Numerical Methods for Fractional Differential Equations

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Abstract

The definition of a Fractional differential type of equations is a branch of mathematics, science which developed from derivative operators and the calculus integral traditional definition. It's so much like the fractional exponents were grown from the exponents having integer number. In this paper, will intend to study the ways which are in turn used for solution approximation in fractional differential equations through and how. This paper will also include the Riemann-Liouville differential operator for the basic theorem of the initial value problem for the fractional differential equations. On the same regard, the classical approach will be employed. The theory involving concepts such as local existence, inequalities, global existence of solutions external solutions, comparison results going to be referred.

Keywords: Fractional Differential Equations**,** Riemann- Liouville operator, initial value problem, Numerical Methods, Gamma function.

الخلاصة

يعرف التفاضل الكسري للمعادلات من ضمن فروع الرياضيات الذي أنتج (تطور) من خلال العمليات على المشتقات و التكاملات التفاضلية و التقليدية و التي هي مشابهة بعملياتها الى حد كبير للدوال الأسية الكسرية الناتجة من الدوال الأسية للعدد الصحيح. في هذا البحث ننوي الدراسة بصورة مبسطة للطرق التي تستخدم لإيجاد الحل التقريبي و كيفية الحصول عليه, و سوف يتضمن البحث كذلك دراسة العامل التفاضلي ريمان–ليوفيل بإستخدام النظرية الأساسية لحل مشكلة القيمة الإبتدائية للمعادلات التفاضلية الكسرية بإستخدام الطرق الكلاسيكية. و سوف تشمل النظريات او الطرق مفهوم الحل الخاص و عدم المساواة و الحل العام مع مقارنة النتائج.

الكلمات المفتاحية : المعادلات التفاضلية الكسرية, عامل ريمان–ليوفيل, مشكلة القيمة الإبتدائية, الطرق العددية, دالة كاما.

I. Introduction

In spite of the availability of fractional calculus tools in many fields of sciences, the fractional differential equations theory investigation is a corner of the research. In contrast with the ordinary differential equations well-known theory and for modelling some physical problems, the importance of the standalone and deep study of such differential equations must be started such differential equations include Riemann-Liouville operators (differential operators) of fractional carrying order q fall in the domain of [0,1]. Fractional differential equations, basic theory which is involving the differential operators of Riemann– Liouville of order between 0 and 1 will be discussed in this paper. There are other ways to state the solution without deducing the uniqueness and basic existence outcomes from the theory of fixed points, after following the differential equations, classical proofing in a manner one can contrast and compare the differences. Moreover, to understand the complexities that might appear at the investigation. The inequalities fundamental theory may start at the beginning, which makes the results of necessary comparison available which are important to be used in the coming study of quantitative and qualitative Properties of fractional differential equations solutions. The existence of Piano's local resulting solution will be proven as we progress with prior underlying results and external solutions will be considered existed. From this comparison then going to study the global existence in the later part of this.

II. Strict type of inequalities

Assuming the IVP (initial value problem) as the question below:

$$D^q x = f(t,x) \quad , \quad x(0) = x_0 \; \ldots\ldots\ldots [1]$$

When f belongs to $C([0,T],R)$, $D^q\, x$ represents the derivative fractional of equation x. Moreover, q value fall in domain $[0,1]$ .Since function f was considered to be continuous, then equation (1) will become equal to Volterra integral fraction as below:

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f\big(s, x(s)\big) ds \,, \qquad 0 \le t \le T \quad \ldots..[2]$$

As this result, it is clear that the solution of equation (1) will be equivalent to that in equation (2). The function of Gamma is termed to Γ. At the following section, the fundamental results are going to be demonstrated. These results are related to inequalities fractional integral.

III. First method

Let w and v are belonging to C which falls in the $([0,T],R)$, and f is belonging to $([0,T],R \,*\, R)$

$$v(t) \le v(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f\big(s, v(s)\big) ds \;,$$

$$w(t) \le w(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f\big(s, w(s)\big) ds \;\; , \;\; 0 \le t \le T$$

Because the fact, says not all the inequalities is straight forward and it may include a stack components and by assuming that that $f\,(t,x)$ for any value of t, f cannot decrease on x and also,

$$v(0) < w(0) \qquad \ldots\ldots..[3]$$

As a result can get:

$$v(t) < w(t) \;\; , \quad 0 \le t \le T \quad \ldots..[4]$$

The proof: By assuming that the last formula is wrong. And since the formula in [Diethelm & Ford, 2002] is in the continuous condition the function may exist at time slot $t_{\,1}$ in the $[0, T]$ domain.

$$v(t_1) = w(t_1) \quad , \quad v(t) < w(t) \quad , \quad 0 < t \leq t_1 \qquad \ldots\ldots[5]$$

Assuming the strict nature of the inequality in equation 2 and be employing the non-decreased results of equation f at 5. The following formulation can be stated.

$$\mathrm{w}(t_1) > w(0) + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} f\big(s, w(s)\big) ds$$

$$\geq v(0) + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} f\big(s, v(s)\big) ds \ \geq v(t_1)$$

Which is a contradiction in view of (5). Hence the conclusion (4) is valid and the proof is complete. The next result is for non-strict inequalities, which requires a one-sided Lipschitz type condition.

IV. Second method

Suppose the conditions in first method reserved with non-strict inequalities (2) & (1). Assume further as the following:

$$f(t, x) - f(t, y) \leq \frac{L}{1+t^q} (x - y) \qquad \ldots\ldots..[6]$$

Whenever $x \geq y$ and $L > 0$. then, $v(0) \leq w(0)$ and $L < \Gamma(q+1)$ implies

$$v(t) \leq w(t) \quad , \quad 0 \leq t \leq T \qquad \ldots\ldots[7]$$

The proof: Put w_ϵ as the following :

$$w_\epsilon(t) = w(t) + \epsilon(1 + t^q) \ , \ \text{for small } \epsilon > 0$$

Then can gain the below formulation:

$$w_\epsilon(t) \geq w(0) + \frac{1}{\Gamma(q)} \int_0^{t_1} (t - s)^{q-1} f\big(s, w(s)\big) ds + \epsilon(1 + t^q) \ \ldots..[8]$$

Thus, by calling Lipschitz condition for one side, then:

$$w_\epsilon(t) \geq w_\epsilon(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left[f\big(s, w_\epsilon(s)\big) - \epsilon\frac{L(1+s^q)}{1+s^q}\right]ds + \epsilon t^q$$

$$= w_\epsilon(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f\big(s, w_\epsilon(s)\big)ds - \frac{\epsilon L}{\Gamma(q)}\int_0^t (t-s)^{q-1}\,ds + \epsilon t^q$$

…..[9]

Now , since

$\int_0^t (t-s)^{q-1}ds = t^q\int_0^1 (1-\sigma)^{q-1}d\sigma = \frac{\Gamma(q)}{\Gamma(q+1)}t^q$, It can arrive at

$$w_\epsilon(t) > w_\epsilon(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f\big(s, w_\epsilon(s)\big)ds \quad ……[10]$$

Now, add the first method to the (9), (10) & (1) inequalities for getting $v(t) < w(t)$, $0 < t < T$. As the $\epsilon > 0$ becomes an arbitrary, as of that we can put the conclusion that (7) considered as true and that is all.

Notation: Suppose the equation number six is changed by Lipshitz condition (single side):

$$f(t,x) - f(t,y) \leq L(x-y) \quad , x \geq y \ , \ L > 0 \quad ....[11]$$

As a result, the second method will keep working (validity).

V. External conditions and local existence

In here, going to consider the external solution existence and also the local existence for the case of equation IPV (1). Firstly the type of existence called Piano's existence is going to be discussed:

VI. Third theory

Assuming that f is fall into domain $[R, R_0]$; $R_0 = [(t,x) : 0 < t < a \text{ and } |x - x_0| < b]$, and on value R_0 suppose that $f\ (t,x) < M$. Thus the equation IVP (1) related to the Eq. (1) The fractional differential must be owning one solution at least, $x(t)$ on $0 < t < \alpha$, when

$\propto = \min \left(a, \left[\frac{b}{M}\ \Gamma(q+1)\right]^{\frac{1}{q}}\right)$, $0 < q < 1$, Γ is become a Gamma function as it was before.

The proof:

By assuming the function $x(t)$ to continue, then can getting the following formulation:

$$x_\epsilon(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f\big(s, x_\epsilon(s-\epsilon)\big) ds \qquad \ldots\ldots[12]$$

On $[0, \alpha_1]$, where $\alpha_1 = \min \ (\alpha, \epsilon)$. we observe that

$$|x_\epsilon(t) - x_0| \le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \big|f\big(s, x_\epsilon(s-\epsilon)\big)\big|\, ds \le \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1}\, ds$$

$$= \frac{M\alpha^q}{\Gamma(q+1)} \le b \qquad \ldots\ldots\ldots[13]$$

As a result of choosing "α_1". And if $\alpha > \alpha_1$, we can exploit of the equation (12) to be extended as the continuity of the functions are being. Such like $|x(t) - x_0| < b$ is kept.

$$|x_\epsilon(t_1) - x_\epsilon(t_2)| = \frac{1}{\Gamma(q)} \left| \int_0^{t_1} (t_1 - s)^{q-1} f\big(s, x_\epsilon(s-\epsilon)\big) ds - \int_0^{t_2} (t_2 - s)^{q-1} f\big(s, x_\epsilon(s - \epsilon)\big) ds \right|$$

$$= \frac{1}{\Gamma(q)} \left| \int_0^{t_1} [(t_1 - s)^{q-1} - (t_2 - s)^{q-1}] f\big(s, x_\epsilon(s-\epsilon)\big) ds + \int_{t_1}^{t_2} (t_2 - s)^{q-1} f\big(s, x_\epsilon(s - \epsilon)\big) ds \right|$$

$$\le \frac{M}{\Gamma(q)} \left| \int_0^{t_1} [(t_1 - s)^{q-1} - (t_2 - s)^{q-1}] ds + \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds \right|$$

$$= \frac{M}{\Gamma(q+1)} \left[2(t_2 - t_1)^q + t_1^q - t_2^q \right] \le \frac{2M}{\Gamma(q+1)} (t_2 - t_1)^q < \epsilon \ , \qquad \ldots\ldots\ldots[14]$$

That may provide the $\delta > | - t_1 + t_2 |$, next by following equations 14 and 13 that $x(t)$ family will be forming the functions that are bounded uniformly equation continuous. As an application provided by the theorem of Ascoli–Arzela is showing that the existence of ϵn sequence such like $\epsilon n < \ldots . < \epsilon_2 < \epsilon_1$ All are approached to zero as n is an approach to infinity. And the $lim\, n \to infinity$ of $x \in n\ (t)$ is found in uniform on the domain $[\alpha\, , -\delta]$. And because of uniform continues nature of f, then can start $f(x \in n\,(-\epsilon n + t), t)$ is tending to be uniform to $f(x(t), t))$ as n approaches to infinity thus, by using term to term integration of the equation (12) the following term is resulting :

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) \mathrm{d}s.$$

By doing this we proved that $x(t)$ is one of equation (1) solutions (IVP) and that's all. Now we can prove the external solution existence for the equation (1) IVP by employing the second and third methods.

VII. Fourth Method

In the domain of $\alpha_0 > t > 0$, min $\left(a, \left[\frac{b\Gamma(q+1)}{2M+b}\right]^{\frac{1}{q}}\right) = \alpha_0$ provided by assuming that the equation (1) IVP has an existence solution on the third method. Hence the $f(x, t)$ for every t it

is not decreasing in term x.

The proof

The existence of the maximal solution will be only provided because of the minimal solution case is much similarity, supposing that $b/2 > \epsilon > 0$, then the initial conditions of the fractional differential equations will be considered:

$$D^q x = f(t, x) + \epsilon, \quad x(0) = x_0 + \epsilon. \quad \text{.........} [15]$$

We observe that $f_\epsilon(t, x) = f(t, x) + \epsilon$ is defined and continuous on

$$R_\epsilon = \left[0 \leq t \leq a \text{ and } |x - (x_0 + \epsilon)| \leq \frac{b}{2}\right],$$

The R_ϵ term is part of R_0 and the $(b/2)M > |f \in (x, t)|$ on the term R_ϵ, then can deduce base on the third method that IVP equation (15) is having a solution $X((, t)$ on the interval of $\alpha_0 < t < 0$.

At this section, $\epsilon > \epsilon_1 > \epsilon_2 > 0$, then can drive the following:

$$x(0, \epsilon_2) < x(0, \epsilon_1),$$
$$x(t, \epsilon_2) \leq x(0, \epsilon_2) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_{\epsilon_2}(s, x(s, \epsilon_2)) \mathrm{d}s,$$
$$x(t, \epsilon_1) > x(0, \epsilon_1) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [f(s, x(s, \epsilon_1)) + \epsilon_2] \mathrm{d}s.$$

And by applying the second method, then can get the following:

$$x(t, \epsilon_2) < x(t, \epsilon_1), \quad 0 \leq t \leq \alpha_0.$$

By considering the function $[x(\epsilon, t)]$ family on the domain $\alpha_0 > t > 0$. Then can get :

$$|x(t,\epsilon) - x(0,\epsilon)| \le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f_\epsilon(s, x(s,\epsilon))| \mathrm{d}s \le \frac{(2M+b)}{2} \int_0^t (t-s)^{q-1} \mathrm{d}s$$
$$\le \frac{(2M+b)}{2} \frac{\alpha_0}{\Gamma(q+1)} \le \frac{b}{2} \le b,$$

The same is demonstrating the nature of the uniform boundary of the family and the same can be applied when $\alpha_0 > t_2 > t_1 > 0$ and:

$$|x(t_1,\epsilon) - x(t_2,\epsilon)| \le \frac{(2M+b)}{\Gamma(q+1)} (t_2 - t_1)^q.$$

Which is very much similar to the formula (14) by forming little changes; That is proving the equation continuous nature of $x\ (\epsilon, t)$ family. Hence, after the sequence ϵn is existing with ϵn is approach to zero as n is approaching to infinity, also the following uniform limit

$$\eta(t) = \lim_{n \to \infty} x(t, \epsilon_n) \quad \text{............. [16]}$$

On the period of $[\alpha_0, 0\ ]$ is having an existence. Clearly the term $x_0 =$ ⍰(0). As it was shown in the third method the f term is having continuous uniformly nature and provide as arguments as before and showing that term ⍰(t) is having a solution of the equation (1) [the IVP equation].

Now we are going to show that the minimal solution requirement by ⍰(t) from the equation (1).

Supposing that $x(t)$ may be a solution of the equation (1) at the period of $\alpha_0 > t > 0$. Then:

$$x_0 < x_0 + \epsilon = x(0,\epsilon),$$

$$x(t) < x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [f(s, x(s)) + \epsilon] \mathrm{d}s,$$
$$x(t,\epsilon) \ge x_0 + \epsilon + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [f(s, x(s,\epsilon)) + \epsilon] \mathrm{d}s.$$

By using the second method (Samko *et al.*, 1993) we can have $x(\epsilon, t) > x(t)$ on the period of $[\alpha_0\ , 0]$ of any $0 < \epsilon$. The uniqueness which are existing on the maximal solution yield that $x((,t)$ is tending to be □(t) in the uniform fashion in the period $[\alpha_0 , 0]$ where the term ϵ is approaching zero and by this, the proof is over.

VIII. Global type of existence

Before driving on this context, the result yields from the last section need to be compared.

IX. Fifth Method:

By supposing that m belongs to the C when C is on the domain $(+R , [T , 0])$, and g when g in the domain of (u , t) is being not decreasing in any u and for every t as well as the:

$$m(t) \leq m(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, m(s)) \mathrm{d}s, \quad 0 \leq t \leq T. \text{......[17]}$$

And by assuming the maximal solution is being the term □(t) of the:

$$D^q u = g(t, u), \quad u(0) = u_0 \geq 0, \quad \text{........ [18]}$$

Which is in the period of $(T , 0]$ such a like $(0) > m(0)$, so that:

$$m(t) \leq \eta(t), \quad 0 \leq t \leq T. \quad \text{........... [19]}$$

The proof:

By the maximal solution definition for the term □(t), that will be sufficient for concluding and proving the equation (19) such:

$$m(t) < u(t, \epsilon), \quad 0 \leq t \leq T, \quad \text{.......... [20]}$$

When the term of u in the domain of (ϵ , t) is being one of the solutions that related to:

$$D^q u = g(t, u) + \epsilon, \quad u(0) = u_o + \epsilon, \quad \epsilon > 0. \text{ ..... [21]}$$

At this point by following the equation (21) such :

$$u(t, \epsilon) > u_0 + \epsilon + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, u(s, \epsilon)) \mathrm{d}s.$$

And by applying the second method, then can get the equation (20) and as a result of the term
$\eta(t) = \lim_{\epsilon \to 0} u(\epsilon, t)$ is a uniformly distributed on the period of $T > T_0 > t > 0$. By this step, the proof is said to be done. At this point, the global existence can be proven.

X. Sixth method:

Suppose that f is fall in C which is distributed on the domain ($R, R * (,0])$ and g is belong to C where C is distributed on the domain $(+R, +R * (,0])$ and finally g of domain (u, t) is a not desirable in u for every t , moreover the following :

$$|f(t,x)| \leq g(t,|x|) \quad ....[22]$$

Another assumption can be stated that on $x (x_0 , t)$ we may say that local existence is assumed on this solution of

$$D^q x = f(t,x) \quad , \quad x(0) = x_0 \quad ....[23]$$

The term ?(t) having a maximal solution of the following:

$$D^q u = g(t,u) \quad , \quad u(0) = u_0 \geq 0$$

Which is existing in the period of $(,0]$. Then the any larger in the value of integration in the $x(x_0 , t)$ solution existence of the equation (23) much similar to $u_0 > |x_0|$ will be $(,0]$.

The proof

Assume the term $x(x_0 , t)$ being any solution of equation (23) just a like $u_0 > |x_0|$, that lie on the range $infinity > \beta > zero$. And the β value is non increasable any more. The set of m, $|x(x_0 , t)| = m(t)$ for the range $\beta > t > 0$. And then by employing the equation (22) assumption we can have the following:

$$m(t) \leq |x_0| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g\big(s, m(s)\big)\, ds$$

Now, by applying the fourth method (comparative method), (Podlubny, 1999) the following expression can be obtained:

$$m(t) = |x(t, x_0)| \leq \eta(t) \quad , \quad 0 \leq t < \beta$$

And because the term ▯(t) is considered to be existing on the domain $(\,infinity, zero]$. The same is following that:

$$|g(t, \eta(t))| \leq M \;\; , \qquad 0 \leq t \leq \beta$$

By letting the domain $\beta > t_2 > t_1 > 0$. And by making use of the arguments is looking like to estimable equation (14) and by employing the equation (22) and the M value of the bounds of the term$g$, thus the following can be state:

$$|x(t_1, x_0) - x(t_2, x_0)| \leq \frac{2M}{\Gamma(q+1)} (t_2 - t_1)^q$$

Let the term t_2, t_1 approaching to $-\beta$ as well as the *Cauchy* criterion is going to be called, then it will bbe the following $\lim_{t \to -\beta} x(x_0, t)$ is existing and may define the term $\lim_{t \to -\beta} x(x_0, t) = x(x_0, \beta)$ and then new form of equation (1) IVP is going to be considered as ffollows

$$D^q x = f(t, x) \, , \qquad x(\beta) = x(\beta, x_0)$$

Now, the term of $x(x_0, t)$ is being possible to consider below the value of β by assuming that local existence is there. Then the each solution of the equation (23) of $x(x_0, t)$ is being existed ointhe range of (infinity, zero]. And by that our proof is over.

XI. Conclusion

This study was concentrating on the methods of differential equations of zero or one order differential operator aiming to state the suitable solutions for different variables that might be existing in the fields of interests such as (Physics, Image processing, etc.). According to the above demonstration, the main agenda was for paving the way for a real-life solution, hence the good understanding of the underlying process of each method was provided independently in this paper, which may be helpful for easing the solutions and avoid the complexity that may face during the real-life solutions. The details of each method have been proven within the above demonstration.

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