On Almost J - injectivity and J –Regularity of Rings

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Abstract

As a generalization of right J- injective rings , we introduce the notion of right almost J-injective rings , that is if for any $a\in J(R)$, there exists a left ideal X_a of R such that $lr(a)=Ra\oplus X_a$. In this paper , we give some characterizations and properties of almost J-injective rings and it is show that R is J-regular if and only if R is a right almost J-injective and JPP ring . We also prove that , if R is right quasi duo , then R is J-regular if and only if every simple right R-module is almost J-injective .

1. Introduction

Throughout this paper , R will be an associative ring with identity, and all modules are unitary right R - module. For $a\in R$, $r(a)$(res.$l(a)$) denote the right (res.left) annihilator of a, respectively. We write $J(R)$, $N(R)$, $Z(R)$ $(Y(R))$ for the Jacobson radical of R , the set of nilpotent elements of R the left (right) singular ideal of R respectively . $X\leq M$ denoted that X is a submodule of a module M .

Generalization of injectivity have been discussed in many papers (see **[1] , [2] , [13]**). A ring R is called principally injective (or p-injective) , if every homomorphism from a principally right ideal of R to R can be extended to an endomorphism of R. Equivalently $l_R(r_R(a))=Ra$ for all $a\in R$. They also continued to study rings with some other kind of injectivity , namely , nil-injective rings **[9]** . A ring R is called right nil-injective , if for each $a\in N(R)$, $l_R(r_R(a))=Ra$. In **[5]** , Zhao and Du introduced an almost nil-injective module . Let M be a module with $S=End(M_R)$. The module M is called right almost nil-injective , if for any $k\in N(R)$ there exists an S-submodule X_k of M such that $l_M r_R(k)=Mk\oplus X_k$ as left S-modules . If R is almost nil-injective R-module then we call R a right almost nil-injective ring . In **[6]** Zhao and Zhou introduced an J-injective module . The module M_R is called right J-injective , if every right R- homomorphism from a principal right ideal aR with $a\in J(R)$ to M extends to one from R to M . R is called a J-injective ring , if R_R is right J-injective .Equivalently $l_R(r_R(a))=Ra$ for any $a\in J(R)$. Note that right p-injective rings is J-injective .

In **[3]** introduced an AP-injective , let M be a right R-module with $S=End(M_R)$. The module M is called AP-injective , if for any $a\in R$, there exists an S-submodule X_a of M such that $l_M r_R(a)=Ma\oplus X_a$ as left S-modules . If R_R is AP-injective , then we call R a right AP-injective ring .

In this paper , we consider rings which are more general than J-injective rings, an idea parallel to the notion almost nil-injective rings . In the second section , we give some characterizations of almost J-injective rings , for example : let R be a right almost J-injective ring, if $a\in J(R)$ and $(aR)_R$ is projective , than $aR=eR$ with $e^2=e\in R$. In the third section , we study regularity of right almost J-injective rings . For example : If every simple right R-module is almost J – injective , then R is a right J - weakly regular ring .

2. Almost J - Injective Rings

Definition 2.1

Let M_R be a module with $S=End(M_R)$. The module M is called right almost J-injective , if for any $a\in J(R)$, there exists an S-submodule X_a of M such that $l_M r_R(a)=Ma\oplus X_a$ as left S-module . If R_R is almost J-injective then we call R is a right almost J-injective ring .

Remark :

Every J-injective ring is almost J-injective . But the converse is not true as the following example :

Example :

1) The ring Z of integers is a right J-injective which is not P-injective .

2) Let $R=\begin{bmatrix}0 & Z_2\\ 0 & Z_2\end{bmatrix}$, where Z_2 is a field . The Jacobson radical of R is $J(R)=\begin{bmatrix}0 & Z_2\\ 0 & 0\end{bmatrix}$, Let $a=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, $Ra=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ and $lr(a)=R\neq Ra$.

Therefore R is not J-injective but $Ra+R=lr(a)$ and $Ra\cap R=(0)$, therefore $lr(a)=Ra\oplus R$, so R is right almost J-injective .

The following proposition gives a necessary and suffient condition for a ring to be almost J-injective .

Proposition 2.2

The following conditions are equivalent for a ring R:

1) R is a right almost J-injective ring .

2) If $a\in J(R)$, then $lr(a)=Ra\oplus X_a$.

3) If $k\in J(R), a\in R$,then $l(aR\cap r(u))=(X_{ka})_l+Rk$ with $ka\in J(R)$ and $(X_{ka}:a)_l\cap Rk\subset l(a)$ for all $a\in R$, where $(X_{ka}:a)_l=\{x\in R: Xa\in X_{ka}\}$ if $Ka\neq 0$ and $(X_{ka}:a)_l=l(aR)$ if $Ka=0$.

Proof .

$\mathbf{1\rightarrow 2}$ is clear .

2 → 3 and 3→ 1 ; it can be proved by the same method as **[5,Theorem 2.3]** .■

Before we introduce the next theorem we must recall the following lemma :

Lemma 2.3 [7]

Suppose M is a right R-module with $S = End(M_R)$. If $l_M r_R(a) = Ma \oplus X_a$, where X_a is a left S-submodule of M_R . Set $f : aR \to M$ is a right R-homomorphism , then $f(a) = ma + x$ with $m \in M$, $x \in X_a$.■

Theorem 2.4

If R is a right almost J-injective , then the following statements hold :

1) If $a \in J(R)$, and $(aR)_R$ is projective , then $aR = eR$ with $e^2 = e$.

2) $J(R) \subseteq Y(R)$.

Proof .

1) Since aR is projective, $r(a) = fR$, $f^2 = f \in R$ [6].By hypothesis, $R(1-f) = l(fR) = lr(a) = Ra \oplus X_a$. Write $(1-f) = ba + x$, where $x \in X_a$ and $b \in R$. Then $a = a(1-f) = aba + ax$, $(a - aba) = ax \in Ra \cap X_a = 0$, so $a = aba$. Let $e = ab$, then $a = ea$, $e^2 = e$ and $aR = eR$.

2) If there exists $b \in J(R)$ with $b \notin Y(R)$. Then there exists anon zero right ideal I of R such that $I \cap r(b) = 0$. Let $0 \neq a \in I$, then $ba \neq 0$. Evidently $ba \in J(R)$. So $lr(ba) = Rba \oplus X_{ba}$ where X_{ba} is a right ideal of R . Set $baR \to R$ via $bar \to ar$, $r \in R$. Then f is a well defined right R- homomorphism , thus $a = f(ba) = uba + x$ by **Lemma 2.3** , where $u \in R$, $x \in X_{ba}$ and so $ba = buba + bx$, $(1-bu)ba = bx \in Rba \cap X_{ba} = 0$. Hence $(1-bu)ba = 0$, but $(1-bu)$ is invertible , thus $ba = 0$, which is a contradiction . Hence $J(R) \subseteq Y(R)$.■

Recall that an right R-module M is said to be Wjcp-injective [**9**] , if for each $a \notin Y(R)$, there exists a positive integer n such that $a^n \neq 0$ and every R-homomorphism from $a^n R$ to M can be extended to one of R to M . If R is Wjcp-injective , we call R is a right Wjcp-injective ring .

We recall the following lemma before we introduce the next result .

Lemma 2.5 [9]

If R is a right Wjcp-injective , then $Y(R) \subseteq J(R)$.■

From **Theorem 2.4** and **Lemma 2.5** we get the following result :

Corollary 2.6

If R is a right Wjcp - injective and almost J-injective , then $Y(R) = J(R)$.

3. J - regular Rings

(Von Neumann) regular rings have been studied extensively by many authors (for example , **[12]** and **[14]**) . A ring R is called regular if for any $a \in R$, there exists $b \in R$ such that $a = aba$.

Definition 3.1 [6]

R is called J-regular , if for any $a \in J(R)$, a is regular element .

Cleary , every regular ring is J-regular .

Following **[11]** , a ring R is called a right JPP ring , if for any $a \in J(R)$, aR is projective .

A ring R is called right pp if every principal right ideal is projective . It is clear that right pp ring are right Jpp , but the converse is not true as example :

Example :[11]

Let $R = \begin{pmatrix} Z & Z \\ 0 & Z \end{pmatrix}$. Then $J = e_{12}R$, where $e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Note that Z/2Z is not a projective Z-module . Hence R is not a right pp ring . Let $0 \neq x \in J(R)$. Then it is easy to verify that $r(x) = e_{11}R$ is a summand of R_R , where $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. So R is a right Jpp ring.

Remark :

Every J-regular is JPP ring , but the converse is not true [**Example 2.12** , **9**] .

A ring R is called ZI **[1]** , if for every $a, b \in R$, $ab = 0$ implies $aRb = 0$.

It well know that R is a ZI ring if and only if $l(a)(r(a))$ is an ideal of R for every $a \in R$ **[1]** . A ring R is called abelian if every idempotent of R is central **[1]** .

Lemma 3.2

Let R be an abelian ring and $a \in J(R)$. Then :

1) If M is a maximal left ideal of R such that $l(a) + Ra \subseteq M$, then M is essential .

2) If K is a maximal right ideal of R such that $r(a) + aR \subseteq K$, then K is essential .

Proof .

It can be proved by the same methods in [**4** , **Lemma 2.3**] .■

Theorem 3.3

Let R be a ZI ring . The following conditions are equivalent :

1) J - regular .

2) Every maximal essential left ideal of R is J – injective .

3) Every maximal essential right of R is J – injective .

Proof .

Clearly **1** → **2** and **1** → **3** .

2 → **1** , Let $a \in J(R)$ such that $l(a) + Ra \neq R$, then there exists a maximal left ideal M of R containing $l(a) + Ra$. Since a ZI ring is abelian by **Lemma 3.2** , M is essential , so that by hypothesis M is J - injective . Therefore for any $a \in J(R)$ and every left R - homomorphism from Ra to M extended to one

from R to M . It follow that there exists some $b\in R$ such that $a=ab$. This gives $1-b\in r(a)\subseteq M$ which yields $1\in M$, a contradiction . Therefore all $a\in J(R)$, $l(a)+Ra=R$. This proves that R is J – regular .
Similarly **3 → 1** .■

The following is characterization for almost J-injective ring .

Theorem 3.4

Let R be a ring . Then R is J-regular if and only if R is a right almost J-injective and JPP ring .

Proof .

Let $a\in J(R)$. Since R is right JPP ring , $r(a)=eR$, $e^2=e$. Since R is a right almost J-injective , $lr(a)=Ra\oplus X_a$. Hence $(1-e)R=lr(a)$,$(1-e)=ra+x$, where $r\in R$, $x\in X_a$. So $a=a(1-e)=ara+ax$, $(1-ra)a=ax\in Ra\cap X_a=0$ and $a=ara$. Hence R is J-regular .

Conversely :

By [**6** , **Theorem 3.2**] .■

Recall that a ring R is called right quasi duo [**1**] , if every maximal right ideal of R is a two sided ideal . A ring R is called reduced if R has no non zero nilpotent element .

Proposition 3.5

If R is a right quasi duo ring. Then the following statements are equivalent :

1) R is J-regular .
2) Every simple right R-module is almost J-injective.

Proof.

1 → 2 is obvious by [**6** , **Theorem 3.2**] .

2 → 1 ; If there exists $0\neq a\in J(R)$ such that $aR+r(a)\neq R$, then there exists a maximal right ideal M containing $aR+r(a)$.Thus R/M is almost J-injective, and $l_{R/M}\, r_R(a)=(R/M)a\oplus X_a$, $X_a\leq R/M$. Let $f:aR\to R/M$ be defined by $f(ab)=b+M$, then f is a well defined R-homomorphism so there exists $b\in R$, $x\in X_a$ such that $1+M=ba+M+x$, $1-ba+M=x\in R/M\cap X_a=0$. Hence $1-ba\in M$. Since $a\in M$ and R is right quasi duo , then $ba\in M$, hence $1\in M$, a contradiction . Therefore a is regular element .■

In the next theorem ,we show that in the class of quasi_duo ring the following conceps are equivalent :

Theorem 3.6

If R is right quasi duo , the following condition are equivalent for a ring R :

1) Every right R-module is J-injective .
2) Every right R-module is almost J-injective .
3) Every cyclic right R-module is almost J-injective
4) Every simple right R-module is almost J-injective .
5) Every element of J(R) is strongly regular .
6) R is J-regular .

Proof.

Obviously **1 → 2 → 3 → 4** and **4 → 6** (**Theorem 3.3**) .

Now , we prove that **4 → 5** . For any $0\neq a\in J(R)$, we will show that $aR+r(a)=R$. Suppose not , then there exists a maximal right ideal K of R containing $aR+r(a)$. Since R/K is almost J-injective, $l_{R/K}\, r_R(a)=(R/K)a\oplus X_a$, $X_a\leq R/K$. Let $f:aR\to R/K$ be defined by $f(ab)=b+K$. Since $aR+r(a)\subseteq K$, f is a well defined R-homomorphism so there exists $c\in R$, $x\in X_a$ such that $1+K=ca+K+x$ by **Lemma 2.3**, $1-ca+K=x\in R/K\cap X_a=0$, $1-ca\in K$ and $ca\in K$ (R is quasi duo) and so $1\in K$, which is a contradiction . Therefore $aR+r(a)=R$. In particular $ax+b=1$, $x\in R$, $b\in r(a)$, so $a^2x+ab=a$. So a is strongly regular element .

5 → 6 is trivial .

6 → 1 by [**6** ,**Theorem 3.2**] .■

A ring R is called SXM if , for any $a\in R$ there exists a positive integer n such that $a^n\neq 0$ and $r(a^n)=r(a)$, $l(a^n)=l(a)$. **[10]**

Theorem 3.7

If R is a SXM right almost J-injective ring , then R is J-regular .

Proof.

For any $a\in J(R)$, then $a^2\in J(R)$. Since R is right SXM , $r(a^2)=r(a)$. By almost J-injective , there exists a left ideal X of R , such that $Ra\oplus X=l(r(a))=l(r(a^2))=Ra^2\oplus X$. Hence $a\in l(r(a))=l(r(a^2))=Ra^2\oplus X$, then $a=ba^2+x$ for any $b\in R$, $x\in X$. Therefore $a^2=aba^2+ax$, which implies that $a^2-aba^2=ax\in Ra\cap X=0$. Hence $a^2=aba^2$, and so $(1-ab)\in l(a^2)=l(a)$. Thus we have that $a=aba$. Therefore R is J-regular .■

Following **[9]**, a left ideal L of a ring R is a generalized weak ideal (GW- ideal) , if for every $a\in L$, there exists a positive integer n such that $a^nR\subseteq L$. A right ideal K of R is defined similarly to be generalized weak ideal .

By [**8**] , we have that { ideal } ⊂ { one side - ideal } , and GW- ideal need not be ideal , one side - GW - ideal .

Lemma 3.8 : [4]

The following conditions are equivalent for a ring R :

1) $l(e)$ is a GW- ideal of R for every $e\in I(R)$ (set of all idempotent elements) .
2) $r(e)$ is a GW- ideal of R for every $e\in I(R)$ (set of all idempotent elements) .

Proposition 3.9

Let R be a ring such that $l(a)$ is GW-ideal of R for every $a\in R$. If R satisfies one of the following conditions , then J(R) is reduced :

1) Every simple singular left R-module is almost J-injective .

2) Every simple singular right R-module is almost J-injective .

Proof .

1) If J(R) not reduced , then there exists $0\neq b\in J$ such that $b^2=0$ and a maximal left ideal such that $l(b)\subseteq M$. By **Lemma 3.8** and **Lemma 3.2** , M is essential and so R/M is simple singular left R-module . By hypothesis , R/M is almost J-injective , so $l_{R/M}\, l_R(b)=b(R/M)\oplus X_b$, $X_b\leq R/M$. Let $f:Rb\to R/M$ be defined by $f(rb)=r+M$. Note that f is a well defined R-homomorphism ($l(b)\subseteq M$). Then $1+M=f(a)=bc+M+x$, $c\in R$, $x\in X_b$, $1-bc\in M=x\in R/M\cap X_b=0$, $1-bc\in M$. If $bc\notin M$, then $M+Rbc=R$ which gives $z+ybc=1$ for some $z\in M$, $y\in R$. Since $cyb\in l(b)$ and $l(b)$ is a GW-ideal of R, there exists a positive integer n such that $(cyb)^n c\in l(b)$ .Therefore $(1-z)^{n+1}=(ybc)^{n+1}=yb(cyb)^n c\in M$. It follows that $1\in M$, A contradiction to $M\neq R$. Hence J is reduced .

2) Suppose that $0\neq b\in J$ such that $b^2=0$. Then there exists an right ideal K of R such that $r(b)\subseteq K$. By **Lemma 3.8** and **Lemma 3.2** , K is essential . Then proof is similar to that of (1) we get $1-cb\in K$. Now l(b) is GW – ideal of R and $b\in l(b)$, so there exists a positive integer n such that $(cb)^n c\in l(b)$. Then , similar to the proof of [**4 , Proposition 2.6(2)]** we can complete proof . ■

Before we end this section , we present the connection between almost J-injective and J-weakly regular .

Following **[1]** , a ring R is said to be right(left) weakly regular , if for every $a\in R$, $a\in aRaR$ ($a\in RaRa$) .

Definition 3.10

R is called right J- weakly regular , if for any $a\in J(R)$, a is right weakly regular element .

Remark :

A weakly regular rings is clearly J – weakly regular , but J – weakly regular ring need not be weakly regular .

Example :

Let Z be the ring of integers . Then R is J-weakly regular but not weakly regular .

Theorem 3.11

If every simple right R-module is almost J-injective , then R is a right J-weakly regular ring .

Proof .

We will show that $RaR+r(a)=R$ for any $a\in J(R)$. Suppose that $RbR+r(b)=R$ for any $b\in J(R)$, then there exists a maximal right ideal M of R containing $RbR+r(b)$. Thus R/M is almost J-injective, then $l_{R/M}\, r_R(b)=(R/M)b\oplus X_b, X_b\leq R/M$. Let $f:bR\to R/M$ be defined by $f(bz)=z+M$ for every $z\in R$. Note f is a well defined . So $1+M=f(b)=cb+M+x$, $c\in R$, $x\in X_b$, $1-cb+M=x\in R/M\cap X_a=0$, $1-cb\in M$. Now $cb\in RaR\subseteq M$, where $1\in M$. This is a contradiction. Hence $RaR+r(a)=R$ for any $a\in J(R)$ and so $a=ad$ for some $d\in RaR$. Hence a is right weakly regular . Therefore R is J-weakly regular ring.■

Corollary 3.12

Let R be a ZI ring . If every simple right R-module is almost J-injective , then R is J-weakly regular ring .■

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حول الحلقات الغامرة من النمط - J التقريبية والمنتظمة من النمط - J

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الملخص

كتعميم لمفهوم الحلقات الغامرة من النمط – J اليمنى ، قدمنا تعريف الحلقات الغامرة من النمط - J التقريبية اليمنى،لأي $a \in J(R)$ فإنه يوجد مثالي أيسر X_a للحلقة R بحيث إن $lr(a) = Ra \oplus X_a$. في هذا البحث ، قدمتُ تميزاً لبعض الخواص للحلقات الغامرة من النمط - J التقريبية وأثبتُ إنه تكون R حلقة منتظمة من النمط – J إذا وفقط إذا كان كل مقاس أيمن في R غامراً من النمط – J تقريبياً وحلقة من النمط JPP . وكذلك أثبتُ إن ، إذا كانت R حلقة كوازي ديو فإنه تكون R حلقة منتظمة من النمط – J إذا وفقط إذا كان كل مقاس أيمن بسيط غامراً من النمط – J تقريبياً .