Solving Higher Order Ordinary Differential Using Lie group Method

Ahmed Sabah Ahmed

University of Babylon / College of Education / Ibn Hayan Department of mathematic

Abstract

In this Paper we will develop an approach to the solution of differential equation based on finding a group invariant by defining Lie group via the transformation , we are giving the finite form of the group $y_1 = f(x, y, \varepsilon)$ $x_1 = f(x, y, \varepsilon)$

Key word-: Differential equation , Group theoretical methods , lie group

الخلاصة

ان الهدف الرئيسي من هذا البحث هو أيجاد نموذج رياضي لحل معادلات تفاضلية من الرتب العليا عن طريق أيجاد مجموعة ثابتة من خلال تحديد طريقة Lie group وباستخدام تحويل $y_1 = f(x, y, \varepsilon)$. $x_1 = f(x, y, \varepsilon)$ ومن مميزات هذا التحويل يمكننا تحديد مجموعة الحل بأسلوب أكثر دقة .

1- Introduction:-

In mathematics a Lie group is a group which is also a differentiable manifold, with the property that the group operations are compatible with the smooth structure. Lie groups are named after Sophus Lie, who laid the foundations of the theory of continuous transformation groups.[L. Dresner, Applications of Lie's Theory of Ordinary and Partial Dif ferential Equations, Institute of Physics Publishing, Bristol, 1999]

Lie groups represent the best-developed theory of continuous symmetry of mathematical objects and structures, which makes them indispensable tools for many parts of contemporary mathematics, as well as for modern theoretical physics. They provide a natural framework for analyzing the continuous symmetries of differential equations (Differential Galois theory), in much the same way as permutation groups are used in Galois theory for analyzing the discrete symmetries of algebraic equations. An extension of Galois theory to the case of continuous symmetry groups was one of Lie's principal motivations.

According to the most authoritative source on the early history of Lie groups (Hawkins), Sophus Lie himself considered the winter of 1873–1874 as the birth date of his theory of continuous groups. Hawkins, however, suggests that it was "Lie's prodigious research activity during the four-year period from the fall of 1869 to the fall of 1873" that led to the theory's creation (ibid). Some of Lie's early ideas were developed in close collaboration with Felix Klein. Lie met with Klein every day from October 1869 through 1872: in Berlin from the end of October 1869 to the end of February 1870, and in Paris, Göttingen and Erlangen in the subsequent two years (ibid, p. 2). Lie stated that all of the principal results were obtained by 1884. But during the 1870s all his papers (except the very first note) were published in Norwegian journals, which impeded recognition of the work throughout the rest of Europe (ibid, p. 76). In 1884 a young German mathematician, Friedrich Engel, came to work with Lie on a systematic treatise to expose his theory of continuous groups. From this effort resulted the three-volume Theorie der Transformations gruppen, published in 1888, 1890, and 1893.

[P. E. Hydon, Symmetry Methods for Differential Equations, Cambridge University Press, Cambridge, 2000]

Lie's ideas did not stand in isolation from the rest of mathematics. In fact, his interest in the geometry of differential equations was first motivated by the work of Carl Gustav Jacobi, on the theory of partial differential equations of first order and on the equations of classical mechanics. Much of Jacobi's work was published posthumously in the 1860s, generating enormous interest in France and Germany (Hawkins, p. 43). Lie's idée fixe was to develop a theory of symmetries of differential equations that would accomplish for them what Évariste Galois had done for algebraic equations: namely, to classify them in terms of group theory. Lie and other mathematicians showed that the most important equations for special functions and orthogonal polynomials tend to arise from group theoretical symmetries. Additional impetus to consider continuous groups came from ideas of Bernhard Riemann, on the foundations of geometry, and their further development in the hands of Klein. Thus three major themes in 19th century mathematics were combined by Lie in creating his new theory: the idea of symmetry, as exemplified by Galois through the algebraic notion of a group; geometric theory and the explicit solutions of differential equations of mechanics, worked out by Poisson and Jacobi; and the new understanding of geometry that emerged in the works of Plücker, Möbius, Grassmann and others, and culminated in Riemann's revolutionary vision of the subject

The infinitesimal Transformation

By defining a lie group via the transformation $x_1 = f(x,y,\varepsilon)$, $y_1 = f(x,y,\varepsilon)$ we are giving the finite form of group . consider what happens when $\varepsilon \le 1$ since $\varepsilon = 0$

Given the identity transformation , we can Taylor expand to obtain

$$x_1 = x + \varepsilon\left(\frac{dx_1}{d\varepsilon}\right)_{\varepsilon=0} + \ldots.. \quad , \quad y_1 = y + \varepsilon\left(\frac{dy_1}{d\varepsilon}\right)_{\varepsilon=0} + \ldots..$$

If we now introduce the function

$$\xi(x,y) = \left(\frac{dx_1}{d\varepsilon}\right)_{\varepsilon=0} \quad , \quad \eta(x,y) = \left(\frac{dx_1}{d\varepsilon}\right)_{\varepsilon=0} \qquad \ldots(1)$$

And just retain the first tow terms in the taylor series expansions , we obtain $x_1 = x + \varepsilon\xi(x,y), y_1 = y + \varepsilon\eta(x,y)$, this is called the infinitesimal form of the group we will show later that every one-parameter group is associated with a unique infinitesimal group . [Edwards,C. Henry and Penney, David E. Differential Equations: Computing and Modeling. Upper Saddle River, NJ: Prentice Hall, 2000]

For example the transformation $x_1 = x\cos\varepsilon - y\sin\varepsilon, y_1 = x\sin\varepsilon - y\cos\varepsilon$ for the rotation group $R(\varepsilon)$ When $\varepsilon = 0$ this gives the identity transformation using the approximation $\cos\varepsilon = 1 + \ldots, \sin\varepsilon = \varepsilon + \ldots.$ for $\varepsilon \le 1$ we obtain the infinitesimal rotation group as $x_1 \approx x - \varepsilon y, y_1 \approx y + \varepsilon x$ and hence $\xi(x,y) = -y$ and $\eta(x,y) = x$ the transformation $x_1 = e^{\varepsilon}x, y_1 = e^{\varepsilon}y$ forms the magnification group $M(\varepsilon)$ using $e^{\varepsilon} = 1 + \varepsilon + \ldots.$ for $\varepsilon \le 1$ we obtain the infinitesimal magnification group as $x_1 \approx (1+\varepsilon)x, y_1 \approx (1+\varepsilon)y$ so that $\xi(x,y) = x, \eta(x,y) = y$

We will now show that every infinitesimal transformation group is similar or isomorphic to a translation group this means that by using a change of variable we can make any infinitesimal transformation group look like $H(\varepsilon)$ or $V(\varepsilon)$ which we defined earlier consider the equation that define ξ and η and write them in the form

$$\frac{dx_1}{d\varepsilon}=\xi(x_1,y_1)\,,\ \frac{dy_1}{d\varepsilon}=\eta(x_1,y_1)$$

A result that is correct at leading order by virtue of the infinitesimal nature of the transformation and which we shall soon see is exact . we can also write this in the form

$$\frac{dx_1}{\xi}=\frac{dy_1}{\eta}=d\varepsilon \qquad \ldots(2)$$

Integration of this gives solution that are in principle expressible in the form $F_1(x_1,y_1)=C_1$ and $F_2(x_1,y_1)=C_2+\varepsilon$ for some constant C_1 and C_2 sine $\varepsilon=0$ corresponds to the identity transformation we can deduce that $F_1(x_1,y_1)=F_1(x,y)$ and $F_2(x_1,y_1)=F_2(x,y)+\varepsilon$ this means that if we define $u=F_1(x,y)$ and $v=F_2(x,y)$ as new variables, then the group can be represented by $u_1=u$ and $v_1=v+\varepsilon$ so that the original group is isomorphic to the translation group $V(\varepsilon)$

2- Infinitesimal Generators and the Lie Series

Consider the change $\delta\phi$, that occurs in a give smooth function $\phi(x,y)$ under an infinitesimal transformation we find that

$$\delta\phi=\phi(x_1,y_1)-\phi(x,y)=\phi(x+\varepsilon\xi,y+\varepsilon\eta)-\phi(x,y)=\varepsilon\left(\xi\frac{\partial\phi}{\partial x}+\eta\frac{\partial\phi}{\partial y}\right)+.....$$

If we retain just this single term , which is consistent with the way we derived the infinitesimal transformation ,we can see that $\delta\phi$ can be written in terms of the quantity

$$U\phi=\xi\frac{\partial\phi}{\partial x}+\eta\frac{\partial\phi}{\partial y} \text{ or in operator notation } U=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y} \qquad \ldots(3)$$

This is called the infinitesimal generator of the group . Any infinitesimal transformation is completely specified by $U\phi$ for Example ,if

$$U\phi=-y\frac{\partial\phi}{\partial x}+x\frac{\partial\phi}{\partial y} \qquad \ldots(4)$$

$\xi(x,y)-y,\eta(x,y)=x$ so the transformation is given by $x_1=x-\varepsilon y, y_1=y+\varepsilon x$ form the definition (3) $Ux=\xi$, $Uy=\eta$ so that

$$U\phi=Ux\frac{\partial\phi}{\partial x}+Uy\frac{\partial\phi}{\partial y} \qquad \ldots(5)$$

And if the group acts on (x,y) to produce new values (x_1,y_1) then

$$U\phi(x_1,y_1)=Ux_1\frac{\partial\phi}{\partial x_1}+Uy_1\frac{\partial\phi}{\partial y_1} \qquad \ldots(6)$$

Let now consider a group defined in finite from by $x_1=f(x,y:\varepsilon), y_1=g(x,y:\varepsilon)$ and a function $\phi=\phi(x,y)$ if we regard $\phi(x_1,y_1:\varepsilon)$ as a function of ε,with a prime denoting $\frac{d}{d\varepsilon}$ we fine that

$$\phi(x_1,y_1:\varepsilon)=\phi(x_1,y_1:0)+\varepsilon\phi'(x_1,y_1:0)+\frac{1}{2}\varepsilon^2\phi''(x_1,y_1:0)+.... \text{ since}$$

$$\phi(x_1,y_1:0)=\phi(x,y)$$

$$\phi'(x_1, y_1 : 0) = \left.\frac{d\phi}{d\varepsilon}\right|_{\varepsilon=0} = \left(\frac{\partial \phi}{\partial x_1}\frac{dx_1}{d\varepsilon} + \frac{\partial \phi}{\partial y_1}\frac{dy_1}{d\varepsilon}\right)_{\varepsilon=0}$$

$$= \left(\xi \left.\frac{\partial \phi}{\partial x_1}\right|_{\varepsilon=0} + \eta \left.\frac{\partial \phi}{\partial y_1}\right|_{\varepsilon=0}\right) = \xi \frac{\partial \phi}{\partial y} + \eta \frac{\partial \phi}{\partial y} = U\phi$$

And $\phi''(x_1, y_1 : 0) = \left.\frac{d}{d\varepsilon}\left(\frac{d\phi}{d\varepsilon}\right)\right|_{\varepsilon=0} = U^2\phi$ we have

$$\phi(x, y : \varepsilon) = \phi(x, y : 0) + \varepsilon U\phi + \frac{1}{2}\varepsilon^2 U^2 \phi + ...$$

This is Known as Lie series and can be written more compactly in operator from as

$\phi(x, y : \varepsilon) = e^{\varepsilon U}\phi(x, y)$

In particular If we take $\phi(x, y : 0) = x$

$$x_1 = x + \varepsilon Ux + \frac{1}{2}\varepsilon^2 U^2 x + ... = x + \varepsilon\xi + \frac{1}{2}\varepsilon^2 U\xi + ...$$

$$x_1 = x + \varepsilon\xi + \frac{1}{2}\varepsilon^2\left(\xi\frac{\partial \xi}{\partial x} + \eta\frac{\partial \xi}{\partial y}\right) + ...$$ similarly

$$y_1 = y + \varepsilon\eta + \frac{1}{2}\varepsilon^2\left(\xi\frac{\partial \xi}{\partial x} + \eta\frac{\partial \xi}{\partial y}\right) + ... \qquad \ldots (7)$$

These two relation are a representation of the group in finite from . It should now be clear that we can calculate the finite of the group from the infinitesimal group (Via the Lie series) and the infinitesimal group from the finite form of the group (Via expansions for small ε)

$$U\phi = x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} \qquad \ldots. (8)$$

Then $x_1 = x + \varepsilon x + \frac{1}{2!}\varepsilon^2 x + \frac{1}{3!}\varepsilon^3 x + ... = xe^{\varepsilon}$

$$y_1 = y + \varepsilon y + \frac{1}{2!}\varepsilon^2 y + \frac{1}{3!}\varepsilon^3 y + ... = ye^{\varepsilon} \qquad \ldots (9)$$

The finite form is therefore $M(\varepsilon)$, the magnification group as a further example if an infinitesimal group is represented by

$$U\phi = -y\frac{\partial \phi}{\partial x} + x\frac{\partial \phi}{\partial y}$$

Then

$$Ux = -y \quad Uy = x \ , \quad U^2 x = -x \quad U^2 y = -y$$
$$U^3 x = y \quad U^3 y = -x \quad U^4 x = x \quad U^4 y = y \qquad \ldots (10)$$

U is Therefore a cyclic operation with period 4 and the equation of the finite from of the group are

$$x_1 = x + \varepsilon y + \frac{1}{2!}\varepsilon^2 x + \frac{1}{3!}\varepsilon^3 y + \frac{1}{4!}\varepsilon^4 x + ...$$

$$= x\left(1 - \frac{1}{2!}\varepsilon^2 + \frac{1}{4!}\varepsilon^4 - ...\right) - y\left(\varepsilon - \frac{1}{3!}\varepsilon^3 + ...\right) = x\cos\varepsilon - y\sin\varepsilon$$

Similarly

$y_1 = y + \varepsilon x + \frac{1}{2!}\varepsilon^3 y + \frac{1}{3!}\varepsilon^3 x + \frac{1}{4!}\varepsilon^4 y + ...$

And we have the rotation group $R(\varepsilon)$

There is a rather more concise way doing this , using the fact that

$\frac{dx_1}{d\varepsilon} = \xi(x_1, y_1)$, $\frac{dy_1}{d\varepsilon} = \eta(x_1, y_1)$ subject to $x_1 = x$ and $y_1 = y$ at $\varepsilon = 0$

Is an exact relationship according to (1) for the first from first example above , this gives $\frac{dx_1}{d\varepsilon} = x_1$, $\frac{dy_1}{d\varepsilon} = y_1$ with $x = x_1$, $y = y_1$ at $\varepsilon = 0$ this first order system can readily be integrated give $x_1 = xe^{\varepsilon}$, $y_1 = ye^{\varepsilon}$

3-Integration of a first order equation with a Known group invariant

To show more explicitly that this group invariance property will load to a more tractable differential equation than the original lets consider a general first order ordinary differential equation $F(x, y, p) = 0$ that is invariant under the extended group

$U'\phi = \xi \frac{\partial \phi}{\partial x} + \varepsilon\eta \frac{\partial \phi}{\partial y} + \zeta \frac{\partial \phi}{\partial p}$ derived from

$U\phi = \xi \frac{\partial \phi}{\partial x} + \varepsilon\eta \frac{\partial \phi}{\partial y}$

We have seen that a sufficient condition for the invariance property is that $U'\phi = 0$

So we are faced with solving

$$\xi \frac{\partial \phi}{\partial x} + \varepsilon\eta \frac{\partial \phi}{\partial y} + \zeta \frac{\partial \phi}{\partial p} = 0 \qquad \ldots (11)$$

The solution curves of this partial differential equation where ϕ is constant are the two independent solution of the simultaneous system

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dp}{\zeta} \qquad \ldots(12)$$

Let $u(x, y) = \alpha$ be a solution of $\frac{dx}{\xi} = \frac{dy}{\eta}$

And $v(p, x, y) = \beta$ be the other independent solution

We now show that if we Know U ,find v is simply a matter of integration to do this recall the earlier result that any group with one parameter is similar to the translation group , let the change of variables from (x, y) to (x_1, y_1) reduce $U\phi$ to the group of translations parallel to the y_1 axis , and call the infinitesimal generator of this group $U_1 f$, then

$$U_1 f = Ux_1 \frac{\partial f}{\partial x_1} + Uy_1 \frac{\partial f}{\partial y_1} = \frac{\partial f}{\partial y_1} \qquad \ldots (13)$$

From which we see that $Ux_1 = 0$, $Uy_1 = 0$ or more explicitly

$$\xi \frac{\partial x_1}{\partial x} + \eta \frac{\partial x_1}{\partial y} = 0 \quad , \quad \xi \frac{\partial y_1}{\partial x} + \eta \frac{\partial y_1}{\partial y} = 1$$

The first of these equations has the solution $x_1 = u(x, y)$ and the second is equivalent to the simultaneous system

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy_1}{1} \qquad \ldots (14)$$

Again , one solution of this system is $u(x, y) = \alpha$ this can be used to eliminate x from the second independent solution , given by

$$\frac{dy_1}{dy} = \frac{1}{\eta(x, y)}$$

So that by a simple integration we can obtain y_1 as a function of x and y . As the extended group of translation $U_1' f$, is identical to $U_1 f$ the most general differential equation invariant under U_1' in the new x_1, y_1 variable will therefore be a solution of simultaneous system

$$\frac{dx_1}{0} = \frac{dy_1}{1} = \frac{dp_1}{0}$$

This particularly simple system has solution $x_1 =$constant and $p_1 =$constant , so that the differential equation can be put in form $p_1 = f(x_1)$ for some calculable function F in principle it is straightforward to solve equation of this form as they are separable the solution of original equation can than be obtain by returning to the (x, y) variable.

4-Higher order Equation

These ideas can be extended to higher order equation , we obtain with an nth order equation $F(x, y, ..., y^{(n)}) = 0$ As usual we seek an infinitesimal generator

$$X = \xi \frac{\partial}{\partial x} + \eta^{(0)} \frac{\partial}{\partial y^{(0)}} + \eta^{(1)} \frac{\partial}{\partial y^{(1)}} + ... + \eta^{(n)} \frac{\partial}{\partial y^{(n)}} = \xi \frac{\partial}{\partial x} + \sum_{j=0}^{n} \eta^{(j)} \frac{\partial}{\partial y^{(j)}}$$

The function in the prolongation formulas are determined following the procedure demonstrated in above they are recursively related

$$\begin{aligned}
\eta^{(0)}(x, y) &= \eta(x, y) \\
\eta^{(1)}(x, y, y^{(1)}) &= D^{(0)}\eta^{(0)} - y^{(1)} D^{(0)}\xi \\
\eta^{(2)}(x, y, y^{(1)}, y^{(2)}) &= D^{(1)}\eta^{(1)} - y^{(2)} D^{(0)}\xi \qquad ..... (15) \\
\eta^{(3)}(x, y, y^{(1)}, y^{(2)}, y^{(3)}) &= D^{(2)}\eta^{(2)} - y^{(3)} D^{(0)}\xi \\
\vdots \qquad & \vdots \qquad \vdots
\end{aligned}$$

The operator X is used as describe to compute the function $\xi(x, y)$ and $\eta(x, y)$ there will be many linearly independent infinitesimal generators as the corank of the set of linear equation for the taylor series coefficients of these function .
If one or more generator can be constructed a dependent coordinate S can be computed by solving eq (13) the remaining invariant coordinates are obtain from the equations

$$\frac{dx}{\xi} = \frac{dy}{\eta^{(0)}} = \frac{dy^{(1)}}{\eta^{(1)}} = ... = \frac{dy^{(n)}}{\eta^{(n)}} \qquad ....(16)$$

As a result of Lie group can be used to reduce in nth order equation to an (n-1) order equation

5-Examples :-

1- the differential equation $\frac{dy}{dx}=\frac{1}{\sqrt{x+y^2}}$

Is invariant under the transformation $x=e^{-2\varepsilon}x_1$, $y=e^{-\varepsilon}y_1$ the infinitesimal transformation associated with this is $x_1=x+2\varepsilon x$, $y_1=y+\varepsilon y$ so that $\xi(x,y)=2x$ And $\eta(x,y)=y$ if we solve the system

We find that $\frac{y}{x^{1/2}}=e^C$ so that $x_1=\frac{y}{x^{1/2}}$ solving $\frac{dy_1}{dy}=\frac{1}{\eta}=\frac{1}{y}$

Given $y_1=\log y$ some simple calculus then shows that the original differential equation transforms to

$\frac{\frac{1}{2}x_1^{\ 3}\frac{dy_1}{dx_1}}{x_1\frac{dy_1}{dx_1}-1}=\frac{x_1}{\sqrt{x_1^{\ 2}+1}}$ which , on rearranging gives $\frac{dy_1}{dx_1}=\frac{1}{x_1(1-\frac{1}{2}x_1\sqrt{x_1^{\ 2}+1}}$

This is the separable equation that we are promised by the theory we have developed the final integration of the equation can be achieved by the successive substitutions $z=\log x_1$ and $t^2=1+e^{-2z}$

2- Consider the equation $\frac{dy}{dx}=\frac{y}{x+x^2+y^2}$

In this case $\frac{\partial F}{\partial x}=\frac{-(1+2x)y}{(x+x^2+y^2)^2}$, $\frac{\partial F}{\partial y}=\frac{x+x^2-y^2}{(x+x^2+y^2)^2}$

$(x+x^2+y^2)\eta-(1+2x)y\xi=(x+x^2+y^2)^2\eta_x+(\eta_x-\xi_x)y(x+x^2+y^2)-y^2\xi_x$

If now choose $\eta=1$, this reduce to

$(x+x^2+y^2)+(1+2x)y\xi=y(x+x^2+y^2)\xi_x+y^2\xi_x$

It is not easy to solve this equation in general , but after some trial and error we can find the solution $\xi=x/y$. The infinitesimal transformation in this case is $x_1=x+cx/y$, $y_1=y+\varepsilon$ the procedure outlined in the last section we now solve

$$\frac{dx}{x/y}=\frac{dy}{1}$$

To obtain $y/x=$constant we therefore take $x_1=y/x$ and hence $y_1=y$ as our new variable in terms of these variables become

$$\frac{dy_1}{dx_1}=\frac{1}{(1+x_1^{\ 2})}$$

With solution $y_1=-\tan^{-1}x_1+C$, the solution of our original differential equation can therefore in the from $y+\tan^{-1}(y/x)=C$

Conclusion:-

In this paper we have attempted to give a simple, self-contained introduction to the use of Lie group methods for the solution of first-order ODEs and higher order ODEs . The method applied to such equations is particularly nice in that a geometric interpretation can be given. The Lie group method of solving higher order ODEs, and systems of differential equations is more involved, but the basic idea is the same: we find a coordinate system in which the equations are simpler and exploit this simplification

References

Dresner, L. Applications of Lie's Theory of Ordinary and Partial Dif- ferential Equations, Institute of Physics Publishing, Bristol, 1999.

Hydon, P. E Symmetry Methods for Differential Equations, Cambridge University Press, Cambridge, 2000.

Edwards, C. Henry and Penney, David E. Differential Equations: Computing and Modeling. Upper Saddle River, NJ: Prentice Hall, 2000.

Borel, Armand Essays in the history of Lie groups and algebraic groups, History of Mathematics, 21, Providence, R.I.: American Mathematical Society 2001

Hydon, Peter E. Symmetry Methods for Differential Equations: A Begin- ner's Guide. Cambridge, UK: Cambridge University Press, 2008.