The Stability and Bifurcation of Liu chaotic system

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Abstract

In this paper, a three dimensional autonomous chaotic system is considered (Liu system) which is discovered by the scientist Liu in 2004, we begin to study the new system with a rich structure, we obtained some information on stability and bifurcation. We show that this system is unstable at origin while has asymptotically stable under the condition $b < (a+c)/2$ at the other critical points where a,b,c are a positive parameters of this system **.** The system possesses a Hopf bifurcation, finally an illustrative example is given .
Keywords: Liu system, Stability, Hopf bifurcation, Routh-Hurwitz.

1- Introduction:

Dynamic systems described by nonlinear differential equations can be strongly sensitive to initial conditions. This phenomenon is known as deterministic chaos just to mean that, even if the system mathematical description is deterministic, its behavior proves to be unpredictable[8].Chaos as a very interesting complex nonlinear phenomenon has been intensively studied in the last three decades within the science, mathematics and engineering communities [5].In recent years, the study of the nonlinear chaotic dynamics is a popular problem in the field of the nonlinear science and great progress has been made in the research of nonlinear chaotic dynamics. In 1963, Lorenz found the first classical chaotic attractor in three-dimension autonomous system [5]. In 1999, Chen and Ueta found another similar but not topological equivalent chaotic attractor to Lorenz's [6]. In 2002, L$\ddot{u}$ found the critical chaotic attractor between the Lorenz and Chen attractor [7]. In the same year, L$\ddot{u}$ et al. unified above three chaotic systems into a chaotic system which is called unified chaotic system [10].It is noticed that these systems can be classified into three types by the definition of Vanece and Celikovsky [3]: the Lorenz system satisfies the condition $a_{12}a_{21} > 0$, the Chen system satisfies $a_{12}a_{21} < 0$, and the L$\ddot{u}$ system satisfies $a_{12}a_{21} = 0$,where a_{12} and a_{21} are the corresponding elements in the linear part matrix $A = (a_{ij})_{3\times 3}$ of the system.In 2004, Liu etc. discovered another chaotic system by using physical electrical circuits and called Liu system [3,10]. It provides a new domain for the study of chaotic system.The nonlinear differential equations that describe the Liu system are

$$\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= bx - kxz \\ \dot{z} &= -cz + hx^2 \end{aligned} \qquad (1)$$

Where a, b, c, k and h are a positive parameters. It has a chaotic attractor. According to the critical term $a_{12}a_{21}$ which was proposed by Vanece and Celikovsky: The Liu system satisfies the condition $a_{12}a_{21} > 0$. The chaotic attractor obtained from this system is also the butterfly-shaped attractor. This attractor is similar but not equivalent to the Lorenz chaotic attractor. The third differential equation has one quadratic item that can produce folding trajectories [3]. In the following we briefly describe some basic properties of the system (1).

1-Symmetry and invariance:

First, we note that the system (1) has a symmetry S because the transformation

$$S : (x, y, z) \rightarrow (-x, -y, z) \quad (2)$$

Which permits system invariant for all values of the system parameters a, b, c, k and h . Obviously, the z-axis itself is an orbit [4].

2- Dissipative:

The system can be a dissipative system, because the divergence of the vector field, also called the trace of the Jacobian matrix is negative if and only if the sum of the parameters a and c is positive, that is $a + c > 0$

$$\begin{aligned} div\vec{V} &= \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = Tr(J) = -(a+c) \ , \\ V(t) &= V(0)e^{-(a+c)t} \end{aligned} \qquad (3)$$

So, the system will always be dissipative if and only if when $a + c > 0$ with an exponential rate:

$$\frac{d\vec{V}}{dt} = e^{-(a+c)} \ [9] .$$

In [4] studied the stability and bifurcation for a new Lorenz-Like system by method of nonlinear dynamics theory. and [9] studied stability and bifurcation for the system derived from the Lorenz system while in [3] studied the problem of Slow Manifold Analysis and adaptive control for Liu system finally, [10] investigated the method for controlling the uncertain Liu system with known parameters k and h via backstopping control. In this paper, we study the stability and bifurcation for the Liu chaotic system by depended on the roots to determine the stability at origin, while depended on the Routh-Hurwitz method to determine the stability and bifurcation on other critical points, and found this system is unstable always at the origin also we find the critical value b_0 of this system.

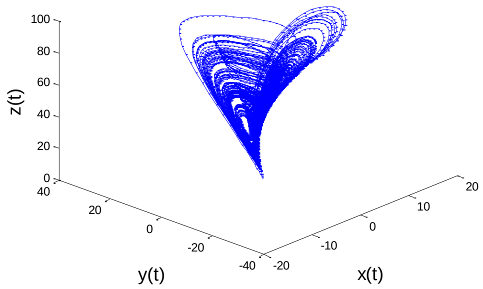

Figure 1: the attractor of Liu system when $a=10, b=40, c=2.5, h=4, k=1$

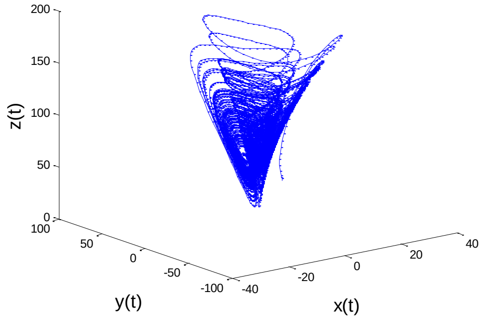

Figure 2: the attractor of Liu system when $a=10, b=68, c=2.5, h=4, k=1$

2- Helping Results:

In the context of ordinary differential equations ODEs the word "bifurcation" has come to mean any marked change in the structure of the orbits of a system (usually nonlinear) as a parameter passes through a critical value[2].The theory of bifurcations of parameterized dynamical system is well known. One consider a vector field

$$\dot{x} = f_\mu(x) \quad \mu \in R , \quad x \in R^n \quad (4)$$

Depending on a parameter μ the critical point of the vector field are those x_0, μ_0 such that

$$f_{\mu 0}(x_0) = 0 \ .$$

Perhaps the most important property of critical point is its stability. In the first approximation, which is determined by stability of its liberalized system around x_0, μ_0

$$\dot{x} = D\, f_{\mu 0}(x_0) \quad \mu_0 \in R , \quad x_0 \in R^n \ (5)$$

Where $D\, f_{\mu 0}(x_0)$ is the Jacobian matrix of f [1].

Theorem1 (Hopf Bifurcation theorem)[1]

Suppose that the system

$\dot{x} = f_\mu(x)$, $\mu \in R$, $x \in R^n$ has critical point (x_0, μ_0) ,then this system has a Hopf bifurcation if the following properties are satisfied:

1- $D\, f_{\mu 0}(x_0)$ has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts.

2- $\frac{d}{d\mu}(\mathrm{Re}(\lambda_{2/3}(\mu)))\big|_{\mu=\mu 0} = d \neq 0$

Remark 1[1]:

Let $\lambda^3 + A\lambda^2 + B\lambda + C = 0$ be the characteristic equation for a three-component system, where A , C indicate the trace and determinant rest, then a Hopf bifurcation takes place of the transit through the surface

$$A.\ B = C \quad \text{if} \quad A, B, C > 0 \ (6)$$

This condition is a necessary condition for a Hopf bifurcation.

3- Main result:

Solving the three equations $\dot{x} = \dot{y} = \dot{z} = 0$ we get that system (1) has three critical points

$$O(0,0,0) \ , \quad A_+\left(\sqrt{\frac{cb}{hk}}, \sqrt{\frac{cb}{hk}}, \frac{b}{k}\right) ,$$

$$A_-\left(-\sqrt{\frac{cb}{hk}}, -\sqrt{\frac{cb}{hk}}, \frac{b}{k}\right)$$

Theorem 2: The critical point $O(0,0,0)$ is always unstable**.**

Proof: At the critical point $O(0,0,0)$, is linearized, the Jacobian matrix of system (1) is defined as;

$$J_o = \begin{bmatrix} -a & a & 0 \\ b & 0 & 0 \\ 0 & 0 & -c \end{bmatrix} \quad (7)$$

The characteristic equation is:

$$f(\lambda) = \lambda^3 + (a+c)\lambda^2 + (ac-ab)\lambda - abc = 0 \quad (8)$$

$$\text{0r} \quad (-\lambda - c).(\lambda^2 + a\lambda - ab) = 0 \quad (9)$$

and three eigenvalues corresponding to the critical point O are:

$$\lambda_1 = -c \ , \ \lambda_{2,3} = \frac{-a \mp \sqrt{a(a+4b)}}{2}$$

So, it's clear that if $-a > \sqrt{a(a+4b)}$, then $\lambda_2 < 0$, $\lambda_3 < 0$, But this is impossible for $b > 0$ since when $-a > \sqrt{a(a+4b)} \Rightarrow a^2 > a(a+4b) \Rightarrow b < 0$ (C!) contraction ,while $-a < \sqrt{a(a+4b)}$ then $\lambda_2 < 0$, $\lambda_3 > 0$.consequently the critical point $O(0,0,0)$ is always unstable. This completes the proof. In the following, we consider the stability of the system (1) at the critical points A_+ and A_- , Because the system is invariant under the transformation, so one only needs to consider the stability of any one of the both . The stability of the system (1) at critical point A_+ is analyzed in this paper. Under the linear transformation

$$(x, y, z) \to (X, Y, Z):$$

$$x = X + \sqrt{\frac{cb}{hk}}$$
$$y = Y + \sqrt{\frac{cb}{hk}} \qquad (10)$$
$$z = Z + \frac{b}{k}$$

the system (1) becomes

$$\dot{X} = -aX + aY$$
$$\dot{Y} = -k\sqrt{\frac{cb}{hk}}Z \qquad (11)$$
$$\dot{Z} = 2h\sqrt{\frac{cb}{hk}}X - cZ$$

The critical point A_+ of the system (1) is swiched to the new critical point $O'(0,0,0)$ of the system (11) under the linear transformation, in the following, the stability of system (11) at the critical point $O'(0,0,0)$ is considered.The Jacobian matrix of the system (11) at $O'(0,0,0)$ is:

$$J(O') = \begin{bmatrix} -a & a & 0 \\ 0 & 0 & -k\sqrt{\frac{cb}{hk}} \\ 2h\sqrt{\frac{cb}{hk}} & 0 & -c \end{bmatrix} \qquad (12)$$

and the characteristic equation is :

$$f(\lambda) = \lambda^3 + A\lambda^2 + B\lambda + C = 0 \qquad (13)$$

Where

$$A = a + c$$
$$B = ac$$
$$C = 2abc$$

Theorem 3:
The critical points $A_+(\sqrt{\frac{cb}{hk}}, \sqrt{\frac{cb}{hk}}, \frac{b}{k})$, $A_-(-\sqrt{\frac{cb}{hk}}, -\sqrt{\frac{cb}{hk}}, \frac{b}{k})$ are:

1) Asymptotically stable if $b < (a+c)/2$,
2) Unstable if $b > (a+c)/2$,
3) Critical cases (bifurcation) if $b = (a+c)/2$.

Proof:
Using Routh-Hurwitz criterion, the equation (13) has all roots with negative real parts if and only if the conditions are satisfied as follows

$$A > 0$$
$$AB > C \qquad (14)$$
$$C > 0$$

Sinse $A = a + c$ and a , b and c are a positive parameters, consequently $A > 0$ always and $C > 0$ also , we must prove that $AB > C$ therefore $(a+c)ac > 2abc$ $\Rightarrow a + c > 2b \Rightarrow \therefore b < (a+c)/2$, the proof of first condition is completed, while if $b > (a+c)/2$ then one of Routh-Hurwitz conditions not satisfied, consequently the system (1) is unstable, finally if $b = (a+c)/2$ then satisfied remark 1, hence the system (1) is critical case (bifurcation) the proof is completed.

Proposition 1: Equation (13) has purely imaginary roots if and only if $b > 0$, $b = \frac{a+c}{2} = b_0$ (b_0 is critical value of Liu system) . In this case the solutions of equation (13) are $\lambda_1 = -(a+c)$, $\lambda_{2,3} = \pm i\sqrt{ac}$.

Proof: If $\lambda_{2,3} = \pm iw$ are the complex solutions and λ_1 the real solution of equation (13) then, from $\lambda_1 + \lambda_2 + \lambda_3 = -(a+c) \Rightarrow \lambda_1 = -(a+c)$. This easily leads to $b > 0$, $b = \frac{(a+c)}{2}$ and $\lambda_1 = -(a+c), \lambda_{2,3} = \pm i\sqrt{ac}$.

We will use the following Corollary, which enables us to find the value λ_b' directly without using transformation.

Corollary1:

$$\lambda_b' = \frac{-\frac{\partial f}{\partial b}}{\frac{\partial f}{\partial \lambda}}$$

, where $b = b_0$ a critical value of system (1)

Hopf bifurcation may appear only at the critical points A_+ or A_- .Due the symmetry of A_+ and A_- . the following we will prove that the system (1) display a Hopf bifurcation at the point A_+. For $b = b_0 = \frac{(a+c)}{2}$ the point A_+ loses its stability.

Theorem 4: If $b = \frac{(a+c)}{2}$, equation (13) has a negative solution $\lambda_1 = -(a+c) < 0$ together with a pair of purely imaginary roots $\lambda_{2,3} = \pm i\sqrt{ac}$ such that $\mathrm{Re}(\lambda_b'(b_0)) \neq 0$, therefore the system (1) displays a Hopf bifurcation at the point A_+ .

Proof: If $b = \frac{(a+c)}{2}$ the equation (13) is transformed into

$$(\lambda + (a+c))(\lambda^2 + ac) = 0$$

with solutions $\lambda_1 = -(a+c), \lambda_{2,3} = \pm i\sqrt{ac}$

$$\lambda_b' = \frac{-2ac}{3\lambda^2 + 2(a+c)\lambda + ac}$$

$$\lambda_b'(b_0) = \frac{-2ac}{3\lambda^2 + 2(a+c)\lambda + ac}\Big|_{b=\frac{a+c}{2}}$$

Substituting $\lambda_{2,3} = \pm i\sqrt{ac}$, the real part and imaginary part of the $\lambda_b'(b_0)$ respectively are:

$$\mathrm{Re}(\lambda_b'(b_0)) = \frac{ac}{ac + (a+c)^2} \neq 0 \quad ,$$

$$\mathrm{Im}(\lambda_b'(b_0)) = \frac{(a+c)\sqrt{ac}}{ac + (a+c)^2} \neq 0 \quad .$$

Consequently, the system (11) displays a Hopf bifurcation at $O'(0, 0, 0)$, so the system (1) displays a Hopf bifurcation at $A_+(\sqrt{\frac{cb}{hk}}, \sqrt{\frac{cb}{hk}}, \frac{b}{k})$.

4 - Illustrative Example*:*

Example: Investigate for stability and Hopf bifurcation of the following Liu system at A_+

$$\dot{x} = 2(y - x)$$
$$\dot{y} = x - 4xz$$
$$\dot{z} = -4z + 6x^2$$

Solution:

Stability at origin:

$a = 2,\ c = 4,\quad b = 1 \quad and \quad b_0 = 3$ and the characteristic equation of Liu system is of the form:

$\lambda^3 + 6\lambda^2 + 6\lambda - 8 = 0$,

So $\lambda_1 = -4$

$\lambda_{2,3} = -1 \pm \sqrt{3}$ therefore $\lambda_2 < 0$, $\lambda_3 > 0$, then the system (1) is unstable.

Stability at A_+:

the characteristic equation of Liu system is of the form: $\lambda^3 + 6\lambda^2 + 8\lambda + 16 = 0$

then the system (1) is asymptotically stable since $b < b_0$, by theorem 3, first condition .

Hopf bifurcation at A_+:

1f $b = 3 \quad and \quad b_0 = 3$ and the characteristic equation is of the form: $\lambda^3 + 6\lambda^2 + 8\lambda + 48 = 0$ then system (1) is a Hopf bifurcation since $b = b_0$, by theorem 3 ,third condition ,and the roots are $\lambda_1 = -6$, $\lambda_{2,3} = \pm i\, 2\sqrt{2}$ by proposition 1.

5 - Conclusion:

In this paper, a new three dimensional Liu chaotic system has been studied, there are obtained some information on stability and bifurcation. and we conclusion that this system is unstable at origin while has asymptotically stable under the condition $b < (a + c)/2$ in the other critical points where a,b,c are a positive parameters of this system. We prove that Hopf bifurcation occurs when the bifurcation parameter passes through the critical value $b_0 = (a + c)/2$,finally we found the parameters k and h do not effected on the stability and bifurcation.

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الاستقرارية والتشعب لنظام المضطرب Liu

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الملخص

في هذا البحث تم التطرق الى نظام مضطرب مستقل ذاتياً ثلاثي الإبعاد هو نظام Liu المكتشف من قبل العالم Liu في عام 2004 ، و قمنا بدراسة النظام الجديد مع تركيبه الغني، وحصلنا على بعض المعلومات في الاستقرارية والتشعب ، إذ تبين بان النظام غير مستقر عندا نقطة الأصل بينما يكون مستقرا محاذي عندا النقاط الحرجة الأخرى تحت الشرط $b<(a+c)/2$ حيث a ، b ، c هي معاملات موجبة لهذا النظام وبالإضافة الى ذلك فان النظام يملك نقطة تشعب , وأخيرا تم إعطاء مثال توضيحي .