Existence and Uniqueness Solution of fractional Nonlinear Integro-Differential Equation of Order (2α) with Boundary Conditions

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Abstract
In this paper we study the existence and uniqueness solution of fractional nonlinear integro-differential equation of order (2α) with boundary conditions, by using the Picard approximation method which is given by [3]. and we extend some results gained by [2].

Introduction
Many Arthur's discussed solving of differential and integral equation of fractional order. Here we refer to some of the works done in this area [2,5].

In this paper we set some definitions and lemmas to be used in the proof of the main theorem, for references [1,4].

Definition 1:
Let f be a function which is defined a. e. (almost every where) on $[a,b]$. For $\alpha > 0$, we define:

$$\mathrm{I}_a^{\alpha} f = \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s)ds$$

Provided this integral (Lebesgue) exists.

Definition 2:
If $\alpha > 0$, then camma's function is denote by (Γ) and defined by the form: $\Gamma(\alpha) = \int_0^{\infty} e^{-s} s^{\alpha-1} ds$

Lemma 1:
If $\{f_n\}_{n=1}^{\infty}$ is a sequences of functions is defined on the set $E \subseteq R$ such that $|f_n| \le M_n$, where M_n is a positive number, then $\sum_{n=1}^{\infty} f_n$ is uniformaly convergent on E if $\sum_{n=1}^{\infty} M_n$.

Lemma 2:
Let $E_\alpha(m;x) = \sum_{m=1}^{\infty} \frac{m^{n-1}x^{n\alpha-1}}{\Gamma(n\alpha)}$, where m=R , then:

1. the series converges for $x \neq 0$ and $\alpha > 0$.
2. the series converges everywhere when $\alpha \ge 0$.
3. if $\alpha = 0$, then $E_1(m,x) = \exp(mx)$.

For the proof see [1].

Lemma 3:
If K_1 and K_2 be a positive constant, and f be a continuous function on $a \le t \le b$, such that:

$$f(t) \le K_1 + K_2 \int_a^t f(s)ds$$

Then

$$f(t) \le K_1 \exp\big(K_2(t-a)\big)$$

Now we are using the Picard approximation method for studying the solution of fractional second order nonlinear integro-differential equations, as the form:

$$x^{2\alpha}(t) = f\left(t, x(t), \dot{x}(t), \int_{-\infty}^{t} w(t,s) g(s, x(s), \dot{x}(s)) ds\right), \cdots \quad (1)$$

$$x^{(\alpha-1)}(0) = x_0, \; 0 < \alpha \le 1$$

with boundary conditions

$x(a) = A$, $x(b) = B$ … (2) where the function $f(t,x,\dot{x},y)$ is a continuous in $t, x, \dot{x}$ and defined on the domain:

$$(t,x,\dot{x},y) \in [0,T] \times G_\alpha \times G_{1\alpha} \times G_{2\alpha} \cdots (3)$$

Where $x \in G_\alpha \subseteq [0,T]$, $\dot{x} \in G_{1\alpha} \subseteq [0,T]$ and $G_{1\alpha}$, $G_{2\alpha}$ are closed and bounded <u>domains and A , B are positive constants.</u>

Suppose that the function $f(t,x,\dot{x},y)$ satisfies the following inequalities:

$$\|f(t,x,\dot{x},y)\| \le M \quad , \; \ldots (4)$$

$$\|f(t,x_1,\dot{x}_1,y_1) - f(t,x_2,\dot{x}_2,y_2)\| \le K_1\|x_1 - x_2\| + K_2\|\dot{x}_1 - \dot{x}_2\| + K_3\|y_1 - y_2\| ,\ldots(5)$$

$$\|g(t,x_1,\dot{x}_1) - g(t,x_2,\dot{x}_2)\| \le L_1\|x_1 - x_2\| + L_2\|\dot{x}_1 - \dot{x}_2\| \quad \ldots (6)$$

for all $t \in [0,T]$ and $x, x_1, x_2 \in G_\alpha$, $\dot{x}, \dot{x}_1, \dot{x}_2 \in G_{1\alpha}$ and $y, y_1, y_2 \in G_{2\alpha}$, where $M, K_1, K_2, K_3, L_1, L_2$, are positive constants and

$$y(t,x,\dot{x}) = \int_{-\infty}^{t} w(t,s) g(s, x(s), \dot{x}(s)) ds.$$

The kernel function $w(t,s)$ was defined and continuous in $-\infty < 0 \le a \le \tau \le s \le t \le b \le T < \infty$, $\|w(t,s)\| \le \delta e^{-\gamma(t-s)}$, where γ, δ be a positive constants.

We define the non-empty sets as follows:

$$\left.\begin{aligned} G_{\alpha f} &= G_\alpha - \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)} M \\ G_{1\alpha f} &= G_{1\alpha} - \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)} M \\ G_{2\alpha f} &= G_{2\alpha} - \frac{\delta}{\gamma}\left[L_1 \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)} + L_2 \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)}\right] M \end{aligned}\right\} \ldots (7)$$

where $\|.\| = \max|.|$.

Furthermore, we suppose that the greatest eigen value of the following matrix:

$$H_{0\alpha} = \begin{pmatrix} \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)}q_1 & \frac{(T-a)t^{\alpha-1}}{\Gamma(\alpha)} + \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)}q_2 \\ \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)}q_1 & \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)}q_2 \end{pmatrix}$$

is less than unity. i.e.

$$h_{\max}(H_{0\alpha}) < 1 \quad \ldots (8)$$

Existence Solution

The study of the existence solution of the problem (1), (2) will be introduced by the following:

<u>Theorem 1</u>:

Let the function $f(t,x,\dot{x},y)$ be defined in the domain (3), continuous in $t,x,\dot{x}$ and satisfy the inequalities (4), (5) and (6), then the sequence of functions:

$$x_{m+1}(t,x_0,\dot{x}_0,y_0) = \frac{[A+(t-a)\dot{x}_m(a)]t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_a^t [(t-s)f(t,x_m(s),\dot{x}_m(s),y_m(s))](t-s)^{\alpha-1}ds$$

.. (9) with

$$x_0 = A + \frac{(t-a)(B-A)}{(b-a)}, \dot{x}_0(t) = \frac{(B-A)}{(b-a)}, \quad m = 0,1,2,\ldots$$

converges uniformly on the domain:

$$(t,x_0,\dot{x}_0,y_0) \in [0,T]\times G_\alpha \times G_{1\alpha} \times G_{2\alpha} \quad \ldots (10)$$

to the limit function $x_\infty(t,x_0,\dot{x}_0,y_0)$ which is satisfying the integral equation:

$$x(t,x_0,\dot{x}_0,y_0) = \frac{[A+(t-a)\dot{x}(a)]t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_a^t [(t-s)f(t,x(s),\dot{x}(s),y(s))](t-s)^{\alpha-1}ds$$

… (11) provided that:

$$\|x_\infty(t,x_0,\dot{x}_0,y_0) - x_0\| \le \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)}M \quad \ldots (12)$$

$$\|\dot{x}_\infty(t,x_0,\dot{x}_0,y_0) - \dot{x}_0\| \le \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)}M \quad \ldots (13)$$

and

$$\begin{pmatrix} \|x_\infty(t,x_0,\dot{x}_0,y_0) - x_m(t,x_0,\dot{x}_0,y_0)\| \\ \|\dot{x}_\infty(t,x_0,\dot{x}_0,y_0) - \dot{x}_m(t,x_0,\dot{x}_0,y_0)\| \end{pmatrix} \quad \ldots (14)$$

$$\le H_{0\alpha}^m (E - H_{0\alpha})^{-1}\Psi_{0\alpha}$$

for $t\in[0,T]$, $x_0 \in G_{\alpha f}$, $\dot{x}_0 \in G_{1\alpha f}$, $y_0 \in G_{2\alpha f}$

where

$$H_{0\alpha} = \begin{pmatrix} \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)}q_1 & \frac{(T-a)t^{\alpha-1}}{\Gamma(\alpha)} + \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)}q_2 \\ \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)}q_1 & \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)}q_2 \end{pmatrix},$$

$$\Psi_{0\alpha} = \begin{pmatrix} \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)}M \\ \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)}M \end{pmatrix},$$

$$q_1 = K_1 + K_3\frac{\delta}{\gamma}L_1 \ , \ q_2 = K_2 + K_3\frac{\delta}{\gamma}L_2 .$$

Where E is identity matrix.

Proof:

Set m=0 and use (9), we get:

$$\|x(t,x_0,\dot{x}_0,y_0) - x_0\| = \left\| \frac{[A+(t-a)\dot{x}_0(a)]t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)f(s,x_0,\dot{x}_0,y_0)(t-s)^{\alpha-1}ds - \frac{[A+(t-a)\dot{x}_0(a)]t^{\alpha-1}}{\Gamma(\alpha)} \right\|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_a^t (T-a)\|f(s,x_0,\dot{x}_0,y_0)\|(t-s)^{\alpha-1}ds \le (T-a)\frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)}M \le \frac{1}{\Gamma(\alpha)}\int_a^t \|(t-s)f(s,x_0,\dot{x}_0,y_0)(t-s)^{\alpha-1}\|ds$$

$$\le (T-a)\frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)}M \quad \|x_1(t,x_0,\dot{x}_0,y_0) - x_0\| \le \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)}M \quad \ldots (15)$$

Moreover on differentiating $x_1(t,x_0,\dot{x}_0,y_0)$, we find:

$$\|\dot{x}_1(t,x_0,\dot{x}_0,y_0) - \dot{x}_0\| =$$

$$\left\| \frac{\dot{x}_0 t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_a^t f(s,x_0,\dot{x}_0,y_0)(t-s)^{\alpha-1}ds - \frac{\dot{x}_0 t^{\alpha-1}}{\Gamma(\alpha)} \right\|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_a^t M(t-s)^{\alpha-1}ds = \frac{1}{\Gamma(\alpha)}M(t-s)^{\alpha}\Big|_a^t = \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)}M$$

so that

$$\|\dot{x}_1(t,x_0,\dot{x}_0,y_0) - \dot{x}_0\| \le \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)}M \quad \ldots (16)$$

So that from (6), (15) and (16), we find

$$\|y_1(s,x_0,\dot{x}_0) - y_0(s)\| = \left\| \int_{-\infty}^t w(t,s)g(s,x_1(s),\dot{x}_1(s))ds - \int_{-\infty}^t w(t,s)g(s,x_0,\dot{x}_0)ds \right\|$$

$$\le \int_{-\infty}^t \|w(t,s)\|\|g(s,x_1(s),\dot{x}_1(s)) - g(s,x_0,\dot{x}_0)\|ds$$

$$\le \int_{-\infty}^t \delta e^{-\gamma(t-s)}[L_1\|x_1(s)-x_0\| + L_2\|\dot{x}_1(s)-\dot{x}_0\|]ds$$

so

$$\|y_1(s,x_0,\dot{x}_0) - y_0(s)\| \le \frac{\delta}{\gamma} \left[L_1\frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)} + L_2\frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)}\right]M \quad \ldots (17)$$

from (15), (16), (17) and the condition (8), we get $x_1(t,x_0,\dot{x}_0,y_0) \in G_\alpha$, $\dot{x}_1(t,x_0,\dot{x}_0,y_0) \in G_{1\alpha}$ for all $t\in[0,T]$, $x_0 \in G_{\alpha f}$, $\dot{x}_0 \in G_{1\alpha f}$, $y_0(s) = \int_{-\infty}^t w(t,s)g(s,x_0,\dot{x}_0)ds \in G_{2\alpha f}$.

Suppose that $x_{m-1}(t,x_0,,\dot{x}_0,y_0) \in G_\alpha$, $\dot{x}_{m-1}(t,x_0,\dot{x}_0,y_0) \in G_{1\alpha}$, we have

$$\|x_m(t,x_0,\dot{x}_0,y_0) - x_0\| \le \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)}M$$

$$\|\dot{x}_m(t,x_0,\dot{x}_0,y_0) - \dot{x}_0\| \le \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)}M$$

$$\|y_m(s,x_0,\dot{x}_0) - y_0(s)\| \le \frac{\delta}{\gamma}\left[L_1\frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)} + L_2\frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)}\right]M$$

where $x_1(t,x_0,\dot{x}_0,y_0) \in G_\alpha$, $\dot{x}_1(t,x_0,\dot{x}_0,y_0) \in G_{1\alpha}$ for all $t\in[0,T]$, $x_0 \in G_{\alpha f}$, $\dot{x}_0 \in G_{1\alpha f}$, $y_0(s) \in G_{2\alpha f}$.

We prove now that the sequence (9) is uniformly convergent in (10). From (9), when m=1 we get:

$$\left\| x_2(t,x_0,\dot{x}_0,y_0) - x_1(t,x_0,\dot{x}_0,y_0) \right\| =$$

$$\left\| \frac{[A+(t-a)\dot{x}_1(a)]t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)\, f(s,x_1(s),\dot{x}_1(s),y_1(s))(t-s)^{\alpha-1}ds - \frac{[A+(t-a)\dot{x}_0(a)]t^{\alpha-1}}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)\, f(s,x_0,\dot{x}_0,y_0)(t-s)^{\alpha-1}ds \right\|$$

$$\leq \frac{(t-a)t^{\alpha-1}}{\Gamma(\alpha)}\left\|\dot{x}_1(a)-\dot{x}_0\right\| + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)\left\| f(s,x_1(s),\dot{x}_1(s),y_1(s)) - f(s,x_0,\dot{x}_0,y_0)\right\| (t-s)^{\alpha-1}ds$$

$$\leq \frac{(t-a)t^{\alpha-1}}{\Gamma(\alpha)}\left\|\dot{x}_1(a)-\dot{x}_0\right\| + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)\left[K_1\|x_1(s)-x_0\| + K_2\|\dot{x}_1(s)-\dot{x}_0\| + K_3\|y_1(s)-y_0\|\right](t-s)^{\alpha-1}ds$$

$$\leq \frac{(t-a)t^{\alpha-1}}{\Gamma(\alpha)}\left\|\dot{x}_1(a)-\dot{x}_0\right\| + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)\left[K_1\|x_1(s)-x_0\| + K_2\|\dot{x}_1(s)-\dot{x}_0\| + K_3\frac{\delta}{\gamma}\left(L_1\|x_1(s)-x_0\| + L_2\|\dot{x}_1(s)-\dot{x}_0\|\right)\right](t-s)^{\alpha-1}ds$$

$$\leq \frac{(t-a)t^{\alpha-1}}{\Gamma(\alpha)}\left\|\dot{x}_1(t)-\dot{x}_0\right\| + \frac{(t-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left[\left(K_1+K_3\frac{\delta}{\gamma}L_1\right)\|x_1(t)-x_0\| + \left(K_2+K_3\frac{\delta}{\gamma}L_2\right)\|\dot{x}_1(t)-\dot{x}_0\|\right]$$

$$= \frac{(t-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(K_1+K_3\frac{\delta}{\gamma}L_1\right)\|x_1(t)-x_0\| + \left[\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(t-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(K_2+K_3\frac{\delta}{\gamma}L_2\right)\right]\|\dot{x}_1(t)-\dot{x}_0\|$$

therefore

$$\left\| x_2(t,x_0,\dot{x}_0,y_0) - x_1(t,x_0,\dot{x}_0,y_0) \right\| \leq \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(K_1+K_3\frac{\delta}{\gamma}L_1\right)_1\|x_1(t)-x_0\| + \left[\frac{(T-a)^{\alpha+1}}{\Gamma(\alpha)} + \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(K_2+K_3\frac{\delta}{\gamma}L_2\right)\right]\|\dot{x}_1(t)-\dot{x}_0\|$$

and

$$\left\|\dot{x}_2(t,x_0,\dot{x}_0,y_0) - \dot{x}_1(t,x_0,\dot{x}_0,y_0)\right\| = \left\| \frac{\dot{x}_0 t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_a^t f(s,x_1(s),\dot{x}_1(s),y_1(s))(t-s)^{\alpha-1}ds - \frac{\dot{x}_0 t^{\alpha-1}}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha)}\int_a^t f(s,x_0,\dot{x}_0,y_0)(t-s)^{\alpha-1}ds \right\|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_a^t \left\| f(s,x_1(s),\dot{x}_1(s),y_1(s)) - f(s,x_0,\dot{x}_0,y_0)\right\|(t-s)^{\alpha-1}ds$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_a^t \left[K_1\|x_1(s)-x_0\| + K_2\|\dot{x}_1(s)-\dot{x}_0\| + K_3\|y_1(s)-y_0\|\right](t-s)^{\alpha-1}ds$$

$$\leq \frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)}\left[\left(K_1+K_3\frac{\delta}{\gamma}L_1\right)\|x_1(t)-x_0\| + \left(K_2+K_3\frac{\delta}{\gamma}L_2\right)\|\dot{x}_1(t)-\dot{x}_0\|\right]$$

then

$$\left\|\dot{x}_2(t,x_0,\dot{x}_0,y_0) - \dot{x}_1(t,x_0,\dot{x}_0,y_0)\right\|$$

$$\leq \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)}\left[\left(K_1+K_3\frac{\delta}{\gamma}L_1\right)\|x_1(t)-x_0\| + \left(K_2+K_3\frac{\delta}{\gamma}L_2\right)\|\dot{x}_1(t)-\dot{x}_0\|\right]$$

$$\left\| y_2(s,x_0,\dot{x}_0) - y_1(s,x_0,\dot{x}_0) \right\| =$$

$$\left\| \int_{-\infty}^t w(t,s)g(s,x_2(s),\dot{x}_2(s))ds - \int_{-\infty}^t w(t,s)g(s,x_1(s),\dot{x}_1(s))ds \right\|$$

$$\leq \int_{-\infty}^t \|w(t,s)\|\left\|g(s,x_2(s),\dot{x}_2(s)) - g(s,x_1(s),\dot{x}_1(s))\right\|ds$$

so

$$\left\| y_2(s,x_0,\dot{x}_0) - y_1(s,x_0,\dot{x}_0) \right\|$$

$$\leq \frac{\delta}{\gamma}\left[L_1\|x_2(t)-x_1(t)\| + L_2\|\dot{x}_2(t)-\dot{x}_1(t)\|\right]$$

Now when m=2 in (9) we get the following:

$$\left\| x_3(t,x_0,\dot{x}_0,y_0) - x_2(t,x_0,\dot{x}_0,y_0) \right\| =$$

$$\left\| \frac{[A+(t-a)\dot{x}_2(a)]t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)\, f(s,x_2(s),\dot{x}_2(s),y_2(s))(t-s)^{\alpha-1}ds - \frac{[A+(t-a)\dot{x}_2(a)]t^{\alpha-1}}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)\, f(s,x_1(s),\dot{x}_1(s),y_1(s))(t-s)^{\alpha-1}ds \right\|$$

$$\leq \frac{(t-a)t^{\alpha-1}}{\Gamma(\alpha)}\left\|\dot{x}_2(a)-\dot{x}_1(a)\right\| + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)\left\| f(s,x_2(s),\dot{x}_2(s),y_2(s)) - f(s,x_1(s),\dot{x}_1(s),y_1(s))\right\| (t-s)^{\alpha-1}ds$$

$$\leq \frac{(t-a)t^{\alpha-1}}{\Gamma(\alpha)}\left\|\dot{x}_2(a)-\dot{x}_1(a)\right\| + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)\left[K_1\|x_2(s)-x_1(s)\| + K_2\|\dot{x}_2(s)-\dot{x}_1(s)\| + K_3\|y_2(s)-y_1(s)\|\right]$$

$$\leq \frac{(t-a)t^{\alpha-1}}{\Gamma(\alpha)}\left\|\dot{x}_2(a)-\dot{x}_1(a)\right\| + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)\left[K_1\|x_2(s)-x_1(s)\| + K_2\|\dot{x}_2(s)-\dot{x}_1(s)\| + K_3\frac{\delta}{\gamma}\left(L_1\|x_2(s)-x_1(s)\| + L_2\|\dot{x}_2(s)-\dot{x}_1(s)\|\right)\right](t-s)^{\alpha-1}ds$$

$$\leq \frac{(t-a)t^{\alpha-1}}{\Gamma(\alpha)}\left\|\dot{x}_2(t)-\dot{x}_1(t)\right\| + \frac{(t-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left[\left(K_1+K_3\frac{\delta}{\gamma}L_1\right)\|x_2(t)-x_1(t)\| + \left(K_2+K_3\frac{\delta}{\gamma}L_2\right)\|\dot{x}_2(t)-\dot{x}_1(t)\|\right]$$

$$= \frac{(t-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(K_1+K_3\frac{\delta}{\gamma}L_1\right)\|x_2(t)-x_1(t)\| + \left[\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(t-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(K_2+K_3\frac{\delta}{\gamma}L_2\right)\right]\|\dot{x}_2(t)-\dot{x}_1(t)\|$$

therefore

$$\left\| x_3(t,x_0,\dot{x}_0,y_0) - x_2(t,x_0,\dot{x}_0,y_0) \right\| \leq \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(K_1+K_3\frac{\delta}{\gamma}L_1\right)\|x_2(t)-x_1(t)\| + \left[\frac{(T-a)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(K_2+K_3\frac{\delta}{\gamma}L_2\right)\right]\|\dot{x}_2(t)-\dot{x}_1(t)\|$$

and

$$\left\|\dot{x}_3(t,x_0,\dot{x}_0,y_0) - \dot{x}_2(t,x_0,\dot{x}_0,y_0)\right\| = \left\| \frac{\dot{x}_0 t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_a^t f(s,x_2(s),\dot{x}_2(s),y_2(s))(t-s)^{\alpha-1}ds - \frac{\dot{x}_0 t^{\alpha-1}}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha)}\int_a^t f(s,x_1(s),\dot{x}_1(s),y_1(s))(t-s)^{\alpha-1}ds \right\|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_a^t \left\| f\left(s, x_2(s), \dot{x}_2(s), y_2(s)\right) - f\left(s, x_1(s), \dot{x}_1(s), y_1(s)\right)\right\| (t-s)^{\alpha-1} ds$$

$$\le \frac{1}{\Gamma(\alpha)}\int_a^t \left[K_1\|x_2(s)-x_1(s)\| + K_2\|\dot{x}_2(s)-\dot{x}_1(s)\| + K_3\|y_2(s)-y_1(s)\|\right](t-s)^{\alpha-1} ds$$

$$= \frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)}\left[\left(K_1 + K_3\frac{\delta}{\gamma}L_1\right)\|x_2(t)-x_1(t)\| + \left(K_2 + K_3\frac{\delta}{\gamma}L_2\right)\|\dot{x}_2(t)-\dot{x}_1(t)\|\right]$$

then

$$\|\dot{x}_3(t,x_0,\dot{x}_0,y_0) - \dot{x}_2(t,x_0,\dot{x}_0,y_0)\| \le \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)}\left[\left(K_1 + K_3\frac{\delta}{\gamma}L_1\right)\|x_2(t)-x_1(t)\| + \left(K_2 + K_3\frac{\delta}{\gamma}L_2\right)\|\dot{x}_2(t)-\dot{x}_1(t)\|\right]$$

$$\|y_3(s,x_0,\dot{x}_0) - y_2(s,x_0,\dot{x}_0)\| = \left\|\int_{-\infty}^t w(t,s)g(s,x_3(s),\dot{x}_3(s))ds - \int_{-\infty}^t w(t,s)g(s,x_2(s),\dot{x}_2(s))ds\right\|$$

$$\le \int_{-\infty}^t \|w(t,s)\|\|g(s,x_3(s),\dot{x}_3(s)) - g(s,x_2(s),\dot{x}_2(s))\| ds$$

so

$$\|y_3(s,x_0,\dot{x}_0) - y_2(s,x_0,\dot{x}_0)\| \le \frac{\delta}{\gamma}$$

$$\left[L_1\|x_3(t)-x_2(t)\| + L_2\|\dot{x}_3(t)-\dot{x}_2(t)\|\right]$$

By induction we have:

$$\|x_{m+1}(t,x_0,\dot{x}_0,y_0) - x_m(t,x_0,\dot{x}_0,y_0)\|$$

$$\le \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)} q_1 \|x_m(t) - x_{m-1}(t)\| +$$

$$+\left[\frac{(T-a)^{\alpha-1}}{\Gamma(\alpha 1)} + \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)} q_2\right]\|\dot{x}_m(t) - \dot{x}_{m-1}(t)\| \dots \quad (18)$$

$$\|\dot{x}_{m+1}(t,x_0,\dot{x}_0,y_0) - \dot{x}_m(t,x_0,\dot{x}_0,y_0)\|$$

$$\le \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)} q_1 \|x_m(t) - x_{m-1}(t)\| \quad \dots (19)$$

$$+ \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)} q_2 \|\dot{x}_m(t) - \dot{x}_{m-1}(t)\|$$

Rewrite inequalities (18) and (19) in vector from:

$$\Psi_{m+1}(t,x_0,\dot{x}_0,y_0) \le H(t)\ \Psi_m(t,x_0,\dot{x}_0,y_0) \quad \dots (20)$$

$$\Psi_{m+1}(t,x_0,\dot{x}_0,y_0) = \begin{pmatrix} \|x_{m+1}(t,x_0,\dot{x}_0,y_0) - x_m(t,x_0,\dot{x}_0,y_0)\| \\ \|\dot{x}_{m+1}(t,x_0,\dot{x}_0,y_0) - \dot{x}_m(t,x_0,\dot{x}_0,y_0)\| \end{pmatrix}$$

$$H(t) = \begin{pmatrix} \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)} q_1 & \frac{(T-a)t^{\alpha-1}}{\Gamma(\alpha)} + \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)} q_2 \\ \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)} q_1 & \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)} q_2 \end{pmatrix}$$

$$\Psi_m(t,x_0,\dot{x}_0,y_0) = \begin{pmatrix} \|x_m(t,x_0,\dot{x}_0,y_0) - x_{m-1}(t,x_0,\dot{x}_0,y_0)\| \\ \|\dot{x}_m(t,x_0,\dot{x}_0,y_0) - \dot{x}_{m-1}(t,x_0,\dot{x}_0,y_0)\| \end{pmatrix}$$

It follows from inequality (20) that:

$$\Psi_{m+1}(t) \le H_{0\alpha}\ \Psi_m(t) \quad \dots\dots (21)$$

where

$$H_{0\alpha} = \max_{t\in[0,T]} H(t)$$

By iterating inequality (21), we have

$$\Psi_{m+1}(t) \le H_{0\alpha}^m\ \Psi_m(t) \quad \dots (22)$$

where

$$\Psi_{0\alpha} = \begin{pmatrix} \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)} M \\ \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)} M \end{pmatrix}$$

this leads to the estimation:

$$\sum_{i=1}^m \Psi_i \le \sum_{i=1}^m H_{0\alpha}^{i-1}\ \Psi_{0\alpha} \quad \dots\dots (23)$$

since the matrix H_0 has eigenvalues:

$$h_{\max}(H_{0\alpha}) = \frac{1}{2}\left[\frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)} q_1 + \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)} q_2 + \sqrt{\left(\frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)} q_1 + \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)} q_2\right)^2 + 4\frac{(T-a)^{\alpha+1}t^{\alpha-1}}{\Gamma(\alpha)\Gamma(\alpha+1)} q_1}\right]$$

then the series (9) is uniformly convergent in (10), i.e.

$$\lim_{m\to\infty}\sum_{i=1}^m H_{0\alpha}^{i-1}\ \Psi_{0\alpha} = \sum_{i=1}^{\infty} H_{0\alpha}^{i-1}\Psi_{0\alpha} = (E - H_{0\alpha})^{-1}\ \Psi_{0\alpha} \quad \dots (24)$$

The limiting relation (24) signifies a uniform convergent of the sequence $x_m(t,x_0,\dot{x}_0,y_0)$, $\dot{x}_m(t,x_0,\dot{x}_0,y_0)$

$$\left.\begin{aligned} \lim_{m\to\infty} x_m(t,x_0,\dot{x}_0,y_0) &= x_\infty(t,x_0,\dot{x}_0,y_0) \\ \lim_{m\to\infty} \dot{x}_m(t,x_0,\dot{x}_0,y_0) &= \dot{x}_\infty(t,x_0,\dot{x}_0,y_0) \end{aligned}\right\} \quad . (25)$$

By inequality (25), the estimation:

$$\begin{pmatrix} \|x_\infty(t,x_0,\dot{x}_0,y_0) - x_m(t,x_0,\dot{x}_0,y_0)\| \\ \|\dot{x}_\infty(t,x_0,\dot{x}_0,y_0) - \dot{x}_m(t,x_0,\dot{x}_0,y_0)\| \end{pmatrix} \dots (26)$$

$$\le H_{0\alpha}^m (E - H_{0\alpha})^{-1}\Psi_{0\alpha}$$

is true for m=1,2,3,…

Thus $x_\infty(t,x_0,\dot{x}_0,y_0)$ is a solution of integro-differential equations (1), (2).

Uniqueness solution

The study of the uniqueness solution of the problem (1), (2) will be introduced by the following:

<u>Theorem 2</u>:

Let all assumptions and conditions of theorem 1 be given then the problem (1), (2) has a unique solution $x = x_\infty(t,x_0,\dot{x}_0,y_0)$ on the domain (10).

Proof:

We have to show to that $x(t,x_0,\dot{x}_0,y_0)$ is a unique solution of problem (1), (2). On the contrary, we suppose that there is at least two different solutions $x(t,x_0,\dot{x}_0,y_0)$ and $\hat{x}(t,x_0,\dot{x}_0,y_0)$ of the problem (1), (2).

From (11) the following inequalities are holds:

$$\|x(t,x_0,\dot{x}_0,y_0) - \hat{x}(t,x_0,\dot{x}_0,y_0)\| \le \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)} q_1\|x(t)-\hat{x}(t)\| \dots \quad (27)$$

$$+\left[\frac{(T-a)t^{\alpha-1}}{\Gamma(\alpha)} + \frac{(T-a)^{\alpha+1}}{\Gamma(\alpha+1)} q_2\right]\|\dot{x}(t) - \hat{\dot{x}}(t)\|$$

on differentiating (27) we should also obtain:

$$\|\dot{x}(t,x_0,\dot{x}_0,y_0) - \hat{\dot{x}}(t,x_0,\dot{x}_0,y_0)\| \le \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)} \quad \dots (28)$$

$$q_1\|x(t)-\hat{x}(t)\| + \frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)} q_2\|\dot{x}(t) - \hat{\dot{x}}(t)\|$$

Inequalities (27) and (28) would lead to the estimation:

$$\begin{pmatrix}\|x(t,x_0,\dot{x}_0,y_0)-\hat{x}(t,x_0,\dot{x}_0,y_0)\| \\ \|\dot{x}(t,x_0,\dot{x}_0,y_0)-\hat{\dot{x}}(t,x_0,\dot{x}_0,y_0)\|\end{pmatrix} \le H_{0\alpha}\begin{pmatrix}\|x(t,x_0,\dot{x}_0,y_0)-\hat{x}(t,x_0,\dot{x}_0,y_0)\| \\ \|\dot{x}(t,x_0,\dot{x}_0,y_0)-\hat{\dot{x}}(t,x_0,\dot{x}_0,y_0)\|\end{pmatrix}$$

By iterating we should find that:

$$\begin{pmatrix}\|x(t,x_0,\dot{x}_0,y_0)-\hat{x}(t,x_0,\dot{x}_0,y_0)\| \\ \|\dot{x}(t,x_0,\dot{x}_0,y_0)-\hat{\dot{x}}(t,x_0,\dot{x}_0,y_0)\|\end{pmatrix} \le H_{0\alpha}^{m}\begin{pmatrix}\|x(t,x_0,\dot{x}_0,y_0)-\hat{x}(t,x_0,\dot{x}_0,y_0)\| \\ \|\dot{x}(t,x_0,\dot{x}_0,y_0)-\hat{\dot{x}}(t,x_0,\dot{x}_0,y_0)\|\end{pmatrix}$$

But $H_{0\alpha}^{m} \to 0$ as $m \to \infty$, hence proceeding in the last inequality to the limit we should obtain the equalities $x(t,x_0,\dot{x}_0,y_0)=\hat{x}(t,x_0,\dot{x}_0,y_0)$ and $\dot{x}(t,x_0,\dot{x}_0,y_0)=\hat{\dot{x}}(t,x_0,\dot{x}_0,y_0)$ which proves the solution is a unique and this completes the proof of the theorem.

Stability solution

The study of the stability solution of the problem (1), will be introduced by the following theorem:

Theorem 3:

If the inequalities (4), (5) and (6), were satisfied, and $z(t,x_0,y_0)$, which was defined bellow as different solutions for the equation (1), then the solution was stabile if satisfy the inequality:

$$|x_0(0,x_0,y_0)-z(0,x_0,y_0)|<\delta\ ,\ \delta<0$$

Where

$$x(t,x_0,\dot{x}_0,y_0)=\frac{[A+(t-a)\dot{x}_0]t^{\alpha-1}}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)}\int_a^t[(t-s)f(t,x(s),\dot{x}(s),y(s))](t-s)^{\alpha-1}ds$$

$$z(t,x_0,\dot{x}_0,y_0)=\frac{[A+(t-a)\dot{z}_0]t^{\alpha-1}}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)}\int_a^t[(t-s)f(t,z(s),\dot{z}(s),y(s))](t-s)^{\alpha-1}ds$$

proof:

$$\|x(t,x_0,\dot{x}_0,y_0)-z(t,x_0,\dot{x}_0,y_0)\|=$$

$$\left\|\frac{[A+(t-a)\dot{x}_0]t^{\alpha-1}}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)f(s,x(s),\dot{x}(s),y(s))(t-s)^{\alpha-1}ds-\frac{[A+(t-a)\dot{z}_0]t^{\alpha-1}}{\Gamma(\alpha)}-\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)f(s,z(s),\dot{z}(s),y(s))(t-s)^{\alpha-1}ds\right\|$$

$$\le\frac{(t-a)t^{\alpha-1}}{\Gamma(\alpha)}\|\dot{x}_0-\dot{z}_0\|+\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)\|f(s,x(s),\dot{x}(s),y(s))-f(s,z(s),\dot{z}(s),y(s))\|(t-s)^{\alpha-1}ds$$

$$\le\frac{(t-a)t^{\alpha-1}}{\Gamma(\alpha)}\|\dot{x}_0-\dot{z}_0\|+\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)[K_1\|x(s)-z(s)\|+K_2\|\dot{x}(s)-\dot{z}(s)\|+K_3\|y(s)-y(s)\|](t-s)^{\beta-1}ds$$

$$\le\frac{(t-a)t^{\alpha-1}}{\Gamma(\alpha)}\delta+\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha}[K_1\|x(s)-z(s)\|+K_2\|\dot{x}(s)-\dot{z}(s)\|]ds$$

Let $\frac{(t-a)t^{\alpha-1}}{\Gamma(\alpha)}=\delta_1$, and by the lemma 3, we get:

$$\|x(t,x_0,\dot{x}_0,y_0)-z(t,x_0,\dot{x}_0,y_0)\| =\le \delta_1\exp\left[\frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)}\right]$$

Put $\exp\left[\frac{(T-a)^{\alpha}}{\Gamma(\alpha+1)}\right]=\frac{\varepsilon}{\delta_1}$

$$\|x(t,x_0,\dot{x}_0,y_0)-z(t,x_0,\dot{x}_0,y_0)\|\le\delta_1\frac{\varepsilon}{\delta_1}$$

$$\|x(t,x_0,\dot{x}_0,y_0)-z(t,x_0,\dot{x}_0,y_0)\|\le\varepsilon$$

then the solution was stabile in the given domain.

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وجود ووحدانية الحل لمعادلة تكاملية–تفاضلية لاخطية من الرتبة الكسرية (2α) مع شروط حدودية

غادة شكر جميل

قسم الرياضيات ، كلية التربية ، جامعة الموصل ، الموصل ، العراق

الملخص

يتضمن البحث دراسة وجود ووحدانية الحل لمعادلة تكاملية–تفاضلية لاخطية من الرتبة الكسرية (2α) مع شروط حدودية وذلك باستخدام طريقة بيكارد للتقريب المعطاة في المرجع [3]. إذ استطعنا من خلال هذه الدراسة توسيع بعض النتائج المعطاة في المرجع [2].