Weak Convergence of Two Iteration Schemes in Banach Spaces

Zahra M. Mohamed Hasan
Department of Mathematics, College of Education for Pure Sciences (Ibn Al-Haitham), University of Baghdad, Baghdad, Iraq
Salwa.s.a@ihcoedu.uobaghdad.edu.iq

Salwa S. Abed
Department of Mathematics, College of Education for Pure Sciences (Ibn Al-Haitham), University of Baghdad, Baghdad, Iraq

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Abstract- *In this paper, we established weak convergence theorems by using appropriate conditions for approximating common fixed points and equivalence between the convergence of the Picard-Mann iteration scheme and Liu et al iteration scheme in Banach spaces. As well as, numerical examples are given to show that Picard-Mann is faster than Liu et al iteration schemes.*

Keywords- *Banach space, weak convergence, common fixed points.*

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1. Introduction and Preliminaries

Let B be a nonempty subset of a Banach space M. A map T on B is called nonexpansive map if $\|Ta - Tb\| \leq \|a - b\|$ $for\ all\ a, b \in B$. It is called quasi-nonexpansive map [1] if $\|Ta - b\| \leq \|a - p\|$ $for\ all\ a \epsilon B\ and\ for\ all\ p \epsilon F(T)$, denote by $F(T)$ the set of all fixed point of T.

In 2008, a new condition for maps, called condition (C) was introduced by Suzuki [2], which is stronger than quasi-nonexpansive and weaker than nonexpansive, and given some results about fixed point for map satisfying condition (C). Dhompongsa et al [3] and Phuengrattana [4] studied fixed point theorems for a map satisfying condition (C). Weak convergence theorem for a map satisfying condition (C) in uniformly convex Banach space are proved by Kahn and Suzuki [5]. Recently, Garcial-Falset et al [6] introduced two new generalization of condition (C), called condition (E_λ), condition (C_λ) and studied the existence of fixed points and also their asymptotic behavior. For approximating common fixed point of two maps, Takahashi and Tamura [7] studied the following Ishikawa iteration scheme for two nonexpansive maps.

$$a \in B$$
$$a_{n+1} = (1 - \alpha_n)a_n + \alpha_n T b_n$$
$$b_n = (1 - \beta_n)a_n + \beta_n T a_n$$

for all $n \in N, (\alpha_n)\ and\ (\beta_n) in\ [0,1]$.

The aim of this paper is to study weak convergence of the Picard-Mann iteration scheme, Liu et al iteration scheme for approximating common fixed point of generalized nonexpansive and quasi-nonexpansive maps and give some corollaries. Throughout this paper, M will be a uniformly convex Banach space and B a nonempty closed convex subset of M. $F(T, S)$ denotes the set of common fixed point of the maps S and T.

The Picard-Mann [8] iteration scheme for two maps through the sequence (a_n) is defined by:

$$a_{n+1} = Sb_n$$
$$b_n = (1 - \alpha_n)a_n + \alpha_n T a_n, \forall n \geq 0 \quad (1)$$

where $(\alpha_n) \in (0,1)$.

The Liu et al [9] iteration scheme for two maps through the sequence (z_n) is defined by:

$$z_{n+1} = (1 - \alpha_n)S z_n + \alpha_n T u_n$$
$$u_n = (1 - \beta_n)S z_n + \beta_n T z_n, \forall n \geq 0 \quad (2)$$

where $(\alpha_n)\ and\ (\beta_n) \in [0,1]$.

If $S = I$ is called Ishikawa iteration scheme.

Definition (1.1): A Banach space M is called satisfying:

1-Opial's condition [10] if for any sequence (a_n) in M, is weakly convergent to a implies that $\lim_{n\to\infty} inf\ \|a_n - a\| < \lim_{n\to\infty} inf\ \|a_n - b\|$ for all $b \in M\ with\ a \neq b$.

2-Kadec-Klee property [11] if for every sequence (a_n) in M converging weakly to (a) together with $\|a_n\|$ converging strongly to $\|a\|$ imply that (a_n) converges strongly to a point $a \in M$.

Defintion (1.2)[12]: A map $T: B \to M$ is said to be generalized nonexpansive map if there are nonnegative constants $\delta, \mu\ and\ \omega\ with\ \delta + 2\mu + 2\omega \leq 1$ ssuch that $\forall\ a, b \in B$

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2412-0758/University of Technology-Iraq, Baghdad, Iraq

$$\|Ta-Tb\| \le \delta\|a-b\| + \mu\begin{Bmatrix} \|a-Ta\| \\ +\|b-Tb\| \end{Bmatrix} + \omega\begin{Bmatrix} \|a-Tb\| \\ +\|b-Ta\| \end{Bmatrix}$$

Defintion (1.3)[13]: A map $T: B \to B$ is said to satisfying:

1- Condition (C) if $\frac{1}{2}\|a-Ta\| \le \|a-b\| \xrightarrow{yields} \|Ta-Tb\| \le \|a-b\|, \forall\, a,b \in B$.

2-Condition (C_λ) if $\lambda\|a-Ta\| \le \|a-b\| \xrightarrow{yields} \|Ta-Tb\| \le \|a-b\|, \forall\, a,b \in B$ and $\lambda \in (0,1)$.

3-Condition (E_λ) if $\|a-Tb\| \le \lambda\|a-Tb\| + \|a-b\|, \forall\, a,b \in B$ and $\lambda \ge 1$.

Remark (1.4): A map $T: B \to M$ satisfy

1-Condition(C_λ) and T has fixed point, then T is quasi-nonexpansive, but the inverse is false[2].

2-Condition(E_λ) and T has fixed point, then T is quasi-nonexpansive, but the inverse is false[6].

Definition (1.5)[14]: A map $T: B \to M$ is said to be demiclosed with respect to $b \in M$ if for any sequence (a_n) in B, (a_n) converges weakly to a and $T(a_n)$ converges strongly to b. Then $a \in B$ and $T(a) = b$. If $(I-T)$ is demiclosed i.e if (a_n) converges weakly to a in B and $(I-T)$ converges strongly to 0. Then $(I-T)(a) = 0$.

Definition (1.6)[15]: Let M be a Banach space, M is called uniformly convex if for any $\epsilon > 0$ there is $\varsigma > 0$such that $\forall\, a,b \in M$with $\|a\| = \|b\| = 1$ and $\|a-b\| \ge \epsilon$, $\|a+b\| \le 2(1-\varsigma)$holds. Every uniformly convex Banach space is reflexive.

- The modulus of convexity of M is defined by

$$\varsigma_M(\epsilon) = inf\begin{Bmatrix} 1 - \dfrac{\|a+b\|}{2};\ \|a\| = \|b\| \le 1, \\ \|a-b\| \ge \epsilon, \forall\, 0 < \epsilon \le 2 \end{Bmatrix}$$

M is uniformly convex if $\varsigma_M(0) = 0$ and $\varsigma_M(\epsilon) \ge 0, \forall\, 0 < \epsilon \le 2$.

Theorem (1.7)[15]: let M be a uniformly convex Banach space then the modulus of convexity is increasing function.

Remark (1.8)[16]: If $\varsigma(0) = 0$ and has the properties :

$\varsigma(\epsilon) \to 0$ as $\epsilon \to +0$.

$\varsigma: [0,2] \to [0,1]$ is strictly monotone increasing function and surjective

Then $\eta: [0,1] \to [0,2]$ is called strictly monotone increasing function of ς.

Theorem (1.9)[15]: let M be a uniformly convex Banach space. Then for any r and ϵ wih $r \ge \epsilon > 0$ and elements $a,b \in M$ such that $\|a\| \le r, \|b\| \le r, \|a-b\| \ge \epsilon, \exists\, \delta = \delta(\frac{\epsilon}{r}) > 0$ such that

$$\left\|\frac{a+b}{2}\right\| \le r\left[1 - \delta\left(\frac{\epsilon}{r}\right)\right].$$

Proposition (1.10)[16]: Let B be a closed convex set in a Banach space M. If (a_n) converges weakly to a for some sequence (a_n) in M, then $a \in M$.

Lemma (1.11)[17]: Let $(\mu)_{n=0}^{\infty}$ and $(\rho)_{n=0}^{\infty}$ be nonnegative real sequences satisfying the inequality:

$$\mu_{n+1} \le (1-\sigma_n)\mu_n + \rho_n$$

where $\sigma_n \in (0,1), \forall\, n \ge n_0, \sum_{n=1}^{\infty}\sigma_n = \infty$ and $\frac{\rho_n}{\sigma_n} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty}\mu_n = 0$.

Lemma (1.12)[13]: Let M be a uniformly convex Banach space and $0 < L \le t_n \le K < 1, \forall\, n \in N$. Suppose that (a_n) and (b_n) are two sequences of M such that:

$$\lim_{n\to\infty}\|a_n\| \le m,\ \lim_{n\to\infty}\|b_n\| \le m \text{ and } \lim_{n\to\infty}\left\|\begin{matrix} t_n a_n + \\ (1-t_n)b_n \end{matrix}\right\| = m$$

hold for some $m \ge 0$. Then $\lim_{n\to\infty}\|a_n - b_n\| = 0$.

Lemma (1.13)[18]: Let B be a nonempty convex subset of a uniformly convex Banach space. Then there is a strictly increasing continuous function $f: [0,\infty) \to [0,\infty)$ with $f(0) = 0$ such that for each lipschitzain map $T: B \to B$ with lipschitz constant K:

$$\|tTx + (1-t)Ty - T(tx + (1-t)y\| \le Kf^{-1}\begin{pmatrix} \|x-y\| \\ -\dfrac{1}{K}\|Tx - Ty\| \end{pmatrix}$$

$\forall\, x,y \in B$ and $\forall t \in [0,1]$.

Lemma (1.14)[18]: Let M be a uniformly convex Banach space suth that its dual M^* satisfies the Kadec-Klee property. Assume that (a_n) bounded sequence in M such that $\lim_{n\to\infty}\|ta_n + (1-t)p_1 - p_2\|$ exists $\forall\, t \in [0,1]$ and $p_1, p_2 \in W_w(a_n)$, then $p_1 = p_2$.

2.The Main Results

Proposition (2.1): Let B be a closed convex bounded of uniformly convex Banach space, $T: B \to M$ is a generalized nonexpansive map and $a_0, a_1 \in B,\ a_0 \neq a_1 \forall\, t \in [0,1], a_1 = ta_0 + (1-t)a_1$ .If $\forall\, \epsilon > 0, \exists\, a(\epsilon) > 0$ such that

$$\|Ta_0 - a_0\| \leq \epsilon \text{ and } \|Ta_1 - a_1\| \leq \epsilon \quad (3)$$

then $\|Ta_t - a_t\| \leq a(\epsilon) \text{ and } a(\epsilon) \to 0 \text{ as } \epsilon \to +0$.

Proof: Assume that (3) holds with $a_0 \neq a_1 \text{ and } 0 < t < 1$. Then let $i = 0, 1$ such that

$$\|a_i - (a_t + Ta_t)/2\| \geq \|a_i - a_t\|$$

If not, would have the contradiction

$$\|a_1 - a_0\| \leq \sum_{i=0}^{1} \left\| a_i - \frac{a_t + Ta_t}{2} \right\| < \sum_{i=0}^{1} \|a_i - a_t\| = \|a_1 - a_0\|$$

since $a_1 \neq a_0$ we have $r = \|a_t - a_i\| > 0, n = \|a_t - Ta_i\|, m = \|a_i - Ta_t\|$.

Since T is generalized nonexpansive mapping

$$\|Ta_t - a_i\| \leq \|Ta_t - Ta_i\| + \|Ta_i - a_i\| \leq \delta\|a_t - a_i\| + \mu \begin{Bmatrix} \|a_t - Ta_t\| \\ +\|a_i - Ta_i\| \end{Bmatrix} + \omega \begin{Bmatrix} \|a_t - Ta_i\| \\ +\|a_i - Ta_t\| \end{Bmatrix} + \|Ta_i - a_i\| \leq \delta r + \mu(a(\epsilon) + \epsilon) + \omega(n + m) + \epsilon$$

let $w = ar + \mu(a(\epsilon) + \epsilon) + \omega(n + m)$. Then $\|Ta_t - a_i\| \leq w + \epsilon$.

Put $a = a_t, b = Ta_t, c = a_i \text{ and } R = w + \epsilon$.

Let $\eta(.)$ indicate the strictly monotone increasing function to ς(.). The diameter of M denotes by $diam(M)$, by theorem (2.10), we have

$$\|Ta_t - a_t\| \leq sup_{r \in [0, d(M)]} (w + \epsilon)\eta\left(\frac{\epsilon}{w + \epsilon}\right)$$

the $a(\epsilon)$ defined here has desired properties. First $a(\epsilon) \geq \epsilon\eta(1) = 2\epsilon \text{ for } w = 0$.

Forming the supermum separately over the two intervals $\left[0, \sqrt{\epsilon} - \epsilon\right[$ and monotonicity of $\eta(.)$, that

$$a(\epsilon) \leq max\left\{\sqrt{\epsilon}\eta(1), (d(M) + \epsilon)\eta(\sqrt{\epsilon})\right\} \to 0 \text{ as } \epsilon \to 0.$$

Since $a(\epsilon) \geq 2\epsilon, \text{then } \|Ta_t - a_t\| \leq a(\epsilon) \text{ as } a(\epsilon) \to 0 \text{ as } \epsilon \to +0$.

Hence

$(3) \text{holds for the remaining cases } a_1 \neq a_0, t = 0, 1 \text{ and } a_0 = a_1$.

Theorem (2.2): Let B be a closed, bounded and convex subset of uniformly convex M, then the operator $I - T$ is demiclosed on B.

Proof: We show that for any sequence (a_n) in M, if (a_n) converges weakly to $a \text{ and } (I - T)(a_n)$ converges strongly to $0 \text{ as } n \to \infty$, then $a \in M \text{ and } (I - T)(a) = 0$.

By proposition (1.10), we get $a \in M$.

For $\epsilon_0 \in (0,1)$ choose a sequence (ϵ_n) such that

$$\epsilon_n \leq \epsilon_{n-1} \text{ and } a(\epsilon_n) \leq \epsilon_{n-1}, \forall\, n \in N$$

This is possible because $a(\epsilon) \to 0 \text{ as } \epsilon \to 0$. Choosing a subsequence of (a_n) if necessary, we have

$$\|Ta_n - a_n\| \leq \epsilon_n, \forall\, n \in N$$

then

$$\|Tb - b\| \leq \epsilon_0, \forall\, b \in co\{a_n, n \in N\} \quad (4)$$

Now

i) Let $b_1 \in co\{a_m, a_n\} \text{ where } 1 \leq m < n$, by hypothesis

$$\|Ta_m - a_m\| \leq \epsilon_m \text{ and } \|Ta_n - a_n\| \leq \epsilon_n, \epsilon_n \leq \epsilon_m$$

then

$$\|Tb_1 - b_1\| \leq a(\epsilon_m) \leq \epsilon_{m-1} \leq \epsilon_0$$

ii) Let $b_2 \in co\{a_k, a_m, a_n\} \text{ where } 1 \leq k < m < n$. The key is that $b_2 \in co\{a_k, b_1\} \text{ since } b_1 \in co\{a_m, a_n\} \text{ by } (i) \|Tb_1 - b_1\| \leq \epsilon_{m-1}, \epsilon_{m-1} \leq \epsilon_k$, so

$$\|Ta_k - a_k\| \leq \epsilon_k \text{ and } \|Tb_1 - b_1\| \leq \epsilon_k$$

hence

$$\|Tb_2 - b_2\| \leq a(\epsilon_k) \leq \epsilon_{k-1} \leq \epsilon_0$$

if (a_n) converges weakly to $a \text{ as } n \to \infty$, then $a \in co\{a_n, n \in N\}$, by proposition (1.10) and step (4) , we obtain

$$\|Ta - a\| \leq \epsilon_0$$

since ϵ_0 can be any arbitrary small, $Ta - a = 0$. Not only is the map $a \to Ta$ generalized nonexpansive, but for fixed point $b \text{ so is } a \to Ta + b$.This implies that $I - T$ is demiclosed.

Lemma (2.3): Let $T: B \to B$ be a quasi-nonexpansive map and $S: B \to B$ be Lipschitzain and generalized nonexpansive maps. Let

i)(a_n) be as in (1) where $(\alpha_n) \in (0,1)$.

ii)(z_n) be as in (2) where $(\alpha_n) \text{and} (\beta_n) \in [0,1]$.

If $F(T, S) \neq \emptyset$, then $\lim_{n\to\infty}\|a_n - a^*\| \text{ and } \lim_{n\to\infty}\|z_n - a^*\|$ both exist $\text{for all } a^* \in F(T, S)$.

Proof: Let $a^* \in F$.

i)
$$\|a_{n+1} - a^*\| = \|Sb_n - a^*\| \leq \delta\|b_n - a^*\| + \mu \begin{Bmatrix} \|b_n - Sb_n\| \\ +\|a^* - a^*\| \end{Bmatrix} + \omega \begin{Bmatrix} \|b_n - a^*\| \\ +\|a^* - Sb_n\| \end{Bmatrix}$$

$$\leq \delta\|b_n - a^*\| + \mu \begin{Bmatrix} \|b_n - a^*\| \\ +\|Sb_n - a^*\| \end{Bmatrix} + \omega \begin{Bmatrix} \|b_n - a^*\| \\ +\|Sb_n - a^*\| \end{Bmatrix}$$

$$\leq \begin{pmatrix} \delta + \mu + \omega \\ +\mu K + \omega K \end{pmatrix} \|b_n - a^*\| \leq (\delta + 2\mu + 2\omega)\|b_n - a^*\| \leq \|b_n - a^*\| = \|(1 - \alpha_n)a_n + \alpha_n Ta_n - a^*\| \leq (1 - \alpha_n)\|a_n - a^*\| + \alpha_n\|a_n - a^*\| = \|a_n - a^*\|$$

then $\lim_{n\to\infty}\|a_n - a^*\|$ exists $\forall a^* \in F(T,S)$.

ii) $\|u_n - a^*\| \le (1-\beta_n)\|Sz_n - a^*\| + \beta_n\|Tz_n - a^*\|$

$$\le (1-\beta_n)\begin{bmatrix} \delta\|z_n - a^*\| \\ +\mu\begin{Bmatrix}\|z_n - Sz_n\| \\ +\|a^* - a^*\|\end{Bmatrix} + \omega\begin{Bmatrix}\|z_n - a^*\| \\ +\|a^* - Sz_n\|\end{Bmatrix}\end{bmatrix} + \beta_n\|z_n - a^*\|$$

$$\le (1-\beta_n)\begin{bmatrix} \delta\|z_n - a^*\| + \mu\begin{Bmatrix}\|z_n - a^*\| \\ +\|Sz_n - a^*\|\end{Bmatrix} \\ +\omega\begin{Bmatrix}\|z_n - a^*\| \\ +\|a^* - Sz_n\|\end{Bmatrix}\end{bmatrix} +\beta_n\|z_n - a^*\|$$

$$\le (1-\beta_n)\begin{pmatrix}\delta+\mu+ \\ \mu K+\omega+\omega K\end{pmatrix}\|z_n - a^*\| + \beta_n\|z_n - a^*\|$$

$$\le (1-\beta_n)\begin{pmatrix}\delta+2\mu \\ +2\omega\end{pmatrix}\|z_n - a^*\| +\beta_n\|z_n - a^*\|$$

$$\le (1-\beta_n+\beta_n)\|z_n - a^*\| = \|z_n - a^*\|$$

$$\|z_{n+1} - a^*\| \le (1-\alpha_n)\|Sz_n - a^*\| +\alpha_n\|Tu_n - a^*\|$$

$$\le (1-\alpha_n)\begin{bmatrix} \delta\|z_n - a^*\| + \mu\begin{Bmatrix}\|z_n - Sz_n\| \\ +\|a^* - a^*\|\end{Bmatrix} \\ +\omega\begin{Bmatrix}\|z_n - a^*\| \\ +\|a^* - Sz_n\|\end{Bmatrix}\end{bmatrix} +\alpha_n\|u_n - a^*\|$$

$$\le (1-\alpha_n)\left[\begin{pmatrix}\delta+2\mu \\ +2\omega\end{pmatrix}\|z_n - a^*\|\right] +\alpha_n\|u_n - a^*\|$$

$$\le (1-\alpha_n)\|z_n - a^*\| + \alpha_n\|z_n - a^*\| = \|z_n - a^*\|$$

Then $\lim_{n\to\infty}\|z_n - a^*\|$ exists $\forall a^* \in F(T,S)$.

Lemma (2.4): Let $T: B \to B$ be quasi-nonexpansive map and $S: B \to B$ be Lipschitzain, generalized nonexpansive maps and affine and (a_n) be as in (1). Suppose that the following condition $\|a - Tb\| \le \|Sa - Tb\|, \forall\, a,b \in B$ holds. If $F(T,S) \ne \emptyset$, then

$$\lim_{n\to\infty}\|Ta_n - a_n\| = \lim_{n\to\infty}\|Sa_n - a_n\| = 0$$

Proof: Let $a^* \in F(T,S)$.

By lemma(2.3.i) $\lim_{n\to\infty}\|a_n - a^*\|$ exists. Suppose that $\lim_{n\to\infty}\|a_n - a^*\| = c, \forall\, c \ge 0$.

If $c = 0$, there is nothing to proof.

Now suppose $c > 0$,

$$\|a_{n+1} - a^*\| = \|Sb_n - a^*\| \le \|b_n - a^*\|$$

By lemma(2.3.i), we show that $\|b_n - a^*\| \le \|a_n - a^*\|$

this implies to

$$\lim_{n\to\infty} sup\|b_n - a^*\| \le c \qquad (5)$$

moreover $d = \lim_{n\infty\to}\|a_{n+1} - a^*\|$

$$\|a_{n+1} - a^*\| \le \|b_n - a^*\|$$

then

$$c \le \lim_{n\to\infty} inf\|b_n - a^*\| \qquad (6)$$

By (5) and (6), we get

$$\lim_{n\to\infty}\|b_n - a^*\| = c$$

Next consider

$$c = \|b_n - a^*\| \le (1-\alpha_n)\|a_n - a^*\| + \alpha_n\|Ta_n - a^*\|$$

By applying lemma (1.12), we get

$$\lim_{n\to\infty}\|a_n - Ta_n\| = 0$$

$$c = \lim_{n\to\infty}\|a_{n+1} - a^*\| = \lim_{n\to\infty}\|Sb_n - a^*\|$$

$$\|S[(1-\alpha_n)a_n + \alpha_n Ta_n] - a^*\| \le (1-\alpha_n)\|Sa_n - a^*\| + \alpha_n\|STa_n - a^*\|$$

By applying lemma (1.12), we get

$$\lim_{n\to\infty}\|Sa_n - STa_n\| = 0$$

Now

$$\|Sa_n - a_n\| \le \|Sa_n - STa_n\| + \|STa_n - a_n\|$$

By using the hypothesis condition, we have

$$\|Sa_n - a_n\| \le 2\|Sa_n - STa_n\| \to 0 \; as \; n \to \infty.$$

Thus

$$\lim_{n\to\infty}\|Sa_n - a_n\| = 0.$$

Lemma (2.5): Let $T: B \to B$ be a quasi-nonexpansive map, $S: B \to B$ be Lipschitzain and generalized nonexpansive maps and (z_n) be as in (2). Suppose that the following condition $\|a - Tb\| \le \|Sa - Tb\|, \forall\, a,b \in B$ holds. If $F(T,S) \ne \emptyset$, then $\lim_{n\to\infty}\|Tz_n - z_n\| = \lim_{n\to\infty}\|Sz_n - z_n\| = 0$.

Proof: Let $a^* \in F(T,S)$.

By lemma (2.3.ii) $\lim_{n\to\infty}\|z_n - a^*\|$ exists. Suppose that $\lim_{n\to\infty}\|z_n - a^*\| = c, \forall\, c \ge 0$.

If $c = 0$, there is nothing to proof.

Now suppose $c > 0$

$$\lim_{n\to\infty}\|z_{n+1} - a^*\| = c$$

$$c = \|z_{n+1} - a^*\| \le (1-\alpha_n)\|Sz_n - a^*\| + \alpha_n\|Tu_n - a^*\|$$

By applying lemma (1.12), we get

$$\lim_{n\to\infty}\|Sz_n - Tu_n\| = 0$$

$$\|a_{n+1} - a^*\| = \|(1-\alpha_n)Sz_n + \alpha_n Tu_n - a^*\| \le \|Sz_n - a^*\| + \alpha_n\|Sz_n - Tu_n\|$$

this implies to

$$c \le \lim_{n\to\infty} inf\|Sz_n - a^*\| \qquad (7)$$

and $\|Sz_n - a^*\| \le \|z_n - a^*\|$

therefore

$$\lim_{n\to\infty} sup\|Sz_n - a^*\| \le c \qquad (8)$$

By (7) and (8), we have

$$\lim_{n\to\infty}\|Sz_n - a^*\| = c$$

$\|Sz_n - a^*\| \leq \|Sz_n - Tu_n\| + \|Tu_n - a^*\|$
that yields to
$$c \leq \lim_{n\to\infty} inf\|u_n - a^*\| \quad (9)$$
and
$\|u_n - a^*\|$
$\leq (1-\beta_n)\|Sz_n - a^*\| + \beta_n\|Tz_n - a^*\|$
$= \|z_n - a^*\|$
Now
$$\lim_{n\to\infty} sup\|u_n - a^*\| \leq c \quad (10)$$
By (9) and (10), we have
$$\lim_{n\to\infty}\|u_n - a^*\| = c$$
$c = \|u_n - a^*\|$
$\leq (1-\beta_n)\|Sz_n - a^*\| + \beta_n\|Tz_n - a^*\|$
By applying lemma (1.12), we obtain
$$\lim_{n\to\infty}\|Sz_n - Tz_n\| = 0$$
Now
$$\|Sz_n - z_n\| \leq \|Sz_n - Tz_n\| + \|Tz_n - z_n\|$$

By using the hypothesis condition, we get
$$\|Sz_n - z_n\| \leq 2\|Sz_n - Tz_n\| \to 0 \ as \ n \to \infty$$

and
$\|Tz_n - z_n\| \leq \|Tz_n - Sz_n\| + \|Sz_n - z_n\|$
$\leq 2\|Tz_n - Sz_n\| \to 0 \ as \ n \to \infty.$
Hence
$\lim_{n\to\infty}\|Tz_n - z_n\| = \lim_{n\to\infty}\|Sz_n - z_n\| = 0.$

Lemma (2.6): Let $T: B \to B$ be Lipschitzain and quasi-nonexpansive maps and $S: B \to B$ be lipschitzain and generalized nonexpansive maps. Then for $a_1^*, a_2^* \in F(T,S), (a_n) be\ as\ in\ (1)\ and\ (z_n) be\ as\ in\ (2)$ such that
$\lim_{n\to\infty}\|ta_n + (1-t)a_1^* - a_2^*\| \ and$
$\lim_{n\to\infty}\|tz_n + (1-t)a_1^* - a_2^*\| = 0, \forall\, t \in [0,1].$
Proof: Now to prove $\lim_{n\to\infty}\|ta_n + (1-t)a_1^* - a_2^*\|$ exists and equal to zero, by lemma (2.3.i) $\lim_{n\to\infty}\|a_n - a^*\|$ exists, $\forall\, a^* \in F(T,S) and\ (a_n)$ is bounded.
Then there is a real number $L > 0$ such that $(a_n) \subseteq D = \overline{B_r(0)} \cap B$, so that $D \neq \emptyset$ is a closed convex bounded subest of B.
Put $\gamma_n(t) = \|ta_n + (1-t)a_1^* - a_2^*\|$.
Notice that $\gamma_n(0) = \|a_1^* - a_2^*\| \ and\ \gamma_n(1) = \|a_n - a_2^*\|$ esists by lemma(2.3.i).

Defin $R_n: D \to D, \forall\, n \in N, R_n a = S[(1-\alpha_n)a_n + \alpha_n T a_n]\ \forall\, a \in D.$
$\|R_n a - R_n b\|$
$= \left\| \begin{matrix} S[(1-\alpha_n)a_n + \alpha_n T a_n] \\ -S[(1-\alpha_n)b_n + \alpha_n T b_n] \end{matrix} \right\|$
$\leq (1-\alpha_n)\|a_n - b_n\| + \alpha_n\|Ta_n - Tb_n\|$
$\leq (1-\alpha_n)\|a_n - b_n\| + \alpha_n\|Ta_n - a^*\| + \alpha_n\|Tb_n - a^*\|$
$\leq (1-\alpha_n)\|a_n - a^*\| + (1-\alpha_n)\|b_n - a^*\| + \alpha_n\|a_n - a^*\| + \alpha_n\|b_n - a^*\|$
$= \|a_n - a^*\| + \|b_n - a^*\|$
Set $W_{n,m} = R_{n+m}R_{n+m-1} \ldots . R_n$ and
$$b_{n,m} = \left\|W_{n,m}(ta_n + (1-t)a_1^*) - (tW_{n,m}a_n + (1-t)a_1^*\right\|, \forall\, n, m \in N.$$

Then
$\left\|W_{n,m}a - W_{n,m}b\right\|$
$\leq \left\|W_{n,m}a - a^*\right\| + \left\|W_{n,m}b - a^*\right\|$
$\leq \|a - a^*\| + \|b - a^*\|$
and $\left\|W_{n,m}a - a^*\right\| \leq \|a - a^*\|, W_{n,m}a_n = a_{n+m}\ and\ w_{n,m}a^* = a^*, \forall\, a^* \in F.$
By lemma(1.13) there is a strictly increasing function continuous function $f: [0,\infty) \to [0,\infty) with\ f(0) = 0$ such that
$$b_{n,m} \leq Kf^{-1}\left(\|a_n - a_1^*\| - \frac{1}{k}\left\| \begin{matrix} W_{n,m}a_n \\ -W_{n,m}a_1^* \end{matrix} \right\|\right)$$
$$\leq Kf^{-1}(\|a_n - a_1^*\| - \frac{1}{K}\left\| \begin{matrix} a_{n+m} \\ -a_1^* \end{matrix} \right\|)$$
since $\lim_{n\to\infty}\|a_n - a^*\|$ exists $\forall\, a^* \in F$.

$$\lim_{n\to\infty} supf(b_{n,m}) = 0 \xrightarrow{yields} \lim_{n\to\infty}\lim_{m\to\infty} sup\, b_{n,m} = 0$$
Now,
$\gamma_{n+m}(t) = \|ta_{n+m} + (1-t)a_1^* - a_2^*\|$
$= \left\|tW_{n,m}a_n + (1-t)a_1^* - a_2^*\right\|$

$$= \left\| \begin{matrix} tW_{n,m}a_n + (1-t)a_1^* - a_2^* + \\ W_{n,m}(ta_n + (1-t)a_1^*) \\ -a_2^* + a^* - a^* \\ -W_{n,m}(ta_n + (1-t)a_1^*) + a_2^* \end{matrix} \right\|$$

$\leq b_{n,m} + \left\|W_{n,m}(ta_n + (1-t)a_1^*) - a_2^*\right\|$
$\leq b_{n,m} + \left\|W_{n,m}(ta_n + (1-t)a_1^*) - W_{n.m}a_2^*\right\|$
$\leq b_{n,m} + \gamma_n(t)$
Now
$$\lim_{n\to\infty} sup\gamma_{n+m}(t) \leq \lim_{n\to\infty} supb_{n,m} + \gamma_n(t)$$
then
$$\lim_{n\to\infty} sup\gamma_{n+m}(t) \leq \lim_{n\to\infty} inf\gamma_n\,(t)$$
which implies that $\lim_{n\to\infty}\|ta_n + (1-t)a_1^* - a_2^*\|$ exists $\forall\, t \in [0,1]$.

Now to prove $\lim_{n\to\infty}\|tz_n + (1-t)a_1^* - a_2^*\|$ exists.
By lemma (2.3.ii) $\lim_{n\to\infty}\|z_n - a^*\|$exists,$\forall\, a^* \in F(T,S) and\ (z_n)$ is bounded.
Then there is a real number L>0 such that$(z_n) \subseteq D = \overline{B_r(0)} \cap B$ so that $D \neq \emptyset$ is a closed convex bounded subest of B.
Put $\gamma_n(t) = \|tz_n + (1-t)a_1^* - a_2^*\|$
notice that $\gamma_n(0) = \|a_1^* - a_2^*\| \ and\ \gamma_n(1) = \|a_n - a_2^*\|$ esists by lemma (2.3.ii).

Define $R_n: D \to D, \forall\, n \in N, R_n z = (1-\alpha_n)Sz_n + \alpha_n T\big((1-\beta_n)Sz_n + \beta_n T z_n\big)$

$$\|R_n z - R_n w\|$$
$$= \left\| \begin{matrix} (1-\alpha_n)Sz_n + \alpha_n Tu_n \\ -(1-\alpha_n)Sw_n - \alpha_n Tv_n \end{matrix} \right\|$$
$$\le (1-\alpha_n)\|Sz_n - Sw_n\| + \alpha_n\|Tu_n - Tv_n\|$$
$$\le (1-\alpha_n)\begin{pmatrix}\delta + 2\mu \\ +2\omega\end{pmatrix}\|z_n - w_n\|$$
$$+\alpha_n\|Tu_n - a^*\| + \alpha_n\|Tv_n - a^*\|$$
$$\le (1-\alpha_n)\|z_n - w_n\| + \alpha_n\|u_n - a^*\|$$
$$+\alpha_n\|v_n - a^*\|$$
$$\le (1-\alpha_n)\|z_n - w_n\|$$
$$+\alpha_n\begin{Bmatrix}(1-\beta_n)\|z_n - a^*\| \\ +\beta_n\|z_n - a^*\|\end{Bmatrix}$$
$$+\alpha_n\begin{Bmatrix}(1-\beta_n)\|w_n - a^*\| \\ +\beta_n\|w_n - a^*\|\end{Bmatrix}$$
$$= \|z_n - a^*\| + \|w_n - a^*\|$$

The rest of the proof follows the pattern of the above argument.

Theorem (2.7): Let M be a uniformly convex Banach space satisfying Opial's condition and $T: B \to B$ be quasi-nonexpansive map with $(I-T)$ demiclosed at zero, $S: B \to B$ be Lipschitzain and generalized nonexpansive maps and $(a_n), (z_n)$ as in lemma (2.4) and lemma (2.5), respectively. If $F(T,S) \neq \emptyset$, then (a_n) *and* (z_n) both converge weakly to a common fixed point of S and T.

Proof: Let $a^* \in F(T,s)$. As proved in lemma (2.3) $\lim_{n\to\infty}\|a_n - a^*\|$ *and* $\lim_{n\to\infty}\|z_n - a^*\|$ exist.

Now, must prove that (a_n) converges weakly to a unique weak subsequential limit in F.

Since (a_n) is bounded sequence in M, there exist two convergent subsequences (a_{ni}) *and* (a_{nj}) *of* (a_n).

Let $x_1, x_2 \in B$ be weak limit of (a_{ni}) *and* (a_{nj}) respectively. By lemma (2.4) $\lim_{n\to\infty}\|Sa_n - a_n\| = 0$.

By propsition (2.1) and theorem (2.2), we get $I - S$ is demiclosed to zero.

Then $Se_1 = e_1$ *and by hypothesis* $I - T$ *is demiclosed so,* $Te_1 = e_1$. In the same way, can prove that $e_2 \in F(T,S)$.

To prove the uniquence, assume $e_1 \neq e_2$. Then by Opials condition:

$$\lim_{n\to\infty}\|a_n - e_1\| = \lim_{n\to\infty}\|a_{ni} - e_1\|$$
$$< \lim_{n\to\infty}\|a_n - e_2\|$$
$$= \lim_{n\to\infty}\|a_{nj} - e_2\|$$
$$< \lim_{n\to\infty}\|a_{nj} - e_1\|$$
$$= \lim_{n\to\infty}\|a_n - e_1\|$$

this is contrasiction. Thus (a_n) converges weakly to a point in $F(T,S)$.

By ulitizing the same above argument, we can prove that (z_n) converges weakly to a point in $F(T,S)$.

Theorem (2.8): Let M be a uniformly convex Banach space such that its dual M^* satisfies the Kadec-Klee property. Let $T, S, B, (a_n)$ *and* (z_n) be as in lemma (2.4) and lemma (2.5), respectively. If $F(T,S) \neq \emptyset$, then (a_n)*and* (z_n) converge weakly to a common fixed point of S and T.

Proof: Since (a_n) *and* (z_n) are bounded and M is reflexive . Then, there is a subsequence (a_{ni}) *of* (a_n) that converges weakly to a point $a^* \in B$. By lemma (2.4)

$$\lim_{n\to\infty}\|Sa_{ni} - a_{ni}\| = 0 = \lim_{n\to\infty}\|Ta_{ni} - a_{ni}\|$$

thus $a^* \in F(T,S)$.

To prove (a_n) converges weakly to a point a^*.

Assume that (a_{nk}) is another subsequence of (a_n) that converges weakly to a point $b^* \in B$.

Then by lemma (2.6) $\lim_{n\to\infty}\|ta_n + (1-t)a^* - b^*\|$ exists $\forall\, t \in [0,1]$.

By lemma(1.15) $a^* = b^*$. Then (a_n) converges weakly to the point $a^* \in F(T,S)$.

Ulitizing the same above argument to prove that (z_n) converges weakly to the point $a^* \in F(T,S)$.

The following corollary as a special case of quasi-nonexpansive mapping is now obvious.

Corollary (2.8): Let M be a uniformly convex Banach space satisfying Opial's condition and $T: B \to B$ be satisfying condition (C_λ), $S: B \to B$ be generalized nonexpansive map and $(a_n), (z_n)$ be as in lemma (2.4) and lemma(2.5), respectively. If $F(T,S) \neq \emptyset$, then (a_n) *and* (z_n) converges weaklt to a common fixed point of S and T.

Corollary (2.9): Let M be a uniformly convex banach space and its dual M^* satisfies the Kadec-Klee property and $T: B \to B$ be lipschitzain map and satisfying condition (C_λ) and $S: B \to B$ be lipschitzain and generalized nonexpansive maps and $(a_n), (z_n)$ be as in lemma (2.6). If $F(T,S) \neq \emptyset$, then (a_n) *and* (z_n) converges weakly to a common fixed point of S and T.

Corollary (2.10): Let M be a uniformly convex Banach space satisfying Opial's condition and $T: B \to B$ be satisfying condition (E_λ), $S: B \to B$ be generalized nonexpansive map and $(a_n), (z_n)$ be as in lemma (2.4) and lemma (2.5), respectively. If $F(T,S) \neq \emptyset$, then (a_n) *and* (z_n) converges weaklt to a common fixed point of S and T.

Corollary (2.11): Let M be a uniformly convex banach space and its dual M^* satisfies the Kadec-Klee property and $T: B \to B$ be Lipschitzain map and satisfying condition (E_λ) and $S: B \to B$ be Lipschitzain and generalized nonexpansive maps and $(a_n), (z_n)$ be as in lemma (2.6). If $F(T,S) \neq \emptyset$, then (a_n) and (z_n) converges weakly to a common fixed point of S and T.

3. Equivalance of Iterations

Theorem (3.1): Let B be a nonempty closed convex subset of a Banach space M. Let $T: B \to B$ be a quasi-nonexpansive map, $S: B \to B$ be Lipschitzain and generalized nonexpansive maps and $a^* \in B$ be a common fixed point of S and T. Let (a_n) and (z_n) be the Picard-Mann and Liu et al iteration schemes defined in (1) and (2), respectively. Suppose (α_n) and (β_n) satisfied the following conditions:

1-$(\alpha_n) and\ (\beta_n) \in (0,1), \forall\, n \geq 0$.
2-$\sum \alpha_n = \infty$.
3-$\sum \alpha_n \beta_n < \infty$.

If $z_0 = a_0$ and $R(T), R(S)$ are bounded, then the Picard-Mann iterative sequence (a_n) converges strongly to a^* $(a_n \to a^*)$and the Liu et al iterative sequence (z_n) converges strongly to $a^* (z_n \to a^*)$.

Proof: Since the range of T and S are bounded, let

$$M = sup_{a\in B}\{\|Ta\|\} + \|a_0\| < \infty$$

then

$$\|a_n\| \leq M, \|b_n\| \leq M, \|z_n\| \leq M, \|u_n\| \leq M$$

therefore

$$\|Ta_n\| \leq M, \|Tz_n\| \leq M$$

$$\begin{aligned}
&\|a_{n+1} - z_{n+1}\| \\
&= \|Sb_n - (1-\alpha_n)Sz_n - \alpha_n Tu_n\| \\
&\leq \|Sb_n - Sz_n\| + \alpha_n\|Sz_n - Tu_n\| \\
&\leq \|Sb_n - a^*\| + \|Sz_n - a^*\| + \alpha_n\|Sz_n - a^*\| \\
&\quad +\alpha_n\|Tu_n - a^*\| \\
&\leq \delta\|b_n - a^*\| + \mu\begin{Bmatrix}\|b_n - Sb_n\| \\ +\|a^* - a^*\|\end{Bmatrix} \\
&+\omega\begin{Bmatrix}\|b_n - a^*\| \\ +\|a^* - Sb_n\|\end{Bmatrix} + \delta\|z_n - a^*\| \\
&+\mu\begin{Bmatrix}\|b_n - Sb_n\| \\ +\|a^* - a^*\|\end{Bmatrix} + \omega\begin{Bmatrix}\|b_n - a^*\| \\ +\|a^* - Sb_n\|\end{Bmatrix} \\
&+\alpha_n\|z_n - a^*\| + \alpha_n\|u_n - a^*\| \\
&\leq \begin{pmatrix}\delta + 2\mu \\ +2\omega\end{pmatrix}\|b_n - a^*\| + \begin{pmatrix}\delta + 2\mu \\ +2\omega\end{pmatrix}\|z_n - a^*\| \\
&\quad +\alpha_n\|z_n - a^*\| + \alpha_n\|u_n - a^*\| \\
&\leq \|b_n - a^*\| + (1+\alpha_n)\|z_n - a^*\| \\
&\quad +\alpha_n\|u_n - a^*\|
\end{aligned}$$

$$\begin{aligned}
&\|b_n - a^*\| \\
&\leq (1-\alpha_n)\|a_n - a^*\| + \alpha_n\|Ta_n - a^*\| \\
&\leq (1-\alpha_n)\|a_n - a^*\| + \alpha_n\{\|Ta_n\| + \|a^*\|\} \\
&\leq (1-\alpha_n)\|a_n - a^*\| + \alpha_n\big\|M + \|a^*\|\big\|
\end{aligned}$$

$$\begin{aligned}
&\|u_n - a^*\| \\
&\leq (1-\beta_n)\|Sz_n - a^*\| + \beta_n\|Tz_n - a^*\| \\
&\leq (1-\beta_n)\begin{pmatrix}\delta + 2\mu \\ +2\omega\end{pmatrix}\|z_n - a^*\| + \beta_n\begin{Bmatrix}\|Tz_n\| \\ +\|a^*\|\end{Bmatrix} \\
&\leq (1-\beta_n)\{M + \|a^*\|\} + \beta_n\{M + \|a^*\|\} \\
&= M + \|a^*\|
\end{aligned}$$

Thus

$$\begin{aligned}
&\|a_{n+1} - z_{n+1}\| \\
&\leq \|b_n - a^*\| + (1+\alpha_n)\|z_n - a^*\| \\
&\quad +\alpha_n\|u_n - a^*\| \\
&\leq (1-\alpha_n)\|a_n - a^*\| + \alpha_n\{M + \|a^*\|\} \\
&\quad +(1+\alpha_n)\|z_n - a^*\| + \alpha_n\{M + \|a^*\|\} \\
&\leq (1-\alpha_n)\|a_n - z_n\| + (1-\alpha_n)\|z_n - a^*\| \\
&\quad +2\alpha_n\{M + \|a^*\|\} + (1+\alpha_n)\|z_n - a^*\| \\
&\leq (1-\alpha_n)\|a_n - z_n\| + \|z_n - a^*\| + 2\alpha_n \\
&\quad \{M + \|a^*\|\} \\
&\leq (1-\alpha_n)\|a_n - z_n\| + (1+2\alpha_n)\{M + \|a^*\|\}
\end{aligned}$$

let $\mu_n = \|a_n - z_n\|$, $\rho_n = (1+2\alpha_n)\{M + \|a^*\|\}, \sigma_n = \alpha_n$ and $\frac{\rho_n}{\sigma_n} \to 0$ as $n \to \infty$.

By applying lemma(1.11), we get

$$\lim_{n\to\infty}\|a_n - z_n\| = 0.$$

If $a_n \to a^* \in F(T,S)$, then

$$\|z_n - a^*\| \leq \|z_n - a_n\| + \|a_n - a^*\| \to 0 \text{ as } n \to \infty.$$

And if $z_n \to a^* \in F(T,S)$, then

$$\|a_n - a^*\| \leq \|a_n - z_n\| + \|z_n - a^*\| \to 0 \text{ as } n \to \infty.$$

4. Numerical examples

In this section, we consider two examples to show that the Picard-Mann iteration scheme converges faster than Liu et al iteration schem.

Example (4.1): Let $T, S: R \to R$ be a map defined by $Ta = \frac{2a}{3}$ and $Sa = \frac{a}{2}$, $\forall\, a \in R$. Choose $\alpha_n = \beta_n = \frac{3}{4}, \forall n$ with initial value $a_1 = 30$. The two iteration scheme converge to the same fixed point $a^* = 0$ .It's clear from table 1, that Picard-Mann converges faster than Liu et al.

Table 1: Numerical results corresponding to $a_1 = 30$ for 20 steps

n	Iteration (1)	Iteration (2)	n	Iteration (1)	Iteration (2)
0	30	20	11	0.0006	0.0034
1	11.2500	13.1250	12	0.0002	0.0015
2	4.2188	5.7422	13	0.0001	0.0006

3	1.5820	2.5122	14	0.0000	0.0003
4	0.5933	1.0991	15	0.0000	0.0001
5	0.2225	0.4809	16	-	0.0001
6	0.0834	0.2104	17	-	0.0000
7	0.0313	0.0920	18	-	0.0000
8	0.0117	0.0403	19	-	0.0000
9	0.0044	0.0176	20		0.0000
10	0.0016	0.0077		-	0.0000

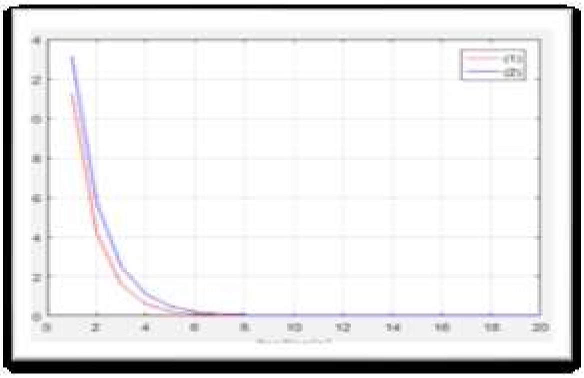

Figure 1: ***Convergence behavior corresponding to $a_1 = 30$ for 20 steps.***

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Example (4.2): Let $B = [-180,180]$, $T, S: B \to B$ be a map defined by $Ta = acosa$ and $Sa = \frac{a}{2}$ $\forall\, a \in B$. Choose $\alpha_n = \frac{1}{3}, \beta_n = \frac{1}{5}$ $\forall\, n$ with initial value $a_1 = 30$. The two iteration scheme converge to the same fixed point $a^* = 0$ .It's clear from table 2, that Picard-Mann converges faster than Liu et al.

Table 2: Numerical results corresponding to $a_1 = 30$ for 30 steps

n	Iteration (1)	Iteration (2)	n	Iteration (1)	Iteration (2)
0	30	30	16	0.0001	0.0014
1	10.7713	12.9255	17	0.0000	0.0008
2	3.1911	7.5904	18	0.0000	0.0005
3	0.5325	3.4317	19	0.0000	0.0003
4	0.2540	0.7150	20	0.0000	0.0002
5	0.1256	0.3940	21	-	0.0001
6	0.0626	0.2304	22	-	0.0001
7	0.0313	0.1370	23	-	0.0000
8	0.0156	0.0819	24	-	0.0000
19	0.0078	0.0491	25	-	-
10	0.0039	0.0295	26	-	-
11	0.0020	0.0177	27	-	-
12	0.0010	0.0106	28	-	-
13	0.0005	0.0064	29	-	-
14	0.0002	0.0038	30	-	-
15	0.0001	0.0023			

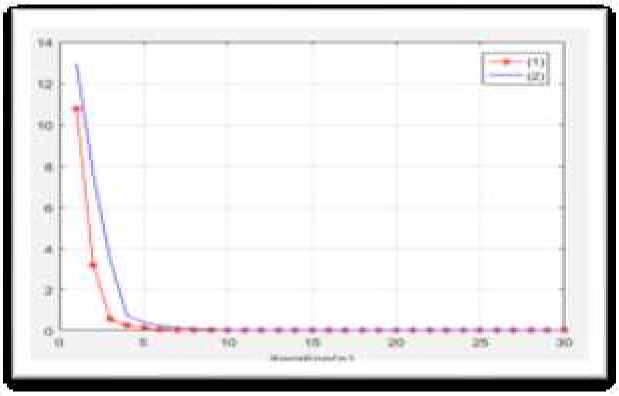

Figure 2: ***Convergence behavior corresponding to $a_1 = 30$ for 30 steps.***

Finally, it is appropriate to ask a question about the possibility of employing the above results in finding solutions to problems such in [19] and [20]

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