(a-b-d-1)(a-c-d)(a-c-d-1)]/[(1+a-c)$$
$$(1+a-b)(1+a-d)(1+\frac{a}{2})(2+a-b)(2+a-c)(2+a-d)$$
$$(a-b-c-d)(a-b-c-d-1)(a-b-c-d-2)]+1+[(2+\alpha)(2+a)$$
$$bcd(a-b-c)(a-b-d)(a-c-d)]/[(1+a-c)(1+a-d)$$
$$(1+a-b)(1+a)(a-b-c-d)(a-b-c-d-1)]\}\leq 2. \quad (14)$$

Proof : By definition of Hadamard product, we have**:**

$$z(H[X]*h(Z))=z-\sum_{N=1}^{\infty}\frac{(a)_n(b)_n(c)_n(d)_n(1+\frac{a}{2})_n}{(\frac{a}{2})_n(1+a-b)_n(1+a-c)_n(1+a-d)_n(1)_n}a_n z^{n+1}.$$

$$=z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}(c)_{n-1}(d)_{n-1}(1+\frac{a}{2})_{n-1}}{(\frac{a}{2})_{n-1}(1+a-b)_{n-1}(1+a-c)_{n-1}(1+a-d)_{n-1}(1)_{n-1}}a_n z^n.$$

Thus by making use of Theorem 2, it is enough to show that :

$$\sum_{n=2}^{\infty}n(n(\alpha+1)-\alpha)\frac{(a)_{n-1}(b)_{n-1}(c)_{n-1}(d)_{n-1}(1+\frac{a}{2})_{n-1}}{(\frac{a}{2})_{n-1}(1+a-b)_{n-1}(1+a-c)_{n-1}(1+a-d)_{n-1}(1)_{n-1}}a_{n-1}\leq 1$$

Thus ,the left hand side of the last expression is equal to :

$$\sum_{n=2}^{\infty}(n+1)((n+1(\alpha+1)-\alpha)\frac{(a)_{n1}(b)_n(c)_n(d)_n(1+\frac{a}{2})_n}{(\frac{a}{2})_n(1+a-b)_n(1+a-c)_n(1+a-d)_n(1)_n}a_n$$

$$=(\alpha+1)\sum_{n=2}^{\infty}n\frac{(a)_{n1}(b)_n(c)_n(d)_n(1+\frac{a}{2})_n}{(\frac{a}{2})_n(1+a-b)_n(1+a-c)_n(1+a-d)_n(1)_{n-1}}a_n+$$

$$(\alpha+2)\sum_{n=2}^{\infty}\frac{(a)_{n1}(b)_n(c)_n(d)_n(1+\frac{a}{2})_n}{(\frac{a}{2})_n(1+a-b)_n(1+a-c)_n(1+a-d)_n(1)_{n-1}}a_n+$$

$$\sum_{n=2}^{\infty}\frac{(a)_{n1}(b)_n(c)_n(d)_n(\frac{a}{2}+1)_n}{(\frac{a}{2})_n(1+a-b)_n(1+a-c)_n(1+a-d)_n(1)_n}a_n$$

$$=\frac{(\alpha+1)bcd(2+a)(a+1)(b+1)(c+1)(d+1)(\frac{a}{2}+2)}{(1+a-b)(1+a-c)(1+a-d)(1+\frac{a}{2})(2+a-b)(2+a-c)(2+a-d)}\times$$

$$\sum_{n=1}^{\infty}\frac{(a+2)_{n-1}(b+2)_{n-1}(c+2)_{n-1}(d+2)_{n-1}(3+\frac{a}{2})_{n-1}}{(\frac{a}{2}+2)_{n-1}(3+a-b)_{n-1}(3+a-c)_{n-1}(3+a-d)_{n-1}(1)_{n-1}}a_n+$$

$$\frac{(\alpha+2)(a+2)bcd}{(1+a-b)(1+a-c)(1+a-d)}\times$$

$$\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}(c+1)_{n-1}(d+1)_{n-1}(2+\frac{a}{2})_{n-1}}{(\frac{a}{2}+1)_{n-1}(2+a-b)_{n-1}(2+a-c)_{n-1}(2+a-d)_{n-1}(1)_{n-1}}a_n+$$

$$\sum_{n=2}^{\infty}\frac{(a)_{n1}(b)_n(c)_n(d)_n(1+\frac{a}{2})_n}{(\frac{a}{2})_n(1+a-b)_n(1+a-c)_n(1+a-d)_n(1)_n}a_n$$

$$=\frac{(\alpha+1)bcd(2+a)(a+1)(b+1)(c+1)(d+1)(\frac{a}{2}+2)}{(1+a-b)(1+a-c)(1+a-d)(1+\frac{a}{2})(2+a-b)(2+a-c)(2+a-d)}\times$$

$$\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n(c+2)_n(d+2)_n(3+\frac{a}{2})_n}{(\frac{a}{2}+2)_n(3+a-b)_n(3+a-c)_n(3+a-d)_n(1)_n}a_{n+1}+$$

$$\frac{(\alpha+2)(a+2)bcd}{(1+a-b)(1+a-c)(1+a-d)}\times$$

$$\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n(c+1)_n(d+1)_n(2+\frac{a}{2})_n}{(\frac{a}{2}+1)_n 2+a-b)_n(2+a-c)_n(2+a-d)_n(1)_n}a_{n+1}+$$

$$\left[\sum_{n=1}^{\infty}\frac{(a)_n(b)_n(c)_n(d)_n(1+\frac{a}{2})_n a_n}{(\frac{a}{2})_n(1+a-b)_n(1+a-c)_n(1+a-d)_n(1)_n}-a_o\right]$$

$$\leq\frac{(\alpha+1)bcd(2+a)(a+1)(b+1)(c+1)(d+1)(\frac{a}{2}+2)M}{(1+a-b)(1+a-c)(1+a-d)(1+\frac{a}{2})(2+a-b)(2+a-c)(2+a-d)}\times$$

$$\left[\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(a-b-c-d-3)}{\Gamma(3+a)\Gamma(a-b-c-1)\Gamma(a-b-d-1)\Gamma(a-c-d-1)}\right]+$$

$$\frac{(2+\alpha)(2+a)bcdM}{(1+a-b)(1+a-c)(1+a-d)}\times$$

$$\left[\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(a-b-c-d-1)}{\Gamma(2+a)\Gamma(a-b-c)\Gamma(a-b-d)\Gamma(a-c-d)}\right]+$$

$$\frac{M\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d))}-1$$

Where $M=\max\{a_n\}_{n\in N}$ and since $a_0=1$

Then the above inequality is less than or equal to 1 if and only if the condition (14) holds.

Theorem 3.2 :

Let $a,b,c,d>0$ and $a>b+c+d-2$.

Then $\int_0^z z_5F_4(t)dt$: Belongs to $SP(\lambda)$ If

$$\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d))}\times$$

$$[\sec\lambda-\frac{2(\sec\lambda-1)(\frac{a}{2}-1)(a-b)(a-c)(a-d)(2+a-b-c-d)}{(a-1)(b-1)(c-1)(d-1)(1+a-b-c)(1+a-c-d)}\times$$

$$\frac{(1+a-b-c-d)}{(1+a-b-d)}+(\sec\lambda-1)\frac{(\frac{a}{2}-1)(a-b)(a-c)(a-d)}{(a-1(b-1)(c-1)(d-1)(\frac{a}{2})}$$

$$\leq 2,|\lambda|<\frac{\pi}{2}. \qquad (15)$$

Proof : We can write ,

$$\int_0^z z_5F_4dt=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}(c)_{n-1}(d)_{n-1}(1+\frac{a}{2})_{n-1}}{(\frac{a}{2})_{n-1}(1+a-b)_{n-1}(1+a-c)_{n-1}(1+a-d)_{n-1}(1)_n}z^n. \qquad (16)$$

By using Theorem 1, we must show that :

$$\sum_{n=2}^{\infty}(1+(n-1)\sec\lambda)\frac{(a)_{n-1}(b)_{n-1}(c)_{n-1}(d)_{n-1}(1+\frac{a}{2})_{n-1}}{(\frac{a}{2})_{n-1}(1+a-b)_{n-1}(1+a-c)_{n-1}(1+a-d)_{n-1}(1)_n}a_{n-1}\leq 1 \qquad (17).$$

The left hand side of (17) is equal to :

$$\sum_{n=2}^{\infty}n\frac{(a)_{n+1}(b)_{n-1}(c)_{n-1}(d)_{n-1}(1+\frac{a}{2})_{n-1}}{(\frac{a}{2})_n(1+a-b)_{n-1}(1+a-c)_{n-1}(1+a-d)_{n-1}(1)_n}\sec\lambda-$$

$$\sum_{n=2}^{\infty}\frac{(a)_{n+1}(b)_{n-1}(c)_{n-1}(d)_{n-1}(1+\frac{a}{2})_{n-1}}{(\frac{a}{2})_n(1+a-b)_{n-1}(1+a-c)_{n-1}(1+a-d)_{n-1}(1)_n}\sec\lambda+$$

$$\sum_{n=2}^{\infty}\frac{(a)_{n+1}(b)_{n-1}(c)_{n-1}(d)_{n-1}(1+\frac{a}{2})_{n-1}}{(\frac{a}{2})_n(1+a-b)_{n-1}(1+a-c)_{n-1}(1+a-d)_{n-1}(1)_n}$$

$$=\sec\lambda\sum_{n=2}^{\infty}\frac{(a)_{n+1}(b)_{n-1}(c)_{n-1}(d)_{n-1}(1+\frac{a}{2})_{n-1}}{(\frac{a}{2})_n(1+a-b)_{n-1}(1+a-c)_{n-1}(1+a-d)_{n-1}(1)_n}-$$

$$(\sec\lambda-1)\frac{(\frac{a}{2}-1)(a-b)(a-c)(a-d)}{(a-1)(b-1)(c-1)(d-1)(\frac{a}{2})}\times$$

$$\sum_{n=2}^{\infty}\frac{(a-1)_n(b-1)_n(c-1)_n(d-1)_n(\frac{a}{2})_n}{(\frac{a}{2}-1)_n(a-b)_n(a-c)_n(a-d)_n(1)_n}.$$

$$(as(\lambda)_{n-1}=\frac{1}{(\lambda-1)}(\lambda-1)_n)$$

$$=\sec\lambda\sum_{n=1}^{\infty}\frac{(a)_n(b)_n(c)_n(d)_n(1+\frac{a}{2})_n}{(\frac{a}{2})_n(1+a-b)_n(1+a-c)_n(1+a-d)_n(1)_n}-$$

$$(\sec\lambda-1)\frac{(\frac{a}{2}-1)(a-b)(a-c)(a-d)}{(a-1)(b-1)(c-1)(d-1)(\frac{a}{2})}\times$$

$$(\sec\lambda-1)\frac{(\frac{a}{2}-1)(a-b)(a-c)(a-d)}{(a-1)(b-1)(c-1)(d-1)(\frac{a}{2})}\times$$

$$[\sum_{n=0}^{\infty}\frac{(a-1)_n(b-1)_n(c-1)_n(d-1)_n(\frac{a}{2})_n}{(\frac{a}{2}-1)_n(a-b)_n(a-c)_n(a-d)_n(1)_n}-1-\frac{(a-1)(b-1)(c-1)(d-1)(\frac{a}{2})}{(\frac{a}{2}-1)(a-b)(a-c)(a-d)}]$$

$$=\sec\lambda[\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d))}-1]-$$

$$(\sec\lambda-1)[\frac{(\frac{a}{2}-1)(a-b)(a-c)(a-d)}{(a-1)(b-1)(c-1)(d-1)(\frac{a}{2})}]\times$$

$$[\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(3+a-b-c-d)}{\Gamma(a)\Gamma(2+a-b-c)\Gamma(2+a-b-d)\Gamma(2+a-c-d))}-1-$$

$$\frac{(a-1)(b-1)(c-1)(d-1)(\frac{a}{2})}{(\frac{a}{2}-1)(a-b)(a-c)(a-d)}]$$

$$=\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d))}\times$$

$$[\sec\lambda-\frac{2(\sec\lambda-1)(\frac{a}{2}-1)(a-b)(a-c)(a-d)(2+a-b-c-d)}{(a-1)(b-1)(c-1)(d-1)(1+a-b-c)(1+a-c-d)}\times$$

$$\frac{(1+a-b-c-d)}{(1+a-b-d)}]+[(\sec\lambda-1)\frac{(\frac{a}{2}-1)(a-b)(a-c)(a-d)}{(a-1)(b-1)(c-1)(d-1)(\frac{a}{2})}-1].$$

The last expression is bounded by 1 if (15) holds true.

4- Some Properties for Generalized Hyper Geometric Functions :

Consider the Generalized Hyper Geometric Function of the form :

$$F(a_1,a_2,...,a_k,b;z)=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n...(a_k)_n}{(b)_n(1)_n}z^n \qquad (18)$$

And $(a)_n$ The Pochhammer symbol such that :

$b\neq 0,-1,-2,...$ Where

$$(a)_{-n}=\frac{(-1)^n}{(1-a)_n},\frac{(a)_n}{(a)_{n-1}}=a+n-1. \qquad (19)$$

And

$$(a)_{n+m}=(a)_m(a+m)_n, \qquad (20)$$

is an integer , positive, negative or zero, n

(18) can written as : $\Re$ N

$$\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n...(a_k)_n}{(b)_n(1)_n}z^n,k\in N,k\geq 2.$$

we consider the above series by the symbol :

which is converges for all ,

$${}_kF_1(a_1,a_2,...,a_k;b;z)$$

In case z = 1, it is converges $|z|>1,$ and diverges in

And diverges if

$$F(a_1,a_2,...,a_k,b;z)=\frac{\Gamma(b)\Gamma(b-a_k-a_{k-1}-...-a_1)}{\Gamma(b-a_1)\Gamma(b-a_2)...\Gamma(b-a_k)}$$

Absolutely if

$\mathrm{Re}(b-a_k-a_{k-1}-...-a_1)\le 0.$

Also we define :

Theorem 4.1 : Let

$b>a_1+a_2+...+a_k+(k-1), k\ge 2, |\lambda|<\frac{\pi}{2}.$ and

If : $SP(\lambda)$ Belongs to $z_k F_1(a_1,a_2,...,a_k;b;z)$

Then

$$\frac{\Gamma(b)\Gamma(b-a_k-...-a_1)}{\Gamma((b-a_1)\Gamma((b-a_2)...\Gamma((b-a_k)}$$

$$[1+[a_1a_2a_3...a_k\sec\lambda]/[(b-a_k-...-a_1-(k-1))$$

$$(b-a_k-...-a_1-(k-2))...(b-a_k-...-a_1-k+(k-1))]]\le 2$$

(21).

Proof : We can write

$$zF(a_1,a_2,...,a_k,b;z)=z+\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(a_2)_{n-1}...(a_k)_{n-1}}{(b)_{n-1}(1)_{n-1}}z^n, k\ge 2.$$

Now by making use of Theorem 1,we have :

$$\sum_{n=2}^{\infty}(1+(n-1)\sec\lambda)\left(\frac{(a_1)_{n-1}(a_2)_{n-1}...(a_k)_{n-1}}{(b)_{n-1}(1)_{n-1}}\right) \quad (22)$$

$$=\sum_{n=1}^{\infty}(1+n\sec\lambda)\left(\frac{(a_1)_n(a_2)_n...(a_k)_n}{(b)_n(1)_n}\right)$$

$$=\sum_{n=1}^{\infty}\frac{(a_1)_n(a_2)_n...(a_k)_n}{(b)_n(1)_{n-1}}\sec\lambda+\sum_{n=1}^{\infty}\frac{(a_1)_n(a_2)_n...(a_k)_n}{(b)_n(1)_{n-1}}=T.$$

Therefore by using (19),(20) we get :

$$T=\frac{a_1a_2...a_k}{b}\sum_{n=0}^{\infty}\frac{(a_1+1)_n(a_2+1)_n...(a_k+1)_n}{(b+1)_n(1)_n}\times$$

$$\sec\lambda+\left[\frac{\Gamma(b)\Gamma(b-a_k-...-a_1)}{\Gamma(b-a_k)\Gamma(b-a_1)}-1\right]$$

$$=\frac{a_1a_2...a_k}{b}\left[\frac{\Gamma(b+1)\Gamma(b-a_k-...-a_1-(k-1))}{\Gamma(b-a_1)\Gamma(b-a_2)...,\Gamma(b-a_k)}\right]\sec\lambda+$$

$$\left[\frac{\Gamma(b)\Gamma(b-a_k-...-a_1)}{\Gamma(b-a_1)...\Gamma(b-a_k)}-1\right]$$

$$=\frac{\Gamma(b)\Gamma(b-a_k-...-a_1)}{\Gamma(b-a_1)\Gamma(b-a_2)...\Gamma(b-a_k)}\times$$

$$\left[\frac{a_1a_2...a_k\sec\lambda}{(b-a_k-...-a_1-1)(b-a_k-...-a_1-2)...(b-a_k-...-a_1-(k-1))}+1\right]-1$$

Then T is bounded by 1 if and only if (21) holds

Theorem 4.2 : Let $a_1,a_2,...,a_k>-1, b>0$ and

Then

$a_1,a_2,...,a_k<0.$

$zF(a_1,...,a_k;b;z)\in UCV(\alpha)$

If and only if :

$$\left(\frac{b(\alpha+1)(a_1+1)...(a_k+1)}{(b-a_k-...-a_1-1)(b-a_k-...-a_1-2)(b-a_k-...-a_1-2(k-1))}+\right.$$

$$\left.\frac{(3+2\alpha)b}{(b-a_k-...-a_1-1)(b-a_k-...-a_1-2)(b-a_k-...-a_1-(k-1)}+b\right)\ge o$$

where $b>a_1+...+a_k+2(k-1).$

Proof : Consider the relation (22), then we have :

$$zF(a_1,...,a_k;b;z)=z-\left|\frac{a_1...a_k}{b}\right|\times$$

$$\sum_{n=2}^{\infty}\frac{(a_1+1)_{n-2}(a_2+1)_{n-2}...(a_k+1)_{n-2}}{(b+1)_{n-2}(1)_{n-1}}z^n. \quad (23)$$

By Theorem 2, it is enough to show that :

$$\sum_{n=2}^{\infty}n(n(\alpha+1)-\alpha)\frac{(a_1+1)_{n-2}...(a_k+1)_{n-2}}{(b+1)_{n-2}(1)_{n-1}}\le$$

$$\frac{b}{a_1a_2...a_k}\le\left|\frac{b}{a_1a_2...a_k}\right|$$

The left side of the last expression is equal to :

$$\sum_{n=2}^{\infty}(n+2)((n+2)(\alpha+1)-\alpha)\frac{(a_1+1)_n...(a_k+1)_n}{(b+1)_n(1)_{n+1}}=T.$$

Then by making use the following :

$(n+2)^2(\alpha+)-(n+2)\alpha=(\alpha+1)(n+1)^2+(\alpha+2)(n+1)+1,$ We have :

$$T=\sum_{n=0}^{\infty}(n+1)\frac{(a_1+1)_n...(a_k+1)_n}{(b+1)_n(1)_n}(\alpha+1)+$$

$$\sum_{n=0}^{\infty}\frac{(a_1+1)_n...(a_k+1)_n}{(b+1)_n(1)_n}(\alpha+2)+\sum_{n=0}^{\infty}\frac{(a_1+1)_n...(a_k+1)_n}{(b+1)_n(1)_n}$$

$$=\frac{(\alpha+1)(a_1+1)...(a_k+1)}{(b+1)}\sum_{n=0}^{\infty}\frac{(a_1+2)_n...(a_k+2)_n}{(b+2)_n(1)_n}+$$

$$(3+2\alpha)\sum_{n=0}^{\infty}\frac{(a_1+1)_n...(a_k+1)_n}{(b+1)_n(1)_n}+\frac{b}{a_1a_2...a_k}\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_k)_n}{(b)_n(1)_n}.$$

$$=\frac{\Gamma(b+1)\Gamma(b-a_k-a_{k-1}-...-a_1-2(k-1))}{\Gamma(b-a_k)\Gamma(b-a_{k-1})...\Gamma(b-a_1)}\times$$

$$[(\alpha+1)(a_1+1)...(a_k+1)]+(3+2\alpha)\times$$

$$\frac{\Gamma(b+1)\Gamma(b-a_k-a_{k-1}-...-a_1-(k-1))}{\Gamma(b-a_k)....\Gamma(b-a_1)}+$$

$$\frac{b}{a_1a_2...a_k}\left[\frac{\Gamma(b)\Gamma(b-a_k-...-a_1)}{\Gamma(b-a_k)...\Gamma(b-a_1)}-1\right].$$

$$=\frac{\Gamma(b)\Gamma(b-a_k-...-a_1)}{\Gamma(b-a_k)...\Gamma(b-a_1)}\times$$

$$[\frac{b(\alpha+1)(a_1+1)...(a_k+1)}{(b-a_k-...-a_1-1)(b-a_k-...-a_1-2)...(b-a_k-...-a_1-2(k-1))}+$$

$$(3+2\alpha)\frac{b}{(b-a_k-...-a_1-1)...(b-a_k-...-a_1-(k-1))}+$$

$$\frac{b}{a_1a_2...a_k}]-\frac{b}{a_1a_2...a_k}.$$

Hence T is bounded above by :

If and only if (23) holds $\left|\frac{b}{a_1a_2...a_k}\right|$

5 – Convolution Properties and Integral Operator II :

Consider the function $h(z)$ which is analytic and u univalent in the defined by:

$$h(z)=z-\sum_{n=2}^{\infty}a_nz^n, \text{ and let}$$

$$H(h)(z)=zF(a_1,a_2,...,a_k;b;)*h(z)=z-$$

$$\sum_{n=1}^{\infty}\frac{(a_1)_n(a_2)_n...(a_k)_n}{(b)_n(1)_n}a_{n+1}z^{n+1}. \ (24)$$

Theorem 5.1 : The function (24) belongs to $UCV(\alpha)$ if and only if :

$$\frac{\Gamma(b)\Gamma(b-a_k-...-a_1)\max\{a_n\}_{n\in N}}{\Gamma(b-a_1)...\Gamma(b-a_k)}\times$$
$$[\frac{(\alpha+1)a_1a_2...a_k(a_1+1)...(a_k+1)}{(b-a_k-...-a_1-1)(b-a_k-...a_1-2)...(b-a_k-...-a_1-2(k-1))}$$
$$+\frac{(\alpha+1)a_1...a_k}{(b-a_k-...-a_1-1)...(b-a_k-...-a_1-(k-1))}+1]\le 2,$$

and : $a_1,a_2,...,a_k>0$ where

$$b>a_1+a_2+...+a_k+2(k-1), k\ge 2. \ (25)$$

proof :

$$H(h)(z)=z-\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(a_2)_{n-1}...(a_k)_{n-1}}{(b)_{n-1}(1)_{n-1}}a_nz^n. \ (26)$$

By Theorem 2, its sufficient to show that :

$$S=\sum_{n=2}^{\infty}n(n(\alpha+1)-\alpha)\frac{(a_1)_{n-1}(a_2)_{n-1}...(a_k)_{n-1}}{(b)_{n-1}(1)_{n-1}}a_n\le 1.$$
$$S=\sum_{n=1}^{\infty}((n+1)^2(\alpha+1)-\alpha(n+1))\left(\frac{(a_1)_n(a_2)_n...(a_k)_n}{(b)_n(1)_n}\right)a_{n+1}.$$

Therefore

$$=(\alpha+1)\sum_{n=1}^{\infty}n\frac{(a_1)_n(a_2)_n...(a_k)_n}{(b)_n(1)_{n-1}}a_{n+1}+(\alpha+2)\times$$
$$\sum_{n=1}^{\infty}n\frac{(a_1)_n(a_2)_n...(a_k)_n}{(b)_n(1)_{n-1}}a_{n+1}+\sum_{n=1}^{\infty}n\frac{(a_1)_n(a_2)_n...(a_k)_n}{(b)_n(1)_n}a_{n+1}.$$
$$(n+1)^2(\alpha+1)-\alpha(n+1)=(\alpha+1)n^2+(\alpha+2)n+1.$$
$$S=\frac{(\alpha+1)a_1...a_k(a_1+1)...(a_k+1)}{b(b+1)}\sum_{n=0}^{\infty}\frac{(a_1+2)_n...(a_k+2)_n}{(b+2)_n(1)_n}a_{n+2}+$$
$$\frac{(\alpha+2)a_1a_2...a_k}{b}\sum_{n=0}^{\infty}\frac{(a_1+1)_n...(a_k+1)_n}{(b+1)_n(1)_n}a_{n+2}+$$
$$\left(\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_k)_n}{(b)_n(1)_n}a_{n+1}\right)-a_1.$$

Take $M=\max\{a_n\}_{n\in N}$

And $:a_1=1$, We get

$$S\le\frac{(\alpha+1)a_1a_2...a_k(a_1+1)...(a_k+1)M}{b(b+1)}\times$$
$$\sum_{n=0}^{\infty}\frac{(a_1+2)_n...(a_k+2)_n}{(b+2)_n(1)_n}+\frac{(\alpha+2)a_n...a_kM}{b}\times$$
$$\sum_{n=0}^{\infty}\frac{(a_1+1)_n...(a_k+1)_n}{(b+1)_n(1)_n}+M\sum_{N=0}^{\infty}\frac{(a_1)_n...(a_k)_n}{(b)_n(1)_n}-1$$
$$=\frac{\Gamma(b)\Gamma(b-a_k-...-a_1)M}{\Gamma(b-a_1)...\Gamma(b-a_k)}\times$$
$$[\frac{(\alpha+1)a_1a_2...a_k(a_1+1)...(a_k+1)}{(b-a_k-...-a_1-1)(b-a_k-...a_1-2)...(b-a_k-...-a_1-2(k-1))}$$
$$+\frac{(\alpha+2)a_1...a_k}{(b-a_k-...-a_1-1)...(b-a_k-...-a_1-(k-1))}+1]-1.$$

Thus, S is bounded by 1 if and only if (25) holds .

Theorem 5.2 :

Let $a_1,...,a_k>0,b,a_1,...,a_k\neq 0$,:

And $b>a_1+...+a_k$. Then

$$\int_0^2F(a_1,...,a_k;b;t)dt=z+\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}...(a_k)_{n-1}}{(b)_{n-1}(1)_n}z^n.$$

Belongs to $SP(\lambda)$ If :

$$\frac{\Gamma(b)\Gamma(b-a_1...-a_k)}{\Gamma(b-a_1)...\Gamma(b-a_k)}\times$$
$$\left[\sec\lambda-(\sec\lambda-1)\frac{(b-a_1-a_2-...-a_k+(k-2)...(b-a_1-a_2-...-a_k)}{(a_1-1)(a_2-1)...(a_k-1)}\right]$$
$$+(\sec\lambda-1)\frac{b-1}{(a_1-1)...(a_k-1)}\le 2.,|\lambda|<\frac{\pi}{2}. \quad (27)$$

Proof : By Theorem 1, we must show that :

$$L=\sum_{n=2}^{\infty}[n\sec\lambda-(\sec\lambda-1)]\frac{(a_1)_{n-1}...(a_k)_{n-1}}{(b)_{n-1}(1)_n}\le 1,|\lambda|<\frac{\pi}{2}.$$

Therefore :

$$L=\sec\lambda\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}...(a_k)_{n-1}}{(b)_{n-1}(1)_{n-1}}-$$
$$(\sec\lambda-1)\frac{(b-1)}{(a_k-1)...(a_1-1)}\sum_{n=2}^{\infty}\frac{(a_1-1)_n...(a_k-1)_n}{(b-1)_n(1)_n}$$
$$=\sec\lambda\sum_{n=1}^{\infty}\frac{(a_1)_n...(a_k)_n}{(b)_n(1)_n}-(\sec\lambda-1)\frac{(b-1)}{(a_k-1)...(a_1-1)}\times$$
$$\left[\sum_{n=0}^{\infty}\frac{(a_1-1)_n...(a_k-1)_n}{(b-1)_n(1)_n}-1-\frac{(a_1-1)...(a_k-1)}{(b-1)}\right]$$
$$=\sec\lambda\left[\frac{\Gamma(b)\Gamma(b-a_1-...-a_k)}{\Gamma(b-a_1)..\Gamma(b-a_k)}-1\right]-(\sec\lambda-1)\frac{b-1}{(a_1-1)...(a_k-1)}\times$$
$$\left[\frac{\Gamma(b)\Gamma(b-a_1-...-a_k+(k-1))}{(b-1)\Gamma(b-a_1)...\Gamma(b-a_k)}-1-\frac{(a_1-1)...(a_k-1)}{(b-1)}\right]$$
$$=\frac{\Gamma(b)\Gamma(b-a_1...-a_k)}{\Gamma(b-a_1)...\Gamma(b-a_k)}\times$$
$$\left[\sec\lambda-\frac{(b-a_1-a_2-...-a_k+(k-2))...(b-a_1-a_2-...-a_k)(\sec\lambda-1)}{(a_1-1)(a_2-1)...(a_k-1)}\right]$$
$$+\left[-1+(\sec\lambda-1)\frac{b-1}{(a_1-1)...(a_k-1)}\right].$$

then the last expression is bounded by 1 if (27) holds

References

[1] M. K. Aouf, H. M. Hossen and A. Y, Lashen, Convex subclass of star like function,[1] Kyungpook Math.j. 40 (2000) , 287- 297.

[2] M. Ghaedi and S. R, Kulkarni, Study of some properties of analytic univalent and multivalent function, Ph.D. Thesis (2005), Pune University,(unpublished),

[3] G. Mutugusundaramoorthy, Studies on classes of analytic function with negative coefficients, Thesis , Univ. Madras, Oct. , 1994.

[4] S. Ponnusamy,Geometric properties for convolutions of hyper geometrics and stochastic [4] analysis , 10 , 2 (1997) , 197-202 .

[5] H. Silverman, Sufficient conditions for spiral likeness, Int. J. Math.Sci. 12. 4 (1989), 641-644
[6] H. Silverman, Star like and convexity properties for hyper geometric functions, J.of Math. Analysis and Applications , 172 (1993), 574-481.
[7] J. Wiley and Sons, Handbook of Hyper geometric integrals, Series : Math. and Its Applications, Chichesater: New York, Brisbane, Toronto (1978).

الدوال الحلزونية و الدوال المحدبة بإنتظام المرتبطة بخواص مع الدوال الزائدية

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الملخص :

الهدف من هذا البحث هو دراسة بعض الخواص المرتبطة مع الدالة الحلزونية (اللولبية) و الدالة المحدبة بانتظام لعدد من الدوال الزائدية إضافة إلى خاصية الالتفاف و المؤثرات تم أخذها بنظر الاعتبار.