Fuzzy Proper Mapping

Habeeb Kareem Abdullah

University of Kufa College of Education for Girls Department of Mathematics

Abstract

The purpose of this paper is to construct the concept of fuzzy proper mapping in fuzzy topological spaces . We give some characterization of fuzzy compact mapping and fuzzy coercive mapping . We study the relation among the concepts of fuzzy proper mapping , fuzzy compact mapping and fuzzy coercive mapping and we obtained several properties.

الخلاصة

الهدف من هذا البحث هو بناء التطبيق السديد الضبابي في الفضاء التوبولوجي الضبابي . ونعطي بعض خصائص التطبيق المتراص الضبابي والتطبيق الاضطراري الضبابي . ودراسةالعلاقة بين المفاهيم التطبيق السديد الضبابي ، والتطبيق المتراص الضبابي والتطبيق الاضطراري الضبابي والحصول على العديد من الخصائص .

1. Introduction

The concept of fuzzy sets and fuzzy set operation were first introduced by (L. A. Zadeh). Several other authors applied fuzzy sets to various branches of mathematics . One of these objects is a topological space .At the first time in 1968 , (C .L. Chang) introduced and developed the concept of fuzzy topological spaces and investigated how some of the basic ideas and theorems of point – set topology behave in this generalized setting . Moreover , many properties on a fuzzy topologically space were prove them by Chang 's definition .

In this paper we introduce and discuss the concepts of fuzzy proper mapping correspondence from a fuzzy topological space to another fuzzy topological space and we obtained several properties and characterization of these mappings by comparing with the other mappings .

2. Preliminaries

First , we present some fundamental definitions and proposition which are needed in the next sections .

Definition 2.1.(M. H. Rashid and D. M. Ali). Let X be a non – empty set and let I be the unit interval , i.e., $I = [0,1]$. A fuzzy set in X is a function from X into the unit interval I (i.e., $A : X \longrightarrow [0,1]$ be a function) .

A fuzzy set A in X can be represented by the set of pairs : $A = \{(x, A(x)) : x \in X\}$. The family of all fuzzy sets in X is denoted by I^X .

Remark 2.2.

(i) 0_X (**the empty set**) is a fuzzy set which has membership defined by $0_X(x) = 0$ for all $x \in X$.

(ii) 1_X (**the universal set**) is a fuzzy set which has membership defined by $1_X(x) = 1$ for all $x \in X$.

Definition 2.3. let A , B and A_i , $i \in I$ be any fuzzy sets in X . Then we put :

(i) $A \leq B$ if and only if $A(x) \leq B(x)$, $\forall\, x \in X$;

(ii) $A = B$ if and only if $A(x) = B(x)$, $\forall\, x \in X$;

(iii) $Z = A \wedge B$ if and only if $Z(x) = min\{A(x), B(x)\}$, $\forall x \in X$; (Z is a fuzzy set in X) ;

(iv) $Z = A \vee B$ if and only if $Z(x) = max\,\{A(x), B(x)\}$, $\forall x \in X$; (Z is a fuzzy set in X) ;

(v) $Z = \vee_{i\in I} A_i$ if and only if $Z(x) = \sup\,\{A_i(x)/\, i \in I\}$, $\forall\, x \in X$ (Z is a fuzzy set in X) ;

(vi) $Z = \wedge_{i\in I} A_i$ if and only if $Z(x) = \inf\{A_i(x)/\, i \in I\}$, $\forall\, x \in X$ (Z is a fuzzy set in X) ;

(vii) $E = A^c$ (the complement of A) if and only if $E(x) = 1 - A(x)$, $\forall\, x \in X$;

$(viii)$ $(A \backslash B)(x) = A(x) \wedge B^c(x)$,$\forall\, x \in X$.

Definition 2.4. (M. H. Rashid and D. M. Ali). Let X and Y be two non – empty sets $f : X \longrightarrow Y$ be function . For a fuzzy set B in Y , the inverse image of B under f is the fuzzy set $f^{-1}(B)$ in X with membership function denoted by the rule :

$f^{-1}(B)(x) = B(f(x))$ for $x \in X$ (i.e., $f^{-1}(B) = B \circ f$).

For a fuzzy set A in X , the image of A under f is the fuzzy set $f(A)$ in Y with membership function $f(A)(y)$, $y \in Y$ defined by

$$f(A)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} A(x) & if\ f^{-1}(y) \neq 0 \\ 0 & if\ f^{-1}(y) = 0 \end{cases}$$

Where $f^{-1}(y) = \{x : f(x) = y\}$.

Definition 2.5.(B. Sikin)and(M. H. Rashid and D. M. Ali). A fuzzy point x_α in X is a fuzzy set defined as follows

$$x_\alpha(y) = \begin{cases} \alpha & if\ y = x \\ 0 & if\ y \neq x \end{cases}$$

Where $0 < \alpha \leq 1$; α is called its value and x is support of x_α .

The set of all fuzzy points in X will be denoted by $FP(X)$.

Definition 2.6.(A. A. Nouh), (M. H. Rashid and D. M. Ali). A fuzzy point x_α is said to belong to a fuzzy set A in X (denoted by : $x_\alpha \in A$) if and only if $\alpha \leq A(x)$.

Definition 2.7. (A. A. Nouh), (M. H. Rashid and D. M. Ali). A fuzzy set A in X is called quasi – coincident with a fuzzy set B in X , denoted by AqB if and only if $A(x) + B(x) > 1$, for some $x \in X$. If A is not quasi –coincident with B , then $A(x) + B(x) \leq 1$, for every $x \in X$ and denoted by $A\tilde{q}B$.

Lemma 2.8.(B. Sikin). Let A and B are fuzzy sets in X . Then :

(i) If $A \wedge B = 0_X$, then $A\tilde{q}B$.

(ii) $A\tilde{q}B$ if and only if $A \leq B^c$.

Proposition 2.9. (B. Sikin). If A is a fuzzy set in X, then $x_\alpha \in A$ if and only if $x_\alpha \tilde{q} A^c$.

Definition 2.10.(C. L. Chang). A fuzzy topology on a set X is a collection T of fuzzy sets in X satisfying :

(i) $0_X \in T$ and $1_X \in T$,

(ii) If A and B belong to T , then $A \wedge B \in T$

(iii) If A_i belongs to T for each $i \in I$ then so does $\vee_{i \in I} A_i$.

If T is a fuzzy topology on X, then the pair (X,T) is called a fuzzy topological space . Members of T are called fuzzy open sets . Fuzzy sets of the forms $1_X - A$, where A is fuzzy open set are called fuzzy closed sets .

Definition 2.11. (M. H. Rashid and D. M. Ali). A fuzzy set A in a fuzzy topological space (X,T) is called quasi-neighborhood of a fuzzy point x_α in X if and only if there exists $B \in T$ such that $x_\alpha qB$ and $B \leq A$.

Definition 2.12.(M. H. Rashid and D. M. Ali). Let (X,T) be a fuzzy topological space and $x_\alpha$be a fuzzy point in X . Then the family $N^Q_{x_\alpha}$ consisting of all quasi-neighborhood (q-neighborhood) of x_α is called the system of quasi-neighborhood of x_α .

Remark 2.13. Let (X,T) be a fuzzy topological space and $A \in FP(X)$. Then A is fuzzy open if and only if A is q – neighbourhood of each its fuzzy point .

Definition 2.14.(A. A. Nouh). A fuzzy topological space (X,T) is called a fuzzy hausdorff (fuzzy T_2- space **)** if and only if for any pair of fuzzy points x_r, y_s such that $x \neq y$ in X , there exists $A \in N^Q_{x_r}$,$B \in N^Q_{y_s}$ and $A \wedge B = 0_X$

Definition 2.15.(D. L. Foster). Let A be a fuzzy set in X and T be a fuzzy topology on X . Then the induced fuzzy topology on A is the family of fuzzy subsets of A which are the intersection with A of fuzzy open set in X . The induced fuzzy topology is denoted by T_A , and the pair (A, T_A) is called **a fuzzy subspace** of X .

Proposition 2.16. Let $A \leq Y \leq X$ .Then :

(i) If A is a fuzzy open set in Y and Y is a fuzzy open set in X, then A is a fuzzy open set in X .

(ii) If A is a fuzzy closed set in Y and Y is a fuzzy closed set in X , then A is a fuzzy closed set in X .

Definition 2.17.(X. Tang**),**(S. M. AL-Khafaji). Let (X,T) be a fuzzy topological space and $A \in I^X$. Then :

(i) The union of all fuzzy open sets contained in A is called the fuzzy interior of A and denoted by A°. i.e., $A^\circ = \vee \{B : B \leq A, B \in T\}$.

(ii) The intersection of all fuzzy closed sets containing A is called the fuzzy closure of A and denoted by $\overline{A}$. i.e., $\overline{A} = \wedge \{B : A \leq B, B^c \in T\}$.

Remarks 2.18. (S. M. AL-Khafaji).

(i) The interior of a fuzzy set A is the largest open fuzzy set contained in A and trivially , a fuzzy set A is fuzzy open set if and only if $A = A^\circ$.

(ii) The closure of a fuzzy set A is te smallest closed fuzzy set containing A and trivially , a fuzzy set A is fuzzy closed if and only if $A = \overline{A}$.

Proposition 2.19. Let (X,T) be a fuzzy topological space and A be a fuzzy set in X . A fuzzy point $x_\alpha \in \overline{A}$ if and only if for every fuzzy open set B in X , if $x_\alpha q B$ then AqB .

Proof :

$\Rightarrow$ Suppose that B is a fuzzy open set in X such that $x_\alpha q B$ and $A\tilde{q}B$. Then $A \leq B^c$. But $x_\alpha \notin B^c$ (since $x_\alpha q B$, then $\alpha > B^c(x)$) and B^c is a fuzzy closed set in X . Thus $x_\alpha \notin \overline{A}$.

$\Leftarrow$ Let $x_\alpha \notin \overline{A}$,then there exists a fuzzy closet set B in X such that $A \leq B$ and $x_\alpha \notin B$, hence by Proposition (2.7) , we have $x_\alpha q B^c$. Since $A \leq B$, then by lemma (2.8 .ii) , $A\tilde{q}B^c$. This completes the proof .

Definition 2.20.(S. M. AL-Khafaji). Let X and Y be fuzzy topological spaces . A map $f: X \longrightarrow Y$ is called fuzzy continuous if and only if for every fuzzy point x_α in X and for every fuzzy open set A such that $f(x_\alpha) \in A$, there exists fuzzy open B of Y such that $x_\alpha \in B$ and $f(B) \leq A$.

Theorem 2.21. (M. H. Rashid and D. M. Ali)**.**Let X ,Ybe fuzzy topological spaces and let $f: X \longrightarrow Y$ be a mapping . then the following statements are equivalent :

(i) f is fuzzy continuous .

(ii) For each fuzzy open set B in Y , $f^{-1}(B)$ is a fuzzy open set in X .

(ii) For each fuzzy closed set Bin Y , then $f^{-1}(B)$ is a fuzzy closed set in X.

(iii) For each fuzzy set B in Y , $\overline{f^{-1}(B)} \leq f^{-1}(\overline{B})$.

(iv) For each fuzzy set A in X , $f(\overline{A}) \leq \overline{f(A)}$.

(v) For each fuzzy set B in Y , $f^{-1}(B^\circ) \leq \left(f^{-1}(B)\right)^\circ$.

Proposition 2.22. If $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ are fuzzy continuous , then $f \circ g: X \longrightarrow Z$ is fuzzy continuous mapping .

Proposition 2.23. Let (X, T) be a fuzzy topological space and A be a non - empty fuzzy subset of X , then the fuzzy inclusion $i_A: (A, T_A) \longrightarrow (X, T)$ is a fuzzy continuous mapping .

Proof Let $B \in T$. Since $i_A{}^{-1}(B) = B \wedge A$, then $i_A{}^{-1}(B) \in T_A$. Therefore i_A is fuzzy continuous .

Proposition 2.24. Let X , Y be fuzzy topological spaces and A be a fuzzy subset of X . If $f: X \longrightarrow Y$ is fuzzy continuous , then the restriction $f_{|A}: A \longrightarrow Y$ is fuzzy continuous . **Proof** Since f is fuzzy continuous and $f \circ i_A = f_{|A}$. Then by proposition (2. 23) and proposition (2.22) , $f_{|A}$ is fuzzy continuous .

Definition 2.25. Let $f: X \longrightarrow Y$ be a mapping of fuzzy spaces . Then f is called fuzzy closed mapping if $f(A)$ is a fuzzy closed set in Y for every fuzzy closed set A in X.

Proposition 2.26 If $f: X \longrightarrow Y$, $g: Y \longrightarrow Z$ are fuzzy closed mapping , then $g \circ f: X \longrightarrow Z$ is a fuzzy closed .

Proposition 2.27. If (X, T) is a fuzzy topological space and A is a fuzzy closed subset of X , then the fuzzy inclusion $i_A: A \longrightarrow X$ is a fuzzy closed mapping .

Proof Let F be a fuzzy closed set in A . Since A is a fuzzy closed in X and $i_A(F) = A \wedge F$, then i_A is a fuzzy closed set in X . Hence the inclusion mapping $i_A: A \longrightarrow X$ is fuzzy closed .

Proposition 2.28. If $f: X \longrightarrow Y$ is a fuzzy closed mapping and A is a fuzzy closed set in X then the restriction mapping $f_{|A}: A \longrightarrow Y$ is a fuzzy closed mapping .

Definition 2.29 .(A. A. Nouh). A fuzzy filter base on X is a nonempty subset $\mathcal{F}$ of I^X such that

(i) $0_X \notin \mathcal{F}$.

(ii) If $A_1, A_2 \in \mathcal{F}$, then $\exists\ A_3 \in \mathcal{F}$ such that $A_3 \leq A_1 \wedge A_2$.

Definition 2.30. A fuzzy point x_α in a fuzzy topological space X is said to be a fuzzy cluster point of a fuzzy filter base $\mathcal{F}$ on X if $x_\alpha \in \bar{B}$, for all $B \in \mathcal{F}$.

Definition 2.31.(A. A. Nouh). A mapping $S: D \longrightarrow FP(X)$ is called a fuzzy net in X and is denoted by $\{S(n): n \in D\}$, where D is a directed set . If $S(n) = x^n_{\alpha_n}$ for each $n \in D$ where $x \in X$, and $\alpha_n \in (0,1]$ then the fuzzy net S is denoted as $\{x^n_{\alpha_n}, n \in D\}$ or simply $\{x^n_{\alpha_n}\}$.

Definition 2.32. (A. A. Nouh). A fuzzy net $\Im = \{y^m_{\alpha_m} : m \in E\}$ in X is called a fuzzy subnet of fuzzy net $S = \{x^n_{\alpha_n}, n \in D\}$ if and only if there is a mapping

$f: E \longrightarrow D$ such that

(i) $\Im = S \circ f$, that is , $y^i_{\alpha_i} = x^{f(i)}_{\alpha_{f(i)}}$ for each $i \in E$.

(ii) For each $n \in D$ there exists some $m \in E$ such that $f(m) \geq n$.

We shall denote a fuzzy subnet of a fuzzy net $\{x^n_{\alpha_n}, n \in D\}$ by $\{x^{f(m)}_{\alpha_{f(m)}}, m \in E\}$.

Definition 2.33. (A. A. Nouh). Let (X, T) be a fuzzy topological space and let $S = \{x^n_{\alpha_n}, n \in D\}$ be a fuzzy net in X and $A \in I^X$. Then S is said to be:

(i) Eventually with A if and only if $\exists\ m \in D$ such that $x^n_{\alpha_n} q\ A$, $\forall\ n \geq m$.

(ii) Frequently with A if and only if $\forall\ n \in D$, $\exists\ m \in D$, $m \geq n$ and $x^m_{\alpha_m} q\ A$.

Definition 2.34. (A. A. Nouh). Let (X, T) be a fuzzy topological space and $S = \{x^n_{\alpha_n} : n \in D\}$ be a fuzzy net in X and $x_\alpha \in FP(X)$. Then S is said to be :

(i) Convergent to x_α and denoted by $S \longrightarrow x_\alpha$, if S is eventually with A , $\forall\ A \in N^Q_{x_\alpha}$, x_α is called a limit point of S.

(ii) Has a cluster point x_α and denoted by $S \propto x_\alpha$, if S is frequently with A , $\forall\ A \in N^Q_{x_\alpha}$.

Remark 2.35. If $S \longrightarrow x_\alpha$, then $S \propto x_\alpha$.

The converse of remark (2. 35) , is not true in general as the following example

Example 2.36. Let $X = \{a\}$ be a set and $T = \{0_X, A, 1_X\}$ be a fuzzy topological space such that $A(a) = \frac{2}{3}$ and let $\{x^n_{\alpha_n}\} = \left\{a_{\frac{1}{2}}^{1}\ ,\ a_{\frac{1}{3}}^{2}, a_{\frac{1}{2}}^{3},\ a_{\frac{1}{3}}^{4}, \ldots\ldots\right\}$ be a fuzzy net on X. Notice that $x^n_{\alpha_n} \propto a_{\frac{1}{2}}$, but $x^n_{\alpha_n} \not\rightarrow a_{\frac{1}{2}}$.

Proposition 2.37. A fuzzy point x_α is a cluster point of a fuzzy net $\{x^n_{\alpha_n} : n \in D\}$, where $(D, \geq)$ is a directed set , in a fuzzy topological space X if and only if it has a fuzzy subnet which converges to x_α.

Proof $\Rightarrow$ Let x_α be a cluster point of the fuzzy net $\{x^n_{\alpha_n} : n \in D\}$, with the directed set $(D, \geq)$. Then for any $U \in N^Q_{x_\alpha}$, there exists $n \in D$ such that $x^n_{\alpha_n} qU$. Let $E = \{(n, U) : n \in D , U \in N^Q_{x_\alpha}\ and\ x^n_{\alpha_n} qU\}$. Then $(E, \geq)$ is directed set where $(m, U) \geq (n, V)$ if and only if $m \geq n$ in D and $U \leq V$ in $N^Q_{x_\alpha}$. Then $\Im : E \longrightarrow FP(X)$ given by $\Im(m, U) = x^m_{\alpha_m}$ is a fuzzy subnet of fuzzy net $\{x^n_{\alpha_n} : n \in D\}$. To show that $\Im : E \longrightarrow x_\alpha$. Let $B \in N^Q_{x_\alpha}$. Then there exists $n \in D$ such that $(n, B) \in E$ and $x^n_{\alpha_n} qB$. Thus for any $(m, U) \in E$ such that $(n, U) \geq (n, B)$, we have $\Im(m, U) = x^m_{\alpha_m} qU \leq B$. Hence $\Im \longrightarrow x_\alpha$.

$\Leftarrow$ If a fuzzy net $\{x^n_{\alpha_n} : n \in D\}$, has not a cluster point . Then for every fuzzy point x_α there is a fuzzy q- neighborhood of x_α and $n \in D$ such that $x^m_{\alpha_m} \tilde{q}\, U$, for all $m \geq n$. Then obviously no fuzzy net converge to x_α .

Theorem 2.38. Let (X, T) be a fuzzy topological space , $x_\alpha \in FP(X)$ and $A \in I^X$. Then $x_\alpha \in \overline{A}$ if and only if there exists a fuzzy net in A convergent to x_α .

Proof $\Rightarrow$ Let $x_\alpha \in \overline{A}$, then for every $B \in N^Q_{x_\alpha}$ there exists

$$x_B(y) = \begin{cases} A(x_\alpha) & if \quad y = x_B \\ 0 & if \quad y \neq x_B \end{cases}$$

Such that $B(x_B) + A(x_B) > 1$ notice that $(N^Q_{x_\alpha}, \geq)$ is a directed set , then $S : N^Q_{x_\alpha} \longrightarrow FP(X)$ is defined as $S(B) = x^A_B$ is a fuzzy net in A . To prove that $S \longrightarrow x_\alpha$. Let $D \in N^Q_{x_\alpha}$. Then there exists $F \in T$ such that $x_\alpha qF$ and $F \leq D$. Since $F(x^A_F) + x^A_F > 1$ and $F \leq D$. Then $D(x^A_F) + x^A_F > 1$. Thus $x^A_F qD$. Let $E \geq F$, then $E \leq F$. Since $E(x^A_E) + x^A_E > 1$ and $F \leq D$, then $D(x^A_F) + x^A_F > 1$. Thus $x^A_E qD$, $\forall\ E \geq F$. Therefore $S \longrightarrow x_\alpha$.

$\Leftarrow$ Let $\{x^n_{\alpha_n} : n \in D\}$ be a fuzzy net in A where $(D, \geq)$ is a directed set such that $x^n_{\alpha_n} \longrightarrow x_\alpha$. Then for every $B \in N^Q_{x_\alpha}$,There exists $m \in D$ such that $x^n_{\alpha_n} qB$ for all

$n \geq m$. Since $x_{\alpha_n}^n \in A$, then by proposition (2.9) , $x_{\alpha_n}^n \tilde{q} A^c$. Thus AqB . Therefore $x_\alpha \in \overline{A}$.

Proposition 2.39. If X is a fuzzy $T_2 -$ space , then every convergent fuzzy net in X has a unique limit point .

Proof :

$\Rightarrow$ Let $x_{\alpha_n}^n$ be a fuzzy net on X such that $x_{\alpha_n}^n \longrightarrow x_\alpha$, $x_{\alpha_n}^n \longrightarrow y_\beta$ and $x \neq y$. Since $x_{\alpha_n}^n \longrightarrow x_\alpha$, we have $\forall\ A \in N_{x_\alpha}^Q$,$\exists$ $m_1 \in D$, such that $x_{\alpha_n}^n qA$, $\forall$ $n \geq m_1$. Also , $x_{\alpha_n}^n \longrightarrow y_\beta$, we have $\forall\ B \in N_{y_\beta}^Q$,$\exists$ $m_2 \in D$, such that $x_{\alpha_n}^n qB$, $\forall$ $n \geq m_2$. Since D is a directed set , then there exists $m \in D$, such that $m \geq m_1$ and $m \geq m_2$,then $x_{\alpha_n}^n q(A \wedge B)$, $\forall n \geq m$. Thus $A \wedge B \neq 0$. This complete the proof .

$\Leftarrow$ Let X be a not fuzzy $T_2 -$ space , then there exists $x_\alpha , y_\beta \in FP(X)$ such that $x \neq y$ and $A \wedge B \neq 0$, $\forall A \in N_{x_\alpha}^Q$, $B \in N_{y_\beta}^Q$. Put $N_{x_\alpha, y_\beta}^Q = \{A \wedge B / A \in N_{x_\alpha}^Q$, $B \in N_{y_\beta}^Q\}$. Thus $\forall\ D \in N_{x_\alpha, y_\beta}^Q$, there exists $x_D qD$, then $\{x_D\}_{D \in N_{x_\alpha}^{y_\beta}}$ is a fuzzy net in X .To prove that $x_D \longrightarrow x_\alpha$ and $x_D \longrightarrow y_\beta$. Let $E \in N_{x_\alpha}^Q$, then $E \in N_{x_\alpha, y_\beta}^Q$ (since $E = E \wedge X \neq 0$) . Thus $x_D qE$,$\forall\ D \geq E$, thus $x_D \longrightarrow x_\alpha$. Also $x_D \longrightarrow y_\beta$, so $\{x_D\}_{D \in N_{x_\alpha}^{y_\beta}}$ has two limit point .

3. Fuzzy compact space

This section contains the definitions , proportions and theorems about fuzzy compact space and we give a new results .

Definition 3.1. A family Λ of fuzzy sets is called a cover of a fuzzy set A if and only if $A \leq \vee\{B_i : B_i \in \Lambda\}$ and Λ is called fuzzy open cover if each member B_i is a fuzzy open set . A sub cover of Λ is a subfamily of Λ which is also a cover of A .

Definition 3.2. Let (X,T) be a fuzzy topological space and let $A \in I^X$. Then A is said to be a fuzzy compact set if for every fuzzy open cover of A has a finite sub cover of A . Let $A = X$, then X is called a fuzzy compact space that is $A_i \in T$ for every $i \in I$ and $\vee_{i \in I} A_i = 1_X$, then there are finitely many indices $i_1, i_2, \dots\dots, i_n \in I$ such that $\vee_{j=1}^n A_{i_j} = 1_X$.

Example 3.3.

If (X,T) is a fuzzy topological space such that T is finite then X is fuzzy compact .

Remark 3.4 Not every fuzzy point of a fuzzy space X is fuzzy compact in general . See the following example :

Example 3.5 Let $X=\{a\}$ be a set and $T=\{0_X,1_X,a_{\frac{1}{2}-\frac{1}{n}}\ /\ n\in Z^+, n\geq 3\}$, where $a\in X$ be a fuzzy topology on X .

Notice that is a $\{\ a_{\frac{1}{2}-\frac{1}{n}}\ /\ n\geq 3\ \}$fuzzy open cover of $a_{\frac{1}{2}}$, but its has no finite sub cover for $a_{\frac{1}{2}}$. Thus $a_{\frac{1}{2}}$ is not fuzzy compact .

Then we will give the following definition .

Definition 3.6 A fuzzy topological space (X,T) is called fuzzy singleton compact space (fuzzy sc – space) if every fuzzy point of X is fuzzy compact .

Example Every fuzzy topological space with finite fuzzy topology is fuzzy sc – space .

Proposition 3.7 Let Y be a fuzzy subspace of a fuzzy topological space X and let $A\in I^Y$. Then A is fuzzy compact relative to X if and only if A is fuzzy compact relative to Y .

Proof $\Longrightarrow$ Let A be a fuzzy compact relative to X and let $\{V_\lambda:\lambda\in\Lambda\}$ be a collection of fuzzy open sets relative to Y , which covers A so that $A\leq\bigvee_{\lambda\in\Lambda}V_\lambda$, then there exist G_λ fuzzy open relative to X , such that $V_\lambda=Y\wedge G_\lambda$ for any $\lambda\in\Lambda$. It then follows that $A\leq\bigvee_{\lambda\in\Lambda}G_\lambda$. So that $\{G_\lambda:\lambda\in\Lambda\}$ is fuzzy open cover of A relative to X . Since A is fuzzy compact relative to X , then there exists a finitely many indices $\lambda_1,\lambda_2,\ldots,\lambda_n\in\Lambda$ such that $A\leq\bigvee_{i=1}^{n}G_{\lambda_i}$. since $A\leq Y$, we have $A=Y\wedge A\leq Y\wedge(G_{\lambda_1}\vee G_{\lambda_2}\vee\ldots\vee G_{\lambda_n})=(Y\wedge G_{\lambda_1})\ldots.(Y\wedge G_n)$, since $Y\wedge G_{\lambda_i}=V_{\lambda_i}\ (i=1,2,\ldots.n)$ we obtain $A\leq\bigvee_{i=1}^{n}V_{\lambda_i}$. Thus show that A is fuzzy compact relative to Y .

$\Longleftarrow$ Let A be fuzzy compact relative to Y and let $\{G_\lambda:\lambda\in\Lambda\}$ be a collection of fuzzy open cover of X , so that $A\leq\bigvee_{\lambda\in\Lambda}G_\lambda$. Since $A\leq Y$, we have $A=Y\wedge A\leq Y\wedge(\bigvee_{\lambda\in\Lambda}G_\lambda)=\bigvee_{\lambda\in\Lambda}(Y\wedge G_\lambda)$. Since $Y\wedge G_\lambda$ is fuzzy open relative to Y , then the collection $\{Y\wedge G_\lambda:\lambda\in\Lambda\}$ is a fuzzy open cover relative to Y . Since A is fuzzy compact relative to Y , we must have $A\leq(Y\wedge G_{\lambda_1})\vee(Y\wedge G_{\lambda_2})\vee\ldots\vee(Y\wedge G_{\lambda_n})$ ……(*) for some choice of finitely many indices $\lambda_1,\lambda_2,\ldots,\lambda_n$. But (*) implies that $A\leq\bigvee_{i=1}^{n}G_{\lambda_i}$. It follows that A is fuzzy compact relative to X .

Theorem 3.8 A fuzzy topological space (X,T) is fuzzy compact if and only if for every collection $\{A_j:j\in J\}$ of fuzzy closed sets of X having the finite intersection property , $\bigwedge_{j\in J}A_j\neq 0_X$.

Proof $\Longrightarrow$ Let $\{A_j:j\in J\}$ be a collection of fuzzy closed sets of X with the finite intersection property . Suppose that $\bigwedge_{j\in J}A_j\neq 0_X$, then $\bigvee_{j\in J}A_j^{\,c}=1_X$. Since X is fuzzy compact , then there exists $j_1, j_2, \dots, j_n$ such that $\bigvee_{i=1}^{n} A_{j_i}{}^{c} = 1_X$ .Then $\bigwedge_{i=1}^{n} A_{j_i} = 0_X$. Which gives a contradiction and therefore $\bigwedge_{j\in J} A_j \neq 0_X$.

$\Leftarrow$ let $\{A_j : j \in J\}$ be a fuzzy open cover of X . Suppose that for every finite $j_1, j_2, \dots, j_n$, we have $\bigvee_{i=1}^{n} A_{j_i} \neq 1_X$. then $\bigwedge_{i=1}^{n} A_{j_i}{}^{c} \neq 0_X$. Hence $\{A_j{}^{c} : j \in J\}$ satisfies the finite intersection property . Then from the hypothesis we have $\bigwedge_{j\in J} A_j{}^{c} \neq 0_X$. Which implies $\bigwedge_{i=1}^{n} A_{j_i} \neq 1_X$ and this contradicting that $\{A_j : j \in J\}$ is a fuzzy open cover of X . Thus X is fuzzy compact .

Theorem 3.9. A fuzzy closed subset of a fuzzy compact space is fuzzy compact .

Proof : Let A be a fuzzy closed subset of a fuzzy space X and let $\{B_i : i \in I\}$ be any family of fuzzy closed in A with finite intersection property , since A is fuzzy closed in X , then by proposition (2. 16 . ii) , B_i are also fuzzy closed in X , since X is fuzzy compact , then by proposition (3 . 8) , $\bigwedge_{i\in I} B_i \neq 0_X$. Therefore A is fuzzy compact .

Theorem 3.10. A fuzzy topological space (X, T) is a fuzzy compact if and only if every fuzzy filter base on X has a fuzzy cluster point .

Proof $\Rightarrow$ Let X be fuzzy compact and let $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a fuzzy filter base on X having no a fuzzy cluster point . Let $x \in X$. Corresponding to each $n \in N$ (N denoted the set of natural numbers), there exists a fuzzy q-neighbourhood U_x^n of the fuzzy point $x_{\frac{1}{n}}$ and an $F_x^n \in \mathcal{F}$ such that $U_x^n \tilde{q} F_x^n$. Since $1 - \frac{1}{n} < U_x^n(x)$, we have $U_x(x) = 1$, where $U_x = \bigvee\{U_{x_n} : n \in N\}$. Thus $\mathcal{U} = \{U_x^n : n \in N , x \in X\}$ is a fuzzy open cover of X . Since X is fuzzy compact , then there exists finitely many members $U_{x_1}^{n_1}, U_{x_2}^{n_2}, \dots\dots, U_{x_k}^{nk}$ of $\mathcal{U}$ such that $\bigvee_{i-1}^{k} U_{x_i}^{n_i} = 1_X$. Since $\mathcal{F}$ is fuzzy filter base , then there exists $F \in \mathcal{F}$ such that $F \leq F_{x_{n1}} \wedge F_{x_{n2}} \wedge \dots\dots \wedge F_{x_{nk}}$. But $U_{x_i}^{n_i} \tilde{q} F_{x_i}^{n_i}$, then $F \tilde{q}\, 1_X$. Consequently , $F = 0_X$ and this contradicts the definition of a fuzzy filter base .

$\Leftarrow$ Let $\beta = \{F_\alpha : \alpha \in \Lambda\}$ be a family of fuzzy closed sets having finite intersection property .Then the set of finite intersections of members of β forms a fuzzy filter base $\mathcal{F}$ on X . So by the condition $\mathcal{F}$ has a fuzzy cluster point say x_s . Thus $x_s \in F_\alpha$. So $x_s \in \bigwedge_{\alpha\in\Lambda} \overline{F_\alpha} = \bigwedge_{\alpha\in\Lambda} F_\alpha$. Thus $\bigwedge\{F , F \in \mathcal{F}\} \neq 0_X$. Hence by theorem (3 . 8) , X is fuzzy compact .

Theorem 3.11. A fuzzy topological space (X, T) is fuzzy compact if and only if every fuzzy net in X has a cluster point .

Proof $\Rightarrow$ Let X be fuzzy compact . Let $\{S(n) : n \in D\}$ be a fuzzy net in X which has no cluster point , then for each fuzzy point x_α , there is a fuzzy q – neighbourhood U_{x_α} of x_α and an $n_{U_{x\alpha}} \in D$ such that $S_m \tilde{q}\, U_{x_\alpha}$ for all $m \in D$ with $m \geq n_{U_{x\alpha}}$. Since $x_\alpha q U_{x_\alpha}$, then $S_m \neq 0$, $\forall\ m \geq n_{U_{x\alpha}}$. Let $\mathcal{U}$ denoted the collection of all U_{x_α} , where x_α runs over all fuzzy points in X . Now to prove that the collection

$V = \{1_X - U_{x_\alpha} : U_{x_\alpha} \in \mathcal{U}\}$ is a family of fuzzy closed sets in X possessing finite intersection property . First notice that there exists $k \geq n_{U_{\alpha_1}}, \ldots, n_{U_{\alpha m}}$ such that $S_p \tilde{q}\, U_{x_{\alpha_i}}$ for $i = 1,2,\ldots, m$ and for all $p \geq k$ $(p \in D)$, i.e. $S_p \in 1_X - \vee_{i=1}^{m} U_{x_{\alpha_i}} = \wedge_{i=1}^{m}(1_X - U_{x_{\alpha_i}})$ for all $p \geq k$. Hence $\wedge\{1_X - U_{x_{\alpha_i}} : i = 1,2,\ldots,m\} \neq 0_X$. Since X is fuzzy compact , by theorem (3 . 5) , there exists a fuzzy point y_β in X such that $y_\beta \in \wedge\{1_X - U_{x_\alpha} : U_{x_\alpha} \in \mathcal{U}\} = 1_X - \vee\{U_{x_\alpha} : U_{x_\alpha} \in \mathcal{U}\}$. Thus $y_\beta \in 1_X - U_{x_\alpha}$, for all $U_{x_\alpha} \in \mathcal{U}$ and hence in particular , $y_\beta \in 1_X - U_{y_\beta}$, i.e., $y_\beta \tilde{q}\, U_{y_\beta}$. But by construction , for each fuzzy point x_α , there exists $U_{x_\alpha} \in \mathcal{U}$ Such that $x_\alpha q U_{x_\alpha}$, and we arrive at a contradiction .

⇐ To prove that converse by theorem (3 . 10) , that every fuzzy filter base on X has a cluster point . Let $\mathcal{F}$ be a fuzzy filter base on X . Then each $F \in \mathcal{F}$ is non empty set , we choose a fuzzy point $x_F \in F$. Let $S = \{x_F : F \in \mathcal{F}\}$ and let a relation " $\geq$ " be defined in $\mathcal{F}$ as follows $F_\alpha \geq F_\beta$ if and only if $F_\alpha \leq F_\beta$ in X, for $F_\alpha, F_\beta \in \mathcal{F}$. Then $(\mathcal{F}, \geq)$ is directed set . Now S is a fuzzy net with the directed set $(\mathcal{F}, \geq)$. By hypothesis the fuzzy net S has a cluster point x_t . Then for every fuzzy q – neighourhood W of x_t and for each $F \in \mathcal{F}$, there exists $G \in \mathcal{F}$ with $G \geq F$ such that $x_G q W$. As $x_G \leq G \leq F$. It follows that FqW for each $F \in \mathcal{F}$, then by proposition

⇐ (2 . 19) , $x_t \in \overline{F}$. Hence x_t is a cluster point of $\mathcal{F}$.

Corollary 3.12. A fuzzy topological space (X, T) is fuzzy compact if and only if every fuzzy net in X has a convergent fuzzy subnet .

Proof : By proposition (2.37) , and theorem (3.11) .

Theorem 3.13. Every fuzzy compact subset of a fuzzy Hausdroff topological space is fuzzy closed .

Proof : Let $x_\alpha \in \overline{A}$, then by theorem (2. 34) , there exists fuzzy net $x_{\alpha_n}^n$ such that $x_{\alpha_n}^n \longrightarrow x_\alpha$. Since A is fuzzy compact and X is fuzzy $T_2 -$ space , then by corollary (3. 12) and proposition (2 .39) , we have $x_\alpha \in A$. Hence A is fuzzy closed set .

Theorem 3.14. In any fuzzy space , the intersection of a fuzzy compact set with a fuzzy closed set is fuzzy compact .

Proof Let A be a fuzzy compact set and B be a fuzzy closed set . To prove that $A \wedge B$ is a fuzzy compact set . Let $x_{\alpha_n}^n$ be a fuzzy net in $A \wedge B$. Then $x_{\alpha_n}^n$ is fuzzy net in A , since A is fuzzy compact , then by corollary (3.12) , $x_{\alpha_n}^n \longrightarrow x_\alpha$ for some $x_\alpha \in FP(X)$ and by proposition (2.38) , $x_\alpha \in \overline{B}$. since B is fuzzy closed , then $x_\alpha \in B$. Hence $x_\alpha \in A \wedge B$ and $x_{\alpha_n}^n \longrightarrow x_\alpha$. Thus $A \wedge B$ is fuzzy compact .

Proposition 3.15 Let X and Y be fuzzy spaces and $f: X \to Y$ be a fuzzy continuous mapping . If U is a fuzzy compact set in X , then $f(U)$ is a fuzzy compact set in Y .

Proof : Let $\{V_i : i \in I\}$ be a fuzzy open cover of $f(U)$ in Y, i.e., $(f(U) \le \vee_{i\in I} G_i)$. Since f is a fuzzy continuous , then $f^{-1}(G_i)$ is a fuzzy open set in X , $\forall i \in I$. Hence the collection $\{f^{-1}(G_i) : i \in I\}$ be a fuzzy open cover of U in X ,i.e., $U \le f^{-1}(f(U)) \le f^{-1}(\vee_{i\in I} G_i) = \vee_{i\in I} f^{-1}(G_i)$. Since U is a fuzzy compact set in X , then there exists finitely many indices $i_1, i_2, \ldots \ldots, i_n$ Such that $U \le \vee_{j=1}^{n} f^{-1}(G_{i_j})$, so that$f(U) \le f(\vee_{j=1}^{n}(f^{-1}(G_{i_j})) = \vee_{j=1}^{n}(f(f^{-1}(G_{i_j}))) \le \vee_{j=1}^{n} G_{i_j}$ .Hence $f(U)$ is a fuzzy compact .

4. Compactly fuzzy closed set .

The section will contain the definition of compactly fuzzy closed set and we give new results .

Definition 4.1. Let X be a fuzzy space . Then a fuzzy subset W of X is called

compactly fuzzy closed set if $W \wedge K$ is fuzzy compact , for every fuzzy compact set K in X .

Example 4.2. Every fuzzy subset of indiscrete fuzzy topological space is compactly

fuzzy closed set .

Proposition 4.3. Every fuzzy closed subset of a fuzzy space X is compactly fuzzy closed

Proof Let A be a fuzzy closed subset of a fuzzy space X and let K be a fuzzy compact set . Then by theorem (3.14) , $A \wedge K$ is a fuzzy compact . Thus A is a compactly fuzzy closed set .

The converse of proposition (4.3) , is not true in general as the following example show :

Example 4.4. Let $X = \{a, b\}$ be a set and T be the indiscrete fuzzy space on X . Notice that $A = \{0.2, 0.3\}$ is compactly fuzzy closed set , but its not fuzzy closed set.

Theorem 4.5. Let X be a fuzzy $T_2 -$ space . A fuzzy subset A of X is compactly fuzzy closed if and only if A is fuzzy closed .

Proof $\Longrightarrow$ Let A be a compactly fuzzy closed set in X and $x_\alpha \in \overline{A}$. Then by proposition (2.34) , there exists a fuzzy net $x_{\alpha_n}^{n}$ in A , such that $x_{\alpha_n}^{n} \longrightarrow x_\alpha$, then by corollary (3.12) , $F = \{x_{\alpha_n}^{n}, x_\alpha\}$ is a fuzzy compact set . Since A is compactly fuzzy closed , then $A \wedge F$ is a fuzzy compact set . But X is a fuzzy $T_2 -$ space , then by theorem (3.13) , $A \wedge F$ is fuzzy closed . Since $x_{\alpha_n}^{n} \longrightarrow x_\alpha$ and $x_{\alpha_n}^{n} \in A \wedge F$, then by proposition (2.38) , $x_\alpha \in A \wedge F$ so $x_\alpha \in A$. Hence $\overline{A} \le A$. Therefore A is a fuzzy closed set .

$\Leftarrow$ By proposition (4.3) .

5. Fuzzy compact mapping.

The section will contain the concept of fuzzy compact mapping and we give new results .

Definition 5.1. Let X and Y be fuzzy spaces. A mapping $f: X \longrightarrow Y$ is called a **fuzzy** compact mapping if the inverse image of each fuzzy compact set in Y , is a fuzzy compact set in X .

Example 5.2. Let (X,T) and (Y,τ) be fuzzy topological spaces , such that T is a finite fuzzy topology , then the mapping $f: X \longrightarrow Y$ is fuzzy compact .

Proposition 5.3. Let X ,Y and Z be fuzzy spaces , If $f: X \longrightarrow Y$, $g: Y \longrightarrow Z$ are fuzzy continuous mapping . Then :-

(i) If f and g are a fuzzy compact mappings , then $g \circ f$ is a fuzzy compact mapping

(ii) If $g \circ f$ is a fuzzy compact mapping , f is onto , then g is a fuzzy compact mapping .

(iii) If $g \circ f$ is a fuzzy compact mapping , g is one to one , then f is a fuzzy compact mapping .

Proof : (i) Let D be a fuzzy compact set in Z . Since g is fuzzy compact mapping , then $g^{-1}(D)$ is a fuzzy compact set in Y . Since f is a fuzzy compact mapping , then $f^{-1}(g^{-1}(D))$ is a fuzzy compact set in X . Hence $g \circ f: X \longrightarrow Z$ is a fuzzy compact mapping .

(ii) Let D be a fuzzy compact set in Z , then $(g \circ f)^{-1}(D)$ is a fuzzy compact set in X , and so $f(g \circ f)^{-1}(D)$ is a fuzzy compact set in Y . Now , since f is onto , then $f(g \circ f)^{-1}(D) = g^{-1}(D)$, therefore f is a fuzzy compact mapping .

(iii) Let D be a fuzzy compact set in Y . Since g is a fuzzy continuous mapping , then $g(D)$ be a fuzzy compact set in Z . Since $g \circ f$ is a fuzzy compact mapping , then $(g \circ f)^{-1}(g(D))$ is a fuzzy compact set in X . Since g is one to one , then $(g \circ f)^{-1}(g(D)) = f^{-1}(D)$. Hence $f^{-1}(D)$ is a fuzzy compact set in X. Then f is a fuzzy compact mapping .

Proposition 5.4. For any fuzzy closed subset F of a fuzzy space X, the inclusion $i_F: F \longrightarrow X$ is a fuzzy compact mapping .

Proof : Let K be a fuzzy compact set in X , then by theorem (3.14) , $F \wedge K$ is a fuzzy compact set in F . But $i_F^{-1}(K) = F \wedge K$, then $i_F^{-1}(K)$ is a fuzzy compact set in F , therefore the inclusion mapping $i_F: F \longrightarrow X$ is fuzzy compact .

Proposition 5.5. Let X and Y be fuzzy space , and $f: X \longrightarrow Y$ be a fuzzy compact mapping . If F is fuzzy closed subset of X , then $f_{|_F}: F \longrightarrow Y$ is fuzzy compact mapping .

Proof : Since F is a fuzzy closed subset of X , then by proposition (5.4) , the inclusion $i_F: F \longrightarrow X$ is a fuzzy compact mapping . But $f_{|_F} = f \circ i_F$, then by proposition (6.3 i) , $f_{|_F}$ is fuzzy compact mapping .

6. Fuzzy Coercive Mapping .

The section will contain the definition of a fuzzy coercive mapping and the relation between fuzzy compact mapping and the fuzzy coercive mapping .

Definition 6.1. Let X and Y be fuzzy space . A mapping $f: X \longrightarrow Y$ is called **a fuzzy coercive** if for every fuzzy compact set $G \leq Y$, there exists a fuzzy compact set $K \leq X$ such that $f(1_X \backslash K) \leq (1_Y \backslash G)$.

Example 6.2. If X is a fuzzy compact space , then the mapping $f: X \longrightarrow Y$ is fuzzy coercive .

Solution : Let G be a fuzzy compact set in Y . Since X is fuzzy compact space and $f(1_X \backslash 1_X) = f(0_X) = 0_X \leq (1_Y \backslash G)$, then f is fuzzy coercive mapping .

Proposition 6.3. Every fuzzy compact mapping is a fuzzy coercive mapping .

Proof Let $f: X \longrightarrow Y$ be a fuzzy compact mapping . To prove that f is a fuzzy coercive . Let G be a fuzzy compact set in Y . Since f is a fuzzy compact mapping , then $f^{-1}(G)$ is a fuzzy compact set in X. Thus $f\left(1_X \backslash f^{-1}(G)\right) \leq (1_Y \backslash G)$. Hence $f: X \longrightarrow Y$ is a fuzzy coercive mapping.

The converse of proposition (6.3) , is not true in general as the following example shows :

Example 6.4 Let $X = \{a\}, Y = \{b\}$ be sets and $T = \{0_X, 1_X, a_{\frac{1}{2}-\frac{1}{n}} \,/\, n \in Z^+, n \geq 3$ } , $\tau = \{0_X, 1_X, b_{\frac{1}{n}} \,/ n \in Z^+\}$ be fuzzy topology on X and Y respectively .

Let $f: X \longrightarrow Y$ be a mapping which is defined by : $f(a) = b$. Notice that f is a fuzzy coercive mapping , but its not fuzzy compact mapping .

Proposition 6.5. Let X and Y be fuzzy spaces such that Y is a fuzzy T_2 - space , and $f: X \longrightarrow Y$ is a fuzzy continuous mapping . Then f is a fuzzy coercive if and only if f is a fuzzy compact mapping .

Proof $\Longrightarrow$ Let G is a fuzzy compact set in Y . To prove that $f^{-1}(G)$ is a fuzzy compact set in X . Since Y a fuzzy T_2 – space and f is a fuzzy continuous mapping , then $f^{-1}(G)$ is a fuzzy closed set in X . Since f is a fuzzy coercive mapping , then there exists a fuzzy compact set K in X , such that $f(1_X\backslash K) \le (1_Y\backslash G)$. Then $f(K^c) \le G^c$, therefore $f^{-1}(G) \le K$, thus $f^{-1}(G)$ is a fuzzy compact set in X . Hence f is a fuzzy compact mapping .

$\Leftarrow$ By proposition (6.3) .

Proposition 6.6. Let X, Y and Z be fuzzy spaces . If $f: X \longrightarrow Y$, $g: Y \longrightarrow Z$ are fuzzy coercive mapping , then $g \circ f: X \longrightarrow Z$ is a fuzzy coercive mapping .

Proof : Let G be a fuzzy compact set in Z . Since $g: Y \longrightarrow Z$ is a fuzzy coercive mapping , then there exists a fuzzy compact set K in Y , such that $g(1_Y\backslash K) \le 1_Z\backslash G)$. Since $f: X \longrightarrow Y$ is a fuzzy coercive mapping and $K$is a fuzzy compact set in Y , then there exists a fuzzy compact set H in X , such that $f(1_X\backslash H) \le (1_Y\backslash K) \longrightarrow g\big(f(1_X\backslash H)\big) \le g(1_Y\backslash K) \le (1_Z\backslash G) \longrightarrow (g \circ f)(1_X\backslash H) \le (1_Z\backslash G)$
. Hence $g \circ f$ is fuzzy coercive mapping .

Corollary 6.7. Let $X, Y$and Z be fuzzy spaces , such that $f: X \longrightarrow Y$ is a fuzzy compact mapping and $g: X \longrightarrow Z$ is a fuzzy coercive napping . Then $g \circ f: X \longrightarrow Z$ is a fuzzy coercive mapping .

Proposition 6.8. Let X and Y be fuzzy space and $f: X \longrightarrow Y$ be a fuzzy coercive mapping . If F is a fuzzy closed subset of X , then the restriction mapping $f_{|F}: F \longrightarrow Y$ is a fuzzy coercive mapping .

Proof : Since F is a fuzzy closed subset of X , then by proposition (5.4) , and proposition (6.3) , the inclusion mapping $i_F: F \longrightarrow X$ is a fuzzy coercive mapping . But $f_{|F} = f \circ i_F$, then by proposition (6.6) , is a fuzzy coercive mapping .

7. Fuzzy proper mapping .

The section will contain the definition of fuzzy proper mapping and addition to studying relation among fuzzy proper mapping , fuzzy compact mapping and fuzzy coercive mapping .

Definition 7.1 A fuzzy continuous mapping $f: X \longrightarrow Y$ is called **fuzzy proper** if

(i) f is fuzzy closed .

(ii) $f^{-1}(\{y_\alpha\})$ is fuzzy compact , for all $y_\alpha \in FP(Y)$.

Example 7.2 Let $X = \{x\}$ be set and T is indiscrete fuzzy topology on X and $f: (X,T) \longrightarrow (X,T)$ be the indentity mapping . Notice that f is fuzzy proper mapping .

Proposition 7.3 Let X ,Y and Z be fuzzy spaces . If $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ are fuzzy proper mappings , then $g \circ f: X \longrightarrow Z$ is fuzzy proper mapping .

Proof : By proposition (2.22) , $g \circ f$ is fuzzy continuous .

(i) Since f an g are fuzzy proper mapping , then $f$and g are fuzzy closed .

Thus by proposition (2.26) , $g \circ f : X \longrightarrow Z$ is a fuzzy closed mapping .

(ii) Let $z_\alpha \in FP(Z)$, then $g^{-1}(\{y_\alpha\})$ is fuzzy compact set in Y, and then $f^{-1}(g^{-1}(\{y_\alpha\})) = (g \circ f)^{-1}(\{y_\alpha\})$ is a fuzzy compact set in X . Therefore by (i) and (ii) , $g \circ f$ is fuzzy proper mapping

Proposition 7.4. Let X ,Y and Z be fuzzy space . If $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ are fuzzy continuous mappings , such that $g \circ f\colon X \longrightarrow Z$ is a fuzzy proper mapping . If f is onto , then g is fuzzy proper .

Proof (i) Let F be a fuzzy closed subset of Y , since f is a fuzzy continuous , then $f^{-1}(F)$ is fuzzy closed in X . Since $g \circ f$ is a fuzzy proper mapping , then $(g \circ f)(f^{-1}(F))$ is fuzzy closed in Z . But f is onto , then $(g \circ f)(f^{-1}(F)) = g(F)$. Hence $g(F)$ is a fuzzy closed in Z .Thus g is fuzzy closed mapping .

(ii) Let $Z_\alpha \in FP(Z)$. Since $g \circ f$ is a fuzzy proper mapping , then $(g \circ f)^{-1}(\{z_\alpha\}) = f^{-1}(g^{-1}(\{z_\alpha\}))$ is fuzzy compact . Since f is fuzzy continuous , then $f(f^{-1}(g^{-1}(\{z_\alpha\})))$ is fuzzy compact set , but f is onto . Then $f(f^{-1}(g^{-1}(\{z_\alpha\}))) = g^{-1}(\{z_\alpha\})$ is fuzzy compact , for every fuzzy point z_α in Z . Thus $g \circ f$ is fuzzy proper .

Proposition 7.5. Let X ,Y and Z be fuzzy space and $f\colon X \longrightarrow Y$, $g\colon Y \longrightarrow Z$ be fuzzy continuous mappings **,** such that $g \circ f\colon X \longrightarrow Z$ is a fuzzy proper mapping . If g is one to one , then f is a fuzzy proper .

Proof (i) Let F be a fuzzy closed subset of X . Then $(g \circ f)(F)$ is a fuzzy closed set in Z . Since $g\colon Y \longrightarrow Z$ is a one to one , fuzzy continuous , mapping , then $g^{-1}\left(g(f(F))\right) = f(F)$ is fuzzy closed in Y. Hence $f\colon X \longrightarrow Y$ is fuzzy closed .

(ii) Let $y_\alpha \in FP(Y)$, then $g(y_\alpha) \in Z$. Now , since $g \circ f\colon X \longrightarrow Z$ is fuzzy proper and g is a one to one , then the set $(g \circ f)^{-1}(g\{y_\alpha\}) = f^{-1}\left(g^{-1}(g(\{y_\alpha\}))\right) = f^{-1}(\{y_\alpha\})$ is fuzzy compact , for every fuzzy point y_α in Y , therefore the mapping f is fuzzy proper .

Proposition 7.6. Let X and Y be fuzzy spaces , and $f\colon X \to Y$ be fuzzy proper mapping . If A is any fuzzy clopen subset of Y , then $f_A\colon f^{-1}(A) \to A$ is a fuzzy proper mapping .

Proof : To prove that $f_A\colon f^{-1}(A) \to A$ is a fuzzy continuous mapping . Let K be a fuzzy open subset of A , then $K = A \wedge B$, for some fuzzy open set B in Y. Since

$f: X \to Y$ is a fuzzy continuous mapping , then by proposition (2.22) , $f_{|_{f^{-1}(A)}}: f^{-1}(A) \to A$ is a fuzzy continuous mapping , then $f^{-1}_{|_{f^{-1}(A)}}(B)$ is a fuzzy open set in $f^{-1}(A)$, but $f_A^{-1}(K) = f^{-1}(A) \wedge f^{-1}_{|_{f^{-1}(A)}}(B)$, thus $f_A^{-1}(K)$ is a fuzzy open set in $f^{-1}(A)$. Hence $f_A: f^{-1}(A) \to A$ is a fuzzy continuous mapping .

(i) By proposition (2.28) , f_T is fuzzy closed .

(ii) Let $t_\alpha \in FP(A)$. Since f is fuzzy proper , then $f^{-1}(\{t_\alpha\})$ is fuzzy compact in Y , since T is fuzzy closed and f is fuzzy continuous , then $f^{-1}(A)$ is fuzzy closed set in X . Thus by theorem (3.14) , $f^{-1}(A) \wedge f^{-1}(\{t_\alpha\})$ is fuzzy compact . But $f^{-1}(\{t_\alpha\}) = f^{-1}(A) \wedge f^{-1}(\{t_\alpha\})$ is a fuzzy compact set in $f^{-1}(A)$. Therefore f_A is fuzzy proper .

Proposition 7.7. Let X and Y be fuzzy spaces . If f: X ⟶ Y is a fuzzy proper mapping , then f is a fuzzy compact mapping .

Proof Let K be a fuzzy compact subset of Y and let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a fuzzy open cover of $f^{-1}(K)$. Since f is a fuzzy proper mapping , then $f^{-1}(\{k_\alpha\})$ is a fuzzy compact set , $\forall$ $k_\alpha \in K$. But $f^{-1}(\{k_\alpha\}) \le f^{-1}(K) \le \vee_{\lambda \in \Lambda} U_\lambda$,

thus there exists n_k , such that $f^{-1}(\{k_\alpha\}) \le \vee_{\lambda=1}^{n_k} U_\lambda$, let $U_{n_k} = \vee_{\lambda=1}^{n_k} U_\lambda$. Thus , for all $k_\alpha \in K$, there exists n_k such that $f^{-1}(k_\alpha) \le U_{n_k}$, then $k_\alpha \le f(U_{n_k})$. Notice that for all $k_\alpha \in K$, $k_\alpha \le \left(1_Y \backslash f(1_X \backslash U_{n_k})\right) \longrightarrow K \le \vee_{k_\alpha \in K}\left(1_Y \backslash f(1_X \backslash U_{n_k})\right)$, but the sets $\left(1_Y \backslash f(1_X \backslash U_{n_k})\right)$ are fuzzy open . Hence there exists $n_{1k}, n_{2k}, \dots\dots, n_{jk}$, such that $K \le \vee_{\lambda=1}^{j}\left(1_Y \backslash f(1_X \backslash U_{n_k})\right) \longrightarrow f^{-1}(K) \le \vee_{\lambda=1}^{j} U_{n_k}$. Therefore $f^{-1}(K)$ is a fuzzy compact set in X . Hence the mapping $f: X \longrightarrow Y$ is a fuzzy compact mapping.

Proposition 7.8 Let X and Y be fuzzy spaces , such that Y is a fuzzy T_2 − space and fuzzy sc - space . If $f: X \longrightarrow Y$ is a fuzzy continuous mapping , then f is a fuzzy proper mapping if and only if f is a fuzzy compact .

Proof $\Longrightarrow$ By proposition (7.7) .

$\Longleftarrow$ To prove that f is a fuzzy proper mapping .

(i) Let F be a fuzzy closed subset of X . To prove that $f(F)$ is a fuzzy closed set in Y , let K be a fuzzy compact set in Y . Then $f^{-1}(K)$ is a fuzzy compact set in X , then by theorem (3.14) , $F \wedge f^{-1}(K)$ is fuzzy compact set in X . Since f is fuzzy continuous , then $f(F \wedge f^{-1}(K))$ is fuzzy compact set in Y .

But $f(F \wedge f^{-1}(K)) = f(F) \wedge K$, then $f(F) \wedge K$ is fuzzy compact , thus $f(F)$ is compactly fuzzy closed set in Y . Since Y is a fuzzy T_2 − space , then by theorem (4.5) , $f(F)$ is a fuzzy closed set in Y . Hence f is a fuzzy closed mapping .

(ii) Let $y_\alpha \in FP(Y)$. Since Y is fuzzy sc - space , then $\{y_\alpha\}$ is fuzzy compact in Y . Since f is a fuzzy compact mapping , then $f^{-1}(\{y_\alpha\})$ is fuzzy compact in X . Thus f is a fuzzy proper mapping .

Proposition 7.9. Let X and Y be fuzzy spaces , such that Y is fuzzy sc – space , fuzzy T_2 − space and $f: X \longrightarrow Y$ be a fuzzy continuous mapping . Then the following statements are equivalent :

(i) f is a fuzzy coercive mapping .

(ii) f is a fuzzy compact mapping .

(iii) f is a fuzzy proper mapping .

Proof

$(i) \longrightarrow (ii)$

By proposition (6.5) .

$(ii) \longrightarrow (iii)$

By proposition (7.8) .

$(iii) \longrightarrow (i)$

Let G be a fuzzy compact set in Y . Since f is fuzzy proper , then by proposition (7.7) , f is fuzzy compact mapping , then $f^{-1}(G)$ is a fuzzy compact set in X. Since $f(1_X \backslash f^{-1}(G)) \leq (1_Y \backslash G)$. Hence $f: X \longrightarrow Y$ is a fuzzy coercive mapping .

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