On Preliminary Group Classification For A Class Of Nonlinear Wave Equations

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Abstract
We perform group classification of one class of nonlinear wave equations with two independent variables and one dependent variable. It is shown that there are one, four and eighteen equations admitting (invariant) one, two and three dimensional Lie algebras respectively.

1- Introduction
The problem of group classification (determining the arbitrary functions) are known as the group classification problem [3], of differential equations is one of the central problems of modern symmetry analysis of differential equations [5]. Many papers on this problem of such equations have been published:

$u_t = u_{xx} + F(t,x,u,u_x)$ [9] & [12]

$u_t = u_{xxx} + F(t,x,u,u_x,u_{xx})$ [5]

$u_{tt} = u_{xx} + F(t,x,u,u_x)$ [10]

$u_t = F(t,x,u,u_x)u_{xx} + G(t,u,u_x)$ [1]

$u_t = F(t,x,u,u_x,u_{xx})u_{xxx} + G(t,x,u,u_x,u_{xx})$ [2]

$u_t = F(t,x,u,u_x)u_{xx} + G(t,x,u,u_x)$ [1]

$i\psi_t = \psi_{xx} + F(t,x,\psi,\psi^*,\psi_x,\psi_x^*)$ [13]

$u_{tx} = g(t,x)u_x + f(t,x,u), g_x \neq 0, f_{uu} \neq 0$ [10]

$u_{tx} = f(t,x,u)$, $f_{uu} \neq 0$ [10]

$u_{tt} = -\lambda u_{xx} + F(u,u_x)$ [7]

In this article, we consider a class of nonlinear wave equation in the form of

$u_{tt} = F(x,u_x)u_{xx} + G(x,u_x)$.

The approach that will be used in the present article is that presented in [12] , being a synthesis of the standard Lie algorithm for finding symmetries and the use of canonical forms for partial differential generators obtained with the equivalence group of the equation at hand. The following notation will be used through out this paper $A_{k.i} = < Q^1, Q^2, ..., Q^k >$, denotes a Lie algebra (a vector space on which there is an additional structure, called a commutator which has the properties of bilinearity, anticommutativity and Jacobi identity). of dimension k, $Q^j (j = 1, 2, ..., k)$ are its basis elements, and the index i denotes the number of the class to which the given Lie algebra belongs [4] .

2- Group Classification
To classify the nonlinear wave equation

$$u_{tt} = F(x,u_x)u_{xx} + G(x,u_x) \quad (1)$$

that admits Lie algebras of dimension upto three. We start from equation admitting one–dimensional Lie algebras, then extending these Lie algebras to describe the admitted of two–, three–dimensional Lie algebras.

2.1 The Most General Infinitesimal Generator
The first step of group classification of partial differential equation (1) is to find the general form of the infinitesimal generator of the Lie group admitted (invariant) by (1), which is according to the Lie algorithm [3] & [4] is of the form:

$$Q = \tau(t,x,u)\frac{\partial}{\partial t} + \zeta(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \quad (2)$$

where t, x are independent variables and $u = (t,x)$ is the dependent variable. Note that τ, ζ, η are real-valued smooth functions. The criterion condition for equation (1) [4] & [6] to be invariant with respect to (2) reads as:

$$\eta_{(tt)} - (\zeta F_x + \eta_{(x)} F_{u_x})u_{xx} - \eta_{(xx)}F - \zeta G_x - \eta_{(x)}G_{u_x} = 0, \quad (3)$$

2.2 Theorem 1:
The infinitesimal generator of the symmetry group of the equation (1), has the following form:

$$Q = a(t)\frac{\partial}{\partial t} + b(x)\frac{\partial}{\partial x} + f(t,x,u)\frac{\partial}{\partial u}, \ldots(4)$$

where $f = \left(\frac{a^{\cdot}}{2} + c(x)\right)u + d(t,x)$;

$a(t), b(x), c(x)$ and $d(t, x)$ are arbitrary smooth functions which satisfy the following classifying equations :

$$f_{tt} + (f_u - 2a^{\cdot})G - bG_x + [(b' - f_u)u_x - f_x]G_{u_x} + [(b'' - 2f_{xu})u_x - f_{xx}]F = 0$$

$$2(b' - \dot{a})F - bF_x + [(b' - f_u)u_x - f_x]F_{u_x} = 0 \quad (5)$$

2.3 The Equivalence Group
There are two different ways for construction the equivalence group, the direct method and the infinitesimal method [6]. But we will use the first one because it gives us enough information about the Jacobi conditions. In order to construct the equivalence group of the class of partial differential equations (1), one has to select from the set of invertable changes of variables of the vector space [6]:

$$\bar{t} = T(t,x,u), \ \bar{x} = X(t,x,u), \ \bar{u} = U(t,x,u), \quad (6)$$

$$where \quad \frac{D(T,X,U)}{D(t,x,u)} = \begin{vmatrix} T_t & T_x & T_u \\ X_t & X_x & X_u \\ U_t & U_x & U_u \end{vmatrix} \neq 0, \quad (7)$$

be those changes of variables which don't alter the form of the class of partial differential equations (1).

2.4 Theorem 2:
The equivalence group of the class of partial differential equations (1) reads as

$\bar{t} = T(t), \bar{x} = X(x), \bar{u} = U(t,x,u),$ (8 a) where $\frac{dT}{dt}=\dot{T}\neq 0$, $\frac{dX}{dx}=X'\neq 0$ and $U_u\neq 0$,

$$\bar{F}(\bar{x},\bar{u}_{\bar{x}})=F(x,u_x)\left((\frac{X'}{\dot{T}})^2\right) \quad (8\text{ b})$$

$$\bar{G}(\bar{x},\bar{u}_{\bar{x}})=G(x,u_x)\frac{U_u}{\dot{T}^2}. \quad (8\text{ c})$$

In order to obtain the functions $\bar{F}(\bar{x},\bar{u}_{\bar{x}})$ and $\bar{G}(\bar{x},\bar{u}_{\bar{x}})$, it suffices to express t, x and u via $\bar{t}, \bar{x}$ and $\bar{u}$ from the equations (8 a)and substitute in (8 b,c).

2.5 Group Classification of the Equation (1)

Here we classify equations of the form (1) that admit symmetry Lie algebras of dimensions one–, two–and three. We start from describing equations admitting one–dimensional Lie algebras, and then proceed to the investigation of the ones invariant with respect to two–and three–dimensional Lie algebras. An intermediate problem which is being solved, while classifying invariant equations of the form (1), is describing all possible realizations of one–,two–and three–dimensional Lie algebras by infinitesimal generators (4) within the equivalence relation .

2.5.1 One-dimensional lie algebras

All inequivalent partial differential equations (1) admitting one–dimensional symmetry Lie algebras having the basis elements of the form (4) are given by the following theorem:

2.5.2 Theorem 3:

There are equivalence transformations (8a) that reduce infinitesimal generator (4) to one of the following generators

$$\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial u} \text{ and } \frac{\partial}{\partial t}+\frac{\partial}{\partial x}. \quad (9)$$

Proof: see [8]

It is clear that the generators $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}+\frac{\partial}{\partial x}.$ cannot be transformed to each other. From these generators we hence can construct four inequivalent one-dimensional Lie algebras namely A_1^1, A_1^2, A_1^3 and A_1^4 having the basis elements of the form (9) (see the appendix).

2.5.3 Nonlinear Wave Equations Invariant under One- Dimensional Lie Algebras

The corresponding invariant equations for each of the Lie algebras A_1^1, A_1^2, A_1^3 & A_1^4 from the class (1) are obtained by inserting the coefficient of these Lie algebras in the classifying equations (5), and then one can solve them for the arbitrary elements F and G. For example, when $Q=\frac{\partial}{\partial x}$ this means that, $b=1$ and $a=f=0$. Substituting in (5) gives that $F_x=G_x=0$, that is, the following class of partial differential equations

$$u_{tt} = F(u_x)u_{xx}+G(u_x),$$

admit the one – dimensional Lie algebra $<\frac{\partial}{\partial x}>$ as well as $<\frac{\partial}{\partial t}+\frac{\partial}{\partial x}>$.While all the equations in the class admit the one–dimensional Lie algebra $<\frac{\partial}{\partial t}>$ and $<\frac{\partial}{\partial u}>$ and this is clear because the variables t and u don't appear explicitly in the equation (1).

Thus, the class is classified into two inequivalent subclasses, which yield that the corresponding invariant equations from class (1) have the form.

$$A_1^2=<\frac{\partial}{\partial x}>: u_{tt} = F(u_x)u_{xx}+G(u_x)$$

2.5.4 Two-Dimensional Lie Algebras

As it is well known, there are abstract two–dimensional Lie algebras [1] namely, the commutative Lie algebras $A_{2.1}=<Q^1, Q^2>, [Q^1, Q^2]=0$ and the solvable one

$$A_{2.2}=<Q^1, Q^2>, [Q^1, Q^2]=Q^2 .$$

So the problem of describing partial differential equations (1) admitting two–dimensional Lie symmetry algebras contains as a subproblem the one of solving the commutation relations above within the class of infinitesimal generator (4) up to the equivalence relation (8a).

Next, one should solve (4) for each realization obtained. Having done this the following theorem is obtained.

2.5.5 Theorem 4:

The list of inequivalent realizations of two–dimensional Lie algebras with the infinitesimal generator (4) and defined within the equivalence transformation (8a) are the Lie algebras $A_{2.1}^1, A_{2.1}^2,..., A_{2.1}^9; A_{2.2}^1, A_{2.2}^2,..., A_{2.2}^{11}$ (see the appendix).

Proof: [8]

Now we drive all nonlinear wave equation (1), that admit two–dimensional Lie algebras as symmetry Lie algebra. Doing this we have to insert the coefficients of the obtained realizations in (5), then solving the later for the arbitrary functions F, G. To this end we have the following nonlinear wave equations corresponding to their realizations

$$A_{2.2}^3: \quad u_{tt} = e^{2x}F(u_x)u_{xx}+e^{2x}G(u_x),$$

$$A_{2.2}^4: \quad u_{tt} = c^1(u_x)^{-2}u_{xx}+c^2,$$

$$A_{2.2}^9: \quad u_{tt} = F(u_x e^x)u_{xx}+u_x G(u_x e^x),$$

$$A_{2.2}^{11}: \quad u_{tt} = c^1(u_x)^{-2}u_{xx}+c^2(u_x)^2 .$$

2.5.6 Three-Dimensional Lie Algebras

The set of abstract three–dimensional Lie algebras are divided into two classes as it was fixed in [1]. The first class contains those Lie algebras which are direct sums of lower dimension ones. The remaining Lie algebras are included into the second class.

The first class of Lie algebras contains two non-isomorphic Lie algebras, namely, $A_{3.1}$, $A_{3.2}$ .what is more,

$A_{3.1}=<Q^1, Q^2, Q^3>, [Q^i, Q^j]=0; (i,j=1,2,3)$, that is, $A_{3.1}=A_1\oplus A_1\oplus A_1=3A_1$, and $A_{3.2}=<Q^1, Q^2, Q^3>$, where $[Q^1, Q^2]=Q^2, [Q^1, Q^3]=[Q^2, Q^3]=0$, that is, $A_{3.2}=A_{2.2}\oplus A_1$.

2.5.6.1 The Lie algebra $A_{3.1}$

For describing all inequivalent realizations of this Lie algebras, we use the results of theorem 4 on classification of inequivalent realizations of the Lie algebras $A_{2.1}$, namely of the realizations, $A^i_{2.1}$, $i=1,2,...,9$.

Let $A_{2.1}= A^1_{2.1}=<\frac{\partial}{\partial t},\frac{\partial}{\partial x}>$, and let Q^1, Q^2, Q^3 of $A_{3.1}=<Q^1, Q^2, Q^3>, [Q^i, Q^j]=0 \quad (i,j=1,2,3)$, to be: $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, a(t)\frac{\partial}{\partial t}+b(x)\frac{\partial}{\partial x}+f(t,x,u)\frac{\partial}{\partial u}$ respectively using the two commutation relations $[Q^1, Q^3]=[Q^2, Q^3]=0$ of $[Q^i, Q^j]$ above, we get that $Q^3=c^1\frac{\partial}{\partial t}+c^2\frac{\partial}{\partial x}+f(u)\frac{\partial}{\partial u}$.

To find the canonical form for Q^3 under the equivalence transformations (8a). However, we must now use only those equivalence transformations (8a) which preserve the form of both Q^1, Q^2. As we know the equivalence transformations $\bar{t}=t+\lambda$, $\bar{x}=X(x)$, $\bar{u}=U(x,u)$ preserves the form of Q^1. Thus we require that $Q^2\rightarrow\bar{Q}^2$ with

$$\bar{Q}^2=X'\frac{\partial}{\partial x}+U_x\frac{\partial}{\partial u}=\frac{\partial}{\partial \bar{x}},$$

which yields that $X'=1, U_x=0$.

Hence we take:

$$\bar{t}=t+\lambda, \quad \bar{x}=x+\lambda^1, \quad \bar{u}=U(u). \quad (10)$$

Under this type of transformations, we find:

$$Q^3\rightarrow\bar{Q}^3=c^1\frac{\partial}{\partial \bar{t}}+c^2\frac{\partial}{\partial \bar{x}}+f(u)U_u\frac{\partial}{\partial \bar{u}}.$$

Taking into account the considerable four cases, where $f\neq0, c^1=c^2=0$; $c^1=0, c^2\neq0, f\neq0$; $c^2=0, c^1\neq0, f\neq0$ *and* $c^1\neq0, c^2\neq0$, $f\neq0$ *in* Q^3 we obtain the realizations $A^1_{3.1}$, $A^2_{3.1}$, $A^3_{3.1}$ & $A^4_{3.1}$ (see the appendix).

Applying the above mentioned same procedure to the Lie algebras $A^i_{2.1}$, $i=2,3,...,9$, we will end with the realizations $A^5_{3.1}$, $A^6_{3.1}$, $\ldots$, $A^{21}_{3.1}$ (see the appendix).

2.5.6.2 The Lie algebra $A_{3.2}$

At last, the corresponding nonlinear wave equations (1), that admit three–dimensional Lie algebras $A^i_{3.1}, i=1,2,\ldots,21$ as symmetry Lie algebra are all repeated or are contradicting the condition $F_{u_x}\neq0$ of equation (1) Let us turn now to the analysis of the realizations of the Lie algebra $A_{3.2}=A_{2.2}\oplus A_1$. In order to describe these, we use the realizations $A^i_{2.2}, i=1,2,...,11$; of the two–dimensional Lie algebra $A_{2.2}$ obtained in theorem 4.

Consider first the case when $A_{2.2}=A^1_{2.2}=<-t\frac{\partial}{\partial t},\frac{\partial}{\partial t}>$,

let $Q^1=-t\frac{\partial}{\partial t}, \quad Q^2=\frac{\partial}{\partial t}$, and

$$Q^3=a(t)\frac{\partial}{\partial t}+b(x)\frac{\partial}{\partial x}+f(t,x,u)\frac{\partial}{\partial u}$$

of $A_{3.2}=<Q^1, Q^2, Q^3>, [Q^1, Q^2]=Q^2, [Q^1, Q^3]=[Q^2, Q^3]=0$. Solving the second commutation relations, yields that $a-a^{\cdot}t=0$, $-t f_t=0$ and form the third commutation relations that $a^{\cdot}=0$, $f_t=0$, and hence

$$Q^3=b(x)\frac{\partial}{\partial x}+f(x,u)\frac{\partial}{\partial u},$$

then

$$\bar{Q}^3=bX'\frac{\partial}{\partial \bar{x}}+(bU_x+f(x,u)U_u)\frac{\partial}{\partial \bar{u}},$$

where the equivalence transformations preserve both Q^1, Q^2 are

$\bar{t}=t$, $\bar{x}=X(x)$, $\bar{u}=U(x,u)$ (11)

There are two cases to be considerd, namely, $b=0, f\neq0$; $b\neq0, f=0$, and thus, we get the realizations $A^1_{3.2}$ & $A^2_{3.2}$ (see the appendix).

Applying the same approach for the remaining two–dimensional Lie algebras, we get the three–dimensional realizations $A^3_{3.2}, A^4_{3.2}, \ldots, A^{30}_{3.2}$ (see the appendix). The corresponding nonlinear wave equations (1) that admit the above three–dimensional Lie algebras $A^1_{3.2}, A^2_{3.2}, \ldots, A^{30}_{3.2}$ as a symmetry Lie algebra are all repeated or are leading to contradiction.

Now we turn to those three–dimensional real Lie algebras $A_3=<Q^1, Q^2, Q^3>$ that cannot be decomposed into a direct sum of lower dimensional Lie algebras. The list of these Lie algebras is given in 2.5.6. While constructing inequivalent realization of these Lie algebras within the class of infinitesimal generators (4), one can use whenever possible the classification results obtained for the lower dimensional Lie algebras.

2.5.6.3 The semi-simple Lie algebra $A_{3.3}$

Consider first the semi-simple Lie algebras

$$A_3=A_{3.3}, \quad A_{3.3}:[Q^1,Q^3]=-2Q^2, \quad [Q^1,Q^2]=Q^1, \quad [Q^2,Q^3]=Q^3$$

Since $[Q^1,Q^2]=Q^1$, one forms a basis of two–dimensional Lie algebra isomorphic to $A_{2.2}$. So choosing $Q^1=Q^{2'}$, $Q^2=-Q^{1'}$, we see that $[Q^{1'},Q^{2'}]=-[Q^2,Q^1]=Q^1=Q^{2'}$. Thus one can use the results on classification of the Lie algebra $A_{2.2}$. According to those results studying the realization of the Lie algebra $A_{3.3}$ reduces to finding the form of the infinitesimal generators Q^3 for each pair of the infinitesimal generators Q^1, Q^2 given below:

$$Q^1=\frac{\partial}{\partial t} \quad , \quad Q^2=t\frac{\partial}{\partial t}\ , \ Q^1=\frac{\partial}{\partial t} \quad , \quad Q^2=t\frac{\partial}{\partial t}-\frac{\partial}{\partial u},$$

$$Q^1=\frac{\partial}{\partial t} \quad , \quad Q^2=t\frac{\partial}{\partial t}-\frac{\partial}{\partial x},$$

$$Q^1=\frac{\partial}{\partial x} \quad , \quad Q^2=x\frac{\partial}{\partial x}\ , \ Q^1=\frac{\partial}{\partial x} \quad , \quad Q^2=x\frac{\partial}{\partial x}-\frac{\partial}{\partial u},$$

$$Q^1=\frac{\partial}{\partial x} \ , \ Q^2=-\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}\ , \ Q^1=\frac{\partial}{\partial u} \quad , \quad Q^2=u\frac{\partial}{\partial u},$$

$$Q^1=\frac{\partial}{\partial u} \ , Q^2=-\frac{\partial}{\partial t}+u\frac{\partial}{\partial u}\ , \ Q^1=\frac{\partial}{\partial u} \ , \ Q^2=-\frac{\partial}{\partial x}+u\frac{\partial}{\partial u},$$

$$Q^1=\frac{\partial}{\partial u} \ , Q^2=-\frac{\partial}{\partial t}-\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}\ , \ Q^1=\frac{\partial}{\partial t}+\frac{\partial}{\partial x} \ , \ Q^2=t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x} \qquad (12)$$

Again one can simplify the form of infinitesimal generator Q^3 by using suitable equivalence transformations.

We consider for example the first pair of (12), where

$$Q^1=\frac{\partial}{\partial t} \ , \ Q^2=t\frac{\partial}{\partial t},$$

we proceed to find the allowable form of Q^3 . So put

$$Q^3=a(t)\frac{\partial}{\partial t}+b(x)\frac{\partial}{\partial x}+f(t,x,u)\frac{\partial}{\partial u} \ .$$

Then it follow from the commutation relations

$[Q^1,Q^3]=-2Q^2 \ , [Q^2,Q^3]=Q^3$,

that $Q^3=-(t)^2\frac{\partial}{\partial t}$.

We used equivalence transformations (11), which preserve both $Q^1=Q^2$. Thus the realization is yielded $A^1_{3.3}$ (see the appendix).

The remaining realizations $A^2_{3.3}, A^3_{3.3}, \dots, A^8_{3.3}$ of the Lie algebra $A_{3.3}$ can be obtained in a similar way(see the appendix).

Again the corresponding nonlinear wave equations that admit the Lie algebra $A_{3.3}$ as symmetry Lie algebra are all repeated or are leading to contradiction.

2.5.6.4 The semi-simple Lie algebra $A_{3.4}$

Turn now to the semi-simple Lie algebra $A_{3.4}$. One can observe that it does not contain a two–dimensional Lie subalgebra $A_{2.1}$ or $A_{2.2}$, so we use the classification results for one–dimensional Lie algebras $A^1_1, A^2_1\ A^3_1,\& A^4_1$.

Given the infinitesimal generator $Q^1=\frac{\partial}{\partial t}$, we verify that there are no infinitesimal generators Q^2, Q^3 of the form (4) satisfying together with Q^1 the commutation relations

$$[Q^1,Q^2]=Q^3 \ , [Q^2,Q^3]=Q^1 \ , [Q^1,Q^3]=Q^2 \ .$$

Consequently, the class of infinitesimal generators (4) does not contain infinitesimal generators Q^2, Q^3 that extend a realization of the one–dimensional Lie algebra $A^1=<Q^1>$ to a realization of the Lie algebra $A_{3.4}$. The same assertion holds true for the remaining realizations of the infinitesimal generator Q^1 . In a nutshell, one can conclude that there is no partial differential equation of the form (1) whose symmetry Lie algebra contains a three–dimensional Lie algebra isomorphic to $A_{3.4}$.

2.5.6.5 The nilpotent Lie algebra $A_{3.5}$

Now turn to the nilpotent Lie algebra $A_{3.5}=<Q^1,Q^2,Q^3>, [Q^1,Q^3]=0, [Q^2,Q^3]=Q^1$, which contains the commuting Lie subalgebra having the basis infinitesimal generators Q^1,Q^2 .Since the latter is isomorphic to the Lie algebra $A_{2.1}$,we can use the results of theorem 4. In view of these we conclude that there are realizations of the Lie algebra $A_{2.1}$, which might be the admitted Lie algebra by the differential equation (1), namely ,

$$A^i_{2.1} \ , \ i=1,2,...,9 \ . \quad (13)$$

Therefore, while considering the Lie algebra $A_{3.5}$, we can suppose that Q^1, Q^2 are given by one of the formulas (13). In order to simplify the form of the infinitesimal generator Q^3, we can use a suitable transformations

Let us take the first pair of (13), where

$$Q^1=\frac{\partial}{\partial t} \ , \ Q^2=\frac{\partial}{\partial x} \ , \text{and}$$

$$\text{let } Q^3=a(t)\frac{\partial}{\partial t}+b(x)\frac{\partial}{\partial x}+f(t,x,u)\frac{\partial}{\partial u} \ ,$$

then analyzing the corresponding commutation relations , yields that the class of infinitesimal generators (4) does not contain an infinitesimal generator Q^3 which forms together with Q^1,Q^2 a basis of the Lie algebra $A_{3.5}$ .The same result will be arising, if we provided that $Q^1=\frac{\partial}{\partial x} \ , \ Q^2=\frac{\partial}{\partial t}$.

Treating with the second pair of (13) in the same approach, where $Q^1=\frac{\partial}{\partial t} \ , \ Q^2=\frac{\partial}{\partial u}$, we arrive to the same result of the first pair. But, if we put in second pair $Q^1=\frac{\partial}{\partial u} \ , \ Q^2=\frac{\partial}{\partial t}$, then analyzing the commutation relations, we get that:

$$Q^3=c^1\frac{\partial}{\partial t}+b(x)\frac{\partial}{\partial x}+(t+f(x))\frac{\partial}{\partial u} \ .$$

Next we transformation Q^3 by the equivalence transformations

$$\bar{t}=t+\lambda \quad , \bar{x}=X(x) \quad , \bar{u}=u+U(x) \quad ,$$

which preserves both Q^1, Q^2 , we will be left with

$$\bar{Q}^3=c^1\frac{\partial}{\partial \bar{t}}+bX'\frac{\partial}{\partial \bar{x}}+(bU_x+\bar{t}+f(x))\frac{\partial}{\partial \bar{u}} \ .$$

Now, there are six cases to be considered. First, $c^1=0$, $b=f=0$ in Q^3 , this gives us that $\bar{Q}^3=\bar{t}\ \frac{\partial}{\partial \bar{u}}$

Thus we get the realization $A^1_{3.5}$ (see the appendix).

The other five cases, namely, when

$c^1 \neq 0, b=f=0$; $b \neq 0, c^1=f=0$; $f \neq 0, c^1=b=0$; $b=0, c^1 \neq 0, f \neq 0$ and $f=0, c^1 \neq 0, b \neq 0$ gives rise to realizations $A_{3.5}^2, A_{3.5}^3, \ldots, A_{3.5}^6$ (see the appendix). Turn to the remaining pairs of (13), we get the realizations $A_{3.5}^7, A_{3.5}^8, \ldots, A_{3.5}^{17}$ (see the appendix).The corresponding nonlinear wave equations (1) that admit the Lie algebra $A_{3.5}$ as symmetry Lie algebra are all repeated or end with contradiction.

2.5.6.6 The solvable Lie algebras $A_{3.i}$, $i=6, \cdots, 11$

Finally, we consider the solvable Lie algebras $A_{3.6}, A_{3.7}, A_{3.8}, A_{3.9}, A_{3.10}, A_{3.11}$. These Lie algebras have a common feature, namely, they contain commutative two–dimensional Lie subalgebras with basis infinitesimal generators Q^1, Q^2 . That is why; analysis of these Lie algebras is similar to that of the Lie algebra $A_{3.5}$.

Realization of these Lie algebras are $A_{3.6}^1, \ldots, A_{3.6}^5$; $A_{3.7}^1, \ldots, A_{3.7}^{15}$; $A_{3.8}^1, \ldots, A_{3.8}^7$; $A_{3.9}^1, \ldots, A_{3.9}^{13}$ (see the appendix).

Note that $A_{3.10}, A_{3.11}$ do not exist, which means that checking the commutation relations yields that there are no infinitesimal generators of forms (4) which enable extending the Lie algebras $A_{2.1}^i$, $i=1,2,....,9$; to the Lie algebras $A_{3.10}, A_{3.11}$.

2.5.6.7 The list of nonlinear wave equations

At last, we list the nonlinear wave equations (1) that admit the Lie algebras $A_{3.6}$, $A_{3.7}, A_{3.8}, A_{3.9}$ as symmetry Lie algebras:

$$A_{3.6}^1: u_{tt} = (u_x)^{-2} F(x) u_{xx} + (u_x)^{-1} G(x) \quad ,$$

$$A_{3.6}^2: u_{tt} = (u_x)^{-2} F^{-1}(u_x e^{-x}) u_{xx} + (u_x)^{-1} G^{-1}(u_x e^{-x}) \quad ,$$

$$A_{3.6}^3: u_{tt} = c^1 e^{-2u_x} u_{xx} + c^2 e^{u_x} \quad ,$$

$$A_{3.7}^{13}: u_{tt} = F(u_x e^{-x}) u_{xx} + (u_x)^2 e^{-x} \quad ,$$

$$A_{3.8}^1: u_{tt} = c^1 (u_x)^{-4} u_{xx} + c^2 (u_x)^{-2} \quad ,$$

$$A_{3.8}^3: u_{tt} = (u_x)^2 F(x) u_{xx} + (u_x)^3 G(x) \quad ,$$

$$A_{3.8}^4: u_{tt} = (u_x)^2 \left(F(u_x e^x)\right)^2 u_{xx} + (u_x)^3 \left(G(u_x e^x)\right)^3 \quad ,$$

$$A_{3.8}^5: u_{tt} = c^1 u_x u_{xx} + c^2 (u_x)^{\frac{1}{2}} \,,$$

$$A_{3.8}^7: u_{tt} = F(u_x) u_{xx} \quad ,$$

$$A_{3.9}^1: u_{tt} = c^1 (u_x)^{-\frac{2(q-1)}{q}} u_{xx} + c^2 (u_x)^{\frac{2}{q}} \,, \quad 0<|q|<1,$$

$$A_{3.9}^2: u_{tt} = c^1 (u_x)^{2(1-q)} u_{xx} + c^2 (u_x)^{2q} \,, \quad 0<|q|<1,$$

$$A_{3.9}^5: u_{tt} = (u_x)^{-\frac{2}{q}} F(x) u_{xx} + (u_x)^{\frac{q-2}{q}} G(x) \quad , 0<|q|<1,$$

$$A_{3.9}^6: u_{tt} = (u_x)^{-2q} F(x) u_{xx} + (u_x)^{1-2q} G(x) \quad , 0<|q|<1,$$

$$A_{3.9}^7: u_{tt} = (u_x)^{-\frac{2}{q}} \left(F\left((u_x)^{\frac{1}{q}} e^{-x}\right)\right)^{-2} u_{xx} + (u_x)^{\frac{q-2}{q}} \left(G\left((u_x)^{\frac{1}{q}} e^{-x}\right)\right)^{q-2}, 0<|q|<1,$$

$$A_{3.9}^8: u_{tt} = (u_x)^{-2q} \left(F(u_x e^{-x})\right)^{-2q} u_{xx} + (u_x)^{1-2q} \left(G(u_x e^{-x})\right)^{1-2q}, 0<|q|<1,$$

$$A_{3.9}^9: u_{tt} = c^1 (u_x)^{-\frac{2}{1-q}} u_{xx} + c^2 (u_x)^{\frac{q}{1-q}}, 0<|q|<1,$$

$$A_{3.9}^{10}: u_{tt} = c^1 (u_x)^{-\frac{2q}{q-1}} u_{xx} + c^2 (u_x)^{-\frac{1}{q-1}}, 0<|q|<1,$$

$$A_{3.9}^{13}: u_{tt} = c^1 (u_x)^{-2q} u_{xx} + c^2 (u_x)^{1-2q}, 0<|q|<1.$$

From the proceeding section, we are in the position to state the following theorem:

2.5.6.8 Theorem 5:

The list of inequivalent realizations of three dimensional Lie algebras, such that they are symmetries of the nonlinear wave equation (1) are the realizations

$A_{3.1}^1, \ldots, A_{3.1}^{21}$; $A_{3.2}^1, \ldots, A_{3.2}^{30}$; $A_{3.3}^1, \ldots$; $A_{3.3}^8, \ldots,$
$A_{3.5}^1, \ldots, A_{3.5}^{17}$; $A_{3.6}^1, \ldots, A_{3.6}^5$; $A_{3.7}^1, \ldots, A_{3.7}^{15}$; $A_{3.8}^1, \ldots,$
$A_{3.8}^7$ and $A_{3.9}^1, \ldots, A_{3.9}^{13}$.

Conclusion

We have derived the preliminary group classification for the nonlinear wave equation of the form (1). One of the evident conclusions is that the complete group classification **(**description of all possible forms of the functions F, G that (4) admits a non-trivial symmetry group**)** of equation (1) still remain open. We hope to return to it in a forthcoming paper.

However, the full solution of this problem needs more powerful algebraic techniques like Live-Maltsev theorem properties of simple, semi–simple and solvable Lie algebras.

Appendix

List of Realizations

$$A_1^1 = <\frac{\partial}{\partial t}>, \quad A_1^2 = <\frac{\partial}{\partial x}>, \quad A_1^3 = <\frac{\partial}{\partial u}> \quad and \quad A_1^4 = <\frac{\partial}{\partial t} + \frac{\partial}{\partial x}>.$$

$$A_{2.1}^1 = <\frac{\partial}{\partial t}, \frac{\partial}{\partial x}>, \quad A_{2.1}^2 = <\frac{\partial}{\partial t}, \frac{\partial}{\partial u}>, \quad A_{2.1}^3 = <\frac{\partial}{\partial t}, \frac{\partial}{\partial t} + \frac{\partial}{\partial u}>,$$

$$A_{2.1}^4 = <\frac{\partial}{\partial t}, \frac{\partial}{\partial t} + \frac{\partial}{\partial x}>, \quad A_{2.1}^5 = <\frac{\partial}{\partial x}, \frac{\partial}{\partial u}>,$$

$$A_{2.1}^6 = <\frac{\partial}{\partial x}, \frac{\partial}{\partial x} + \frac{\partial}{\partial u}>, \quad A_{2.1}^7 = <\frac{\partial}{\partial x}, \frac{\partial}{\partial t} + \frac{\partial}{\partial x}>,$$

$$A_{2.1}^8 = <\frac{\partial}{\partial u}, g(t,x)\frac{\partial}{\partial u}>, \quad A_{2.1}^9 = <\frac{\partial}{\partial u}, \frac{\partial}{\partial t} + \frac{\partial}{\partial x}>.$$

$$A_{2.2}^1 = <-t\frac{\partial}{\partial t}, \frac{\partial}{\partial t}>, \quad A_{2.2}^2 = <-t\frac{\partial}{\partial t} + \frac{\partial}{\partial u}, \frac{\partial}{\partial t}>, \quad A_{2.2}^3 = <-t\frac{\partial}{\partial t} + \frac{\partial}{\partial x}, \frac{\partial}{\partial t}>,$$

$$A_{2..2}^4 = <-x\frac{\partial}{\partial x}, \frac{\partial}{\partial x}>,$$

$$A_{2..2}^5 = <-x\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \frac{\partial}{\partial x}>, \quad A_{2..2}^6 = <\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}, \frac{\partial}{\partial x}>,$$

$$A_{2..2}^7 = <-u\frac{\partial}{\partial u}, \frac{\partial}{\partial u}>, \quad A_{2..2}^8 = <\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, \frac{\partial}{\partial u}>,$$

$$A_{2..2}^9 = <\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, \frac{\partial}{\partial u}>, \quad A_{2..2}^{10} = <\frac{\partial}{\partial t} + \frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, \frac{\partial}{\partial u}>,$$

$$A_{2..2}^{11} = <-t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}, \frac{\partial}{\partial t} + \frac{\partial}{\partial x}>$$

$$A_{3.1}^1 = <\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial u}>, \quad A_{3.1}^2 = <\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial x} + \frac{\partial}{\partial u}>,$$

$$A_{3.1}^3 = <\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial t} + \frac{\partial}{\partial u}>, \quad A_{3.1}^4 = <\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial u}>.$$

$$A_{3.1}^5 = <\frac{\partial}{\partial t}, \frac{\partial}{\partial u}, g(x)\frac{\partial}{\partial u}>, \quad A_{3.1}^6 = <\frac{\partial}{\partial t}, \frac{\partial}{\partial u}, \frac{\partial}{\partial t} + g(x)\frac{\partial}{\partial u}>,$$

$$A_{3.1}^7 = <\frac{\partial}{\partial t}, \frac{\partial}{\partial u}, \frac{\partial}{\partial t} + \frac{\partial}{\partial x}>, \quad A_{3.1}^9 = <\frac{\partial}{\partial t}, \frac{\partial}{\partial t} + \frac{\partial}{\partial u}, \frac{\partial}{\partial t} + g(x)\frac{\partial}{\partial u}>,$$

$$A_{3.1}^{10}=<\frac{\partial}{\partial t},\frac{\partial}{\partial t}+\frac{\partial}{\partial x},\frac{\partial}{\partial x}+\frac{\partial}{\partial u}> \quad , A_{3.1}^{11}=<\frac{\partial}{\partial t},\frac{\partial}{\partial t}+\frac{\partial}{\partial x},\frac{\partial}{\partial t}+\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.1}^{12}=<\frac{\partial}{\partial t},\frac{\partial}{\partial t}+\frac{\partial}{\partial x},\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.1}^{13}=<\frac{\partial}{\partial x},\frac{\partial}{\partial u},g(t)\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.1}^{14}=<\frac{\partial}{\partial x},\frac{\partial}{\partial u},\frac{\partial}{\partial x}+g(t)\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.1}^{15}=<\frac{\partial}{\partial x},\frac{\partial}{\partial u},\frac{\partial}{\partial t}+\frac{\partial}{\partial x}> \quad , A_{3.1}^{16}=<\frac{\partial}{\partial x},\frac{\partial}{\partial x}+\frac{\partial}{\partial u},g(t)\frac{\partial}{\partial u}> ,$$

$$A_{3.1}^{17}=<\frac{\partial}{\partial x},\frac{\partial}{\partial x}+\frac{\partial}{\partial u},\frac{\partial}{\partial x}+g(t)\frac{\partial}{\partial u}> , A_{3.1}^{18}=<\frac{\partial}{\partial x},\frac{\partial}{\partial t}+\frac{\partial}{\partial x},\frac{\partial}{\partial x}+\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.1}^{19}=<\frac{\partial}{\partial x},\frac{\partial}{\partial t}+\frac{\partial}{\partial x},\frac{\partial}{\partial t}+\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.1}^{20}=<\frac{\partial}{\partial x},\frac{\partial}{\partial t}+\frac{\partial}{\partial x},\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.1}^{21}=<\frac{\partial}{\partial u},\frac{\partial}{\partial t}+\frac{\partial}{\partial x},g(t-x)\frac{\partial}{\partial u}> .$$

$$A_{3.2}^{1}=<-t\frac{\partial}{\partial t},\frac{\partial}{\partial t},\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.2}^{2}=<-t\frac{\partial}{\partial t},\frac{\partial}{\partial t},\frac{\partial}{\partial x}> \quad .$$

$$A_{3.2}^{3}=<-t\frac{\partial}{\partial t}+\frac{\partial}{\partial u},\frac{\partial}{\partial t},g(x)\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.2}^{4}=<-t\frac{\partial}{\partial t}+\frac{\partial}{\partial u},\frac{\partial}{\partial t},\frac{\partial}{\partial x}> \quad ,$$

$$A_{3.2}^{5}=<-t\frac{\partial}{\partial t}+\frac{\partial}{\partial x},\frac{\partial}{\partial t},\frac{\partial}{\partial x}> \quad ,$$

$$A_{3.2}^{6}=<-t\frac{\partial}{\partial t}+\frac{\partial}{\partial x},\frac{\partial}{\partial t},\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.2}^{7}=<-t\frac{\partial}{\partial t}+\frac{\partial}{\partial x},\frac{\partial}{\partial t},\frac{\partial}{\partial x}+\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.2}^{8}=<-x\frac{\partial}{\partial x},\frac{\partial}{\partial x},\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.2}^{9}=<-x\frac{\partial}{\partial x}+\frac{\partial}{\partial u},\frac{\partial}{\partial x},g(t)\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.2}^{10}=<-x\frac{\partial}{\partial x}+\frac{\partial}{\partial u},\frac{\partial}{\partial x},\frac{\partial}{\partial t}> \quad ,$$

$$A_{3.2}^{11}=<\frac{\partial}{\partial t}-x\frac{\partial}{\partial x},\frac{\partial}{\partial x},\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.2}^{12}=<\frac{\partial}{\partial t}-x\frac{\partial}{\partial x},\frac{\partial}{\partial x},\frac{\partial}{\partial t}> \quad ,$$

$$A_{3.2}^{13}=<\frac{\partial}{\partial t}-x\frac{\partial}{\partial x},\frac{\partial}{\partial x},\frac{\partial}{\partial t}+\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.2}^{14}=<-u\frac{\partial}{\partial u},\frac{\partial}{\partial u},\frac{\partial}{\partial x}> \quad ,$$

$$A_{3.2}^{15}=<-u\frac{\partial}{\partial u},\frac{\partial}{\partial u},\frac{\partial}{\partial t}> \quad , A_{3.2}^{16}=<-u\frac{\partial}{\partial u},\frac{\partial}{\partial u},\frac{\partial}{\partial t}+\frac{\partial}{\partial x}> \quad ,$$

$$A_{3.2}^{17}=<\frac{\partial}{\partial t}-u\frac{\partial}{\partial u},\frac{\partial}{\partial u},\frac{\partial}{\partial t}> \quad , A_{3.2}^{18}=<\frac{\partial}{\partial t}-u\frac{\partial}{\partial u},\frac{\partial}{\partial u},\frac{\partial}{\partial x}> \quad ,$$

$$A_{3.2}^{19}=<\frac{\partial}{\partial t}-u\frac{\partial}{\partial u},\frac{\partial}{\partial u},e^{-t}g(x)\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.2}^{20}=<\frac{\partial}{\partial t}-u\frac{\partial}{\partial u},\frac{\partial}{\partial u},\frac{\partial}{\partial t}+\frac{\partial}{\partial x}> \quad ,$$

$$A_{3.2}^{21}=<\frac{\partial}{\partial x}-u\frac{\partial}{\partial u},\frac{\partial}{\partial u},\frac{\partial}{\partial t}> \quad ,$$

$$A_{3.2}^{22}=<\frac{\partial}{\partial x}-u\frac{\partial}{\partial u},\frac{\partial}{\partial u},\frac{\partial}{\partial x}> \quad ,$$

$$A_{3.2}^{23}=<\frac{\partial}{\partial x}-u\frac{\partial}{\partial u},\frac{\partial}{\partial u},e^{-x}g(t)\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.2}^{24}=<\frac{\partial}{\partial x}-u\frac{\partial}{\partial u},\frac{\partial}{\partial u},\frac{\partial}{\partial t}+\frac{\partial}{\partial x}> \quad ,$$

$$A_{3.2}^{25}=<\frac{\partial}{\partial t}+\frac{\partial}{\partial x}-u\frac{\partial}{\partial u},\frac{\partial}{\partial u},e^{t-2x}\frac{\partial}{\partial u}> ,$$

$$A_{3.2}^{26}=<\frac{\partial}{\partial t}+\frac{\partial}{\partial x}-u\frac{\partial}{\partial u},\frac{\partial}{\partial u},\frac{\partial}{\partial x}> ,$$

$$A_{3.2}^{27}=<\frac{\partial}{\partial t}+\frac{\partial}{\partial x}-u\frac{\partial}{\partial u},\frac{\partial}{\partial u},\frac{\partial}{\partial t}> ,$$

$$A_{3.2}^{28}=<\frac{\partial}{\partial t}+\frac{\partial}{\partial x}-u\frac{\partial}{\partial u},\frac{\partial}{\partial u},\frac{\partial}{\partial t}+\frac{\partial}{\partial x}> \quad ,$$

$$A_{3.2}^{29}=<-t\frac{\partial}{\partial t}-x\frac{\partial}{\partial x},\frac{\partial}{\partial t}+\frac{\partial}{\partial x},\frac{\partial}{\partial x}> ,$$

$$A_{3.2}^{30}=<-t\frac{\partial}{\partial t}-x\frac{\partial}{\partial x},\frac{\partial}{\partial t}+\frac{\partial}{\partial x},\frac{\partial}{\partial t}> \quad .$$

$$A_{3.3}^{1}=<\frac{\partial}{\partial t},t\frac{\partial}{\partial t},-(t)^2\frac{\partial}{\partial t}> \quad ,$$

$$A_{3.3}^{2}=<\frac{\partial}{\partial t},t\frac{\partial}{\partial t}-\frac{\partial}{\partial u},-(t)^2\frac{\partial}{\partial t}+2t\frac{\partial}{\partial u}> ,$$

$$A_{3.3}^{3}=<\frac{\partial}{\partial t},t\frac{\partial}{\partial t}-\frac{\partial}{\partial u},-(t)^2\frac{\partial}{\partial t}+(2t+e^{-u}g(x))\frac{\partial}{\partial u}>,$$

$$A_{3.3}^{4}=<\frac{\partial}{\partial x},x\frac{\partial}{\partial x},-(x)^2\frac{\partial}{\partial x}> ,$$

$$A_{3.3}^{5}=<\frac{\partial}{\partial x},x\frac{\partial}{\partial x}-\frac{\partial}{\partial u},-(x)^2\frac{\partial}{\partial x}+2x\frac{\partial}{\partial u}> ,$$

$$A_{3.3}^{6}=<\frac{\partial}{\partial x},x\frac{\partial}{\partial x}-\frac{\partial}{\partial u},-(x)^2\frac{\partial}{\partial x}+(2x+e^{-u}g(t))\frac{\partial}{\partial u}>,$$

$$A_{3.3}^{7}=<\frac{\partial}{\partial u},u\frac{\partial}{\partial u},-(u)^2\frac{\partial}{\partial u}> ,$$

$$A_{3.3}^{8}=<\frac{\partial}{\partial t}+\frac{\partial}{\partial x},t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x},-(t)^2\frac{\partial}{\partial t}-(x)^2\frac{\partial}{\partial x}> \quad .$$

$$A_{3.5}^{1}=<\frac{\partial}{\partial u},\frac{\partial}{\partial t},t\frac{\partial}{\partial u}> .$$

$$A_{3.5}^{2}=<\frac{\partial}{\partial u},\frac{\partial}{\partial t},\frac{\partial}{\partial t}+t\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.5}^{3}=<\frac{\partial}{\partial u},\frac{\partial}{\partial t},\frac{\partial}{\partial x}+t\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.5}^{4}=<\frac{\partial}{\partial u},\frac{\partial}{\partial t},(t+g(x))\frac{\partial}{\partial u}> ,$$

$$A_{3.5}^{5}=<\frac{\partial}{\partial u},\frac{\partial}{\partial t},\frac{\partial}{\partial t}+(t+g(x))\frac{\partial}{\partial u}> ,$$

$$A_{3.5}^{6}=<\frac{\partial}{\partial u},\frac{\partial}{\partial t},\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+t\frac{\partial}{\partial u}> .$$

$$A_{3.5}^{7}=<\frac{\partial}{\partial u},\frac{\partial}{\partial x},x\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.5}^{8}=<\frac{\partial}{\partial u},\frac{\partial}{\partial x},\frac{\partial}{\partial t}+x\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.5}^{9}=<\frac{\partial}{\partial u},\frac{\partial}{\partial x},\frac{\partial}{\partial x}+x\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.5}^{10}=<\frac{\partial}{\partial u},\frac{\partial}{\partial x},(x+g(t))\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.5}^{11}=<\frac{\partial}{\partial u},\frac{\partial}{\partial x},\frac{\partial}{\partial x}+(x+g(t))\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.5}^{12}=<\frac{\partial}{\partial u},\frac{\partial}{\partial x},\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+x\frac{\partial}{\partial u}> \quad ,$$

$$A_{3.5}^{13}=<\frac{\partial}{\partial u},\frac{\partial}{\partial t}+\frac{\partial}{\partial x},x\frac{\partial}{\partial u}> ,$$

$$A_{3.5}^{14}=<\frac{\partial}{\partial u},\frac{\partial}{\partial t}+\frac{\partial}{\partial x},\frac{\partial}{\partial t}+x\frac{\partial}{\partial u}> ,$$

$$A_{3.5}^{15}=<\frac{\partial}{\partial u},\frac{\partial}{\partial t}+\frac{\partial}{\partial x},\frac{\partial}{\partial x}+x\frac{\partial}{\partial u}>,$$

$$A_{3.5}^{16}=<\frac{\partial}{\partial u},\frac{\partial}{\partial t}+\frac{\partial}{\partial x},(x+g(t-x))\frac{\partial}{\partial u}>,$$

$$A_{3.5}^{17}=<\frac{\partial}{\partial u},\frac{\partial}{\partial t}+\frac{\partial}{\partial x},\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+x\frac{\partial}{\partial u}>.$$

$$A_{3.6}^{1}=<\frac{\partial}{\partial u},\frac{\partial}{\partial t},t\frac{\partial}{\partial t}+(t+u)\frac{\partial}{\partial u}>,$$

$$A_{3.6}^{2}=<\frac{\partial}{\partial u},\frac{\partial}{\partial t},t\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+(t+u)\frac{\partial}{\partial u}>,$$

$$A_{3.6}^{3}=<\frac{\partial}{\partial u},\frac{\partial}{\partial x},x\frac{\partial}{\partial x}+(x+u)\frac{\partial}{\partial u}>,$$

$$A_{3.6}^{4}=<\frac{\partial}{\partial u},\frac{\partial}{\partial x},\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+(x+u)\frac{\partial}{\partial u}>,$$

$$A_{3.6}^{5}=<\frac{\partial}{\partial u},\frac{\partial}{\partial t}+\frac{\partial}{\partial x},t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+(x+u)\frac{\partial}{\partial u}>.$$

$$A_{3.7}^{1}=<\frac{\partial}{\partial t},\frac{\partial}{\partial x},t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}>,$$

$$A_{3.7}^{2}=<\frac{\partial}{\partial t},\frac{\partial}{\partial x},t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+\frac{\partial}{\partial u}>,$$

$$A_{3.7}^{3}=<\frac{\partial}{\partial t},\frac{\partial}{\partial u},t\frac{\partial}{\partial t}+u\frac{\partial}{\partial u}>,$$

$$A_{3.7}^{4}=<\frac{\partial}{\partial t},\frac{\partial}{\partial u},t\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}>,$$

$$A_{3.7}^{5}=<\frac{\partial}{\partial t},\frac{\partial}{\partial t}+\frac{\partial}{\partial u},t\frac{\partial}{\partial t}+u\frac{\partial}{\partial u}>,$$

$$A_{3.7}^{6}=<\frac{\partial}{\partial t},\frac{\partial}{\partial t}+\frac{\partial}{\partial u},t\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}>,$$

$$A_{3.7}^{7}=<\frac{\partial}{\partial x},\frac{\partial}{\partial u},x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}>,$$

$$A_{3.7}^{8}=<\frac{\partial}{\partial x},\frac{\partial}{\partial u},\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}>,$$

$$A_{3.7}^{9}=<\frac{\partial}{\partial x},\frac{\partial}{\partial x}+\frac{\partial}{\partial u},x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}>,$$

$$A_{3.7}^{10}=<\frac{\partial}{\partial x},\frac{\partial}{\partial x}+\frac{\partial}{\partial u},\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}>,$$

$$A_{3.7}^{11}=<\frac{\partial}{\partial u},g(t,x)\frac{\partial}{\partial u},u\frac{\partial}{\partial u}>,$$

$$A_{3.7}^{12}=<\frac{\partial}{\partial u},g(t,x)\frac{\partial}{\partial u},\frac{\partial}{\partial t}+u\frac{\partial}{\partial u}>,$$

$$A_{3.7}^{13}=<\frac{\partial}{\partial u},g(t,x)\frac{\partial}{\partial u},\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}>,$$

$$A_{3.7}^{14}=<\frac{\partial}{\partial u},g(t,x)\frac{\partial}{\partial u},\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}>,$$

$$A_{3.7}^{15}=<\frac{\partial}{\partial u},\frac{\partial}{\partial t}+\frac{\partial}{\partial x},t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}>.$$

$$A_{3.8}^{1}=<\frac{\partial}{\partial t},\frac{\partial}{\partial x},t\frac{\partial}{\partial t}-x\frac{\partial}{\partial x}>,$$

$$A_{3.8}^{2}=<\frac{\partial}{\partial t},\frac{\partial}{\partial x},t\frac{\partial}{\partial t}-x\frac{\partial}{\partial x}+\frac{\partial}{\partial u}>,$$

$$A_{3.8}^{3}=<\frac{\partial}{\partial t},\frac{\partial}{\partial u},t\frac{\partial}{\partial t}-u\frac{\partial}{\partial u}>,$$

$$A_{3.8}^{4}=<\frac{\partial}{\partial t},\frac{\partial}{\partial u},t\frac{\partial}{\partial t}+\frac{\partial}{\partial x}-u\frac{\partial}{\partial u}>,$$

$$A_{3.8}^{5}=<\frac{\partial}{\partial x},\frac{\partial}{\partial u},x\frac{\partial}{\partial x}-u\frac{\partial}{\partial u}>,$$

$$A_{3.8}^{6}=<\frac{\partial}{\partial x},\frac{\partial}{\partial u},\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}-u\frac{\partial}{\partial u}>,$$

$$A_{3.8}^{7}=<\frac{\partial}{\partial u},\frac{\partial}{\partial t}+\frac{\partial}{\partial x},-t\frac{\partial}{\partial t}-x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}>.$$

$$A_{3.9}^{1}=<\frac{\partial}{\partial t},\frac{\partial}{\partial x},t\frac{\partial}{\partial t}+qx\frac{\partial}{\partial x}>,\quad 0<|q|<1,$$

$$A_{3.9}^{2}=<\frac{\partial}{\partial x},\frac{\partial}{\partial t},qt\frac{\partial}{\partial t},x\frac{\partial}{\partial x}>,\quad 0<|q|<1,$$

$$A_{3.9}^{3}=<\frac{\partial}{\partial t},\frac{\partial}{\partial x},t\frac{\partial}{\partial t}+qx\frac{\partial}{\partial x}+\frac{\partial}{\partial u}>,\quad 0<|q|<1,$$

$$A_{3.9}^{4}=<\frac{\partial}{\partial x},\frac{\partial}{\partial t},qt\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+\frac{\partial}{\partial u}>,\quad 0<|q|<1,$$

$$A_{3.9}^{5}=<\frac{\partial}{\partial t},\frac{\partial}{\partial u},t\frac{\partial}{\partial t}+qu\frac{\partial}{\partial u}>,\quad 0<|q|<1,$$

$$A_{3.9}^{6}=<\frac{\partial}{\partial u},\frac{\partial}{\partial t},qt\frac{\partial}{\partial t}+u\frac{\partial}{\partial u}>,\quad 0<|q|<1,$$

$$A_{3.9}^{7}=<\frac{\partial}{\partial t},\frac{\partial}{\partial u},t\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+qu\frac{\partial}{\partial u}>,\quad 0<|q|<1,$$

$$A_{3.9}^{8}=<\frac{\partial}{\partial u},\frac{\partial}{\partial t},qt\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}>,\quad 0<|q|<1,$$

$$A_{3.9}^{9}=<\frac{\partial}{\partial x},\frac{\partial}{\partial u},x\frac{\partial}{\partial x}+qu\frac{\partial}{\partial u}>,\quad 0<|q|<1,$$

$$A_{3.9}^{10}=<\frac{\partial}{\partial u},\frac{\partial}{\partial x},qx\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}>,\quad 0<|q|<1,$$

$$A_{3.9}^{11}=<\frac{\partial}{\partial x},\frac{\partial}{\partial u},\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+qu\frac{\partial}{\partial u}>,\quad 0<|q|<1,$$

$$A_{3.9}^{12}=<\frac{\partial}{\partial u},\frac{\partial}{\partial x},\frac{\partial}{\partial t}+qx\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}>,\quad 0<|q|<1,$$

$$A_{3.9}^{13}=<\frac{\partial}{\partial u},\frac{\partial}{\partial t}+\frac{\partial}{\partial x},qt\frac{\partial}{\partial t}+q\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}>,\quad 0<|q|<1.$$

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حول التصنيف الزمري التمهيدي لصنف من المعادلات الموجية اللاخطية

عصام رفيق فائق

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الملخص

تم أنجاز التصنيف الزمري لاحد اصناف المعادلات الموجية اللاخطية بمتغير معتمد واحد ومتغيرين مستقلين. تبين بان هناك واحدة ، أربعة و ثمانية عشرة معادلة تتسع لجبور لي ذات الأبعاد واحد ، اثنان و ثلاثة على الترتيب.