SOME NOTES ON U-FACTORIZATIONS MODULES

SINAN O. AL-SALIHI
Department of Mathematics , College of Education for Women , University of Tikrit , Tikrit, Iraq
drsinan2001@yahoo.com

(Received: 26 / 10 / 2013 ---- Accepted: 3 / 12 / 2013)

Abstract

This paper extended the notion of U-factorization to modules. Analogous to the ring case, it is define U-Unique factorization modules, U-Bounded factorization modules and U-Finite factorization modules. It gives an example of a U-Unique factorization module and end by recasting the definition of a U-FF module over a domain given by M. Axtell .

Introduction

[2], [3] study the factorization of nonunits in to atoms is a central theme in algebra . Classically the theory has concentrated on integral domains. Much of this theory generalized to the case ring with zero divisor and modules. In [1],[7] study properties of rings and domains are extended in to U-factorization property of rings, namely bounded factorizations ring and finite factorization ring . In this paper we study new way, in which factorization in modules. Most of the these are well known and can be find in [6]. Let R be a commutative ring and let M be an R-module .

Definition 1[6]

Let m $\in$M. A factorization of m is m=a_1 a_2 ... a_{s-1} n_s where S $\geq$ 1, a_1 , ... a_{s-1} are nonunits of R and $n_s \in$ M. We do allow s=1 in which case we have the trivial factorization m=m .

Definition 2[7]

A factorization of m, m = a_1 a_2 ... a_s b_1 b_2 ... b_t n is a U-factorization if for i=1,2,...,s, a_iR b_1 b_2...b_tn = R b_1 b_2... b_tn but for j =1,...,t, b_jR b_1 b_2...$\hat{b_j}$... b_t n $\neq$ R b_1 b_2 ... $\hat{b_j}$... b_tn. In this case we write m= a_1 a_2... a_s [b_1 b_2 ... b_t]n. Here b_1 ,b_2... b_l , n are called the essential factors and a_1 ,a_2 , ... ,a_s the inessential factors. Note that we allow s or t=0 , in which case we write m= [b_1 b_2... b_l n] or n= a_1 a_2... a_s[n], respectively .

Thus m= [m] is a U-factorization of m . We now record some definitions that appear in [4] and [5] .

Definition 3[4]

Let a $\in$R be a nonunit . Then a is irreducible (strongly irreducible, very strongly irreducible) if a=bc , then either (a)=(b) or (a)=(c) (respectively, either a=ub or a=uc for some unit u of R, either (a)=(b) or (a)=(c) and if a $\neq$ 0, then one of b or c can only be a unit multiple of a). The no unit a is m-irreducible if (a) is a maximal element in the set of proper principal ideals of R . Definition 4. Let 0 $\neq$ m $\in$M. Then m is primitive (respectively, strongly primitive, very strongly primitive) if for a$\in$ R and n $\in$M, m=an implies Rm=Rn (respectively, m=un for some unit u of R, Rm=Rn if m $\neq$ 0 then n can only be a unit multiple of m). And m is superprimitive if bm=an for a, b $\in$R implies a | b in R .

We see [4] for comparison of the various forms of irreducibility and [5] for a thorough discussion about associates and primitives in a module . For the sake of the current discussion it suffices to mention that a primitive element in a module is one that generates a maximal cyclic sub module .

Let S={irreducible, strongly irreducible,very strongly irreducible, m-irreducible, prime} and T= {primitive,strongly primitive, very strongly primitive, superprimitive} .

Definition 5

A factorization m =a_1 a_2. .. a $_{s-1}$ n_s is an α-factorizartion of m if each a_1 is α, $\alpha \in$ S. The same factorization will be called a β-factorization if n is β, $\beta \in$T, and an (α ,β) - factorization if each a_i is α, $\alpha \in$S and n is β, $\beta \in$T.

We similarly define a α-U-factorization, β-U-factorization or (α,β) -U- factorization, i.e. U-factorizations where every factor is of the appropriate type .

Definition 6

A factorization m= a_1 a_2... a_s [b_1 b_2 ... b_l n] is called an essential α-U-factorization(resp., essential β-U-factorization essential (α,β) -U-factorization) if each b_i is α, $\alpha\in$S (resp., n is β, $\beta \in$T, each b_i is α, $\alpha\in$S and n is β, $\beta\in$T) .

We can use the notion of U-factorization to define U-atomicity of a module.

Definition 7

A R-module M is α-atomic (resp.,β-atomic, (α,β)-atomic .

If each 0$\neq$ m$\in$ M has an α -factorization (resp., β – factorization, (α,β)-factorization) . And M is α-U-atomic (resp., β-U-atomic, (α, β)-U-atomic) if each 0$\neq$ m $\in$M has a α-U-factorization (resp. β- U-factorization, (α, β)-U-factorization) . The idea of U-factorization in modules is a generalization of U-factorization in rings. In [1] the authors prove that any factorization of a ring element can be rearranged to be a U-factorization. The same property holds for factorization of a module element .

Theorem 8. A factorization m=(a_1 a_2...a_k) $\overline{m}$ can be rearranged to be a U-factorization .

Proof. If Ra_1(a_2... a_k) $\overline{m}$ = R(a_2... a_k) $\overline{m}$ and Ra_2(a_3 ... a_k)$\overline{m}$ = R(a_3... a_k) $\overline{m}$ then R(a_1a_2)(a_3... a_k)$\overline{m}$ = R(a_3... a_k) $\overline{m}$. The following equations establish that .

R(a_1a_2) (a_3... a_k)$\overline{m}$ = Ra_1(a_2... a_k) $\overline{m}$ = R(a_2... a_k)$\overline{m}$ = Ra_2(a_3... a_k) $\overline{m}$ = R(a_3... a_k) $\overline{m}$ And also R(a_1)(a_3 ... a_k) $\overline{m}$ = R(a_3... a_k)$\overline{m}$ because R(a_1) (a_3... a_k) $\overline{m}$ = a_1 R(a_3... a_k)$\overline{m}$ = (a_1) (a_2... a_k)$\overline{m}$ = R(a_1 a_2... a_k) $\overline{m}$ = R(a_3... a_k) $\overline{m}$.

The above calculations imply that we can test for inessential factors one at a time.

If $m=(a_1 a_2 \dots a_k)(b_1 b_2 \dots b_l)\bar{m}$ with $Ra_i(b_1 b_2 \dots b_l)\bar{m} = R(b_1 b_2 \dots b_l)\bar{m}$ for $1 \le i \le k$ and we find $R(b_1)(b_2 \dots b_l)\bar{m} = R(b_2 \dots b_l)\bar{m}$ then $R(a_1)(b_2 \dots b_l)\bar{m} = R(a_1)(b_1 b_2 \dots b_l)\bar{m} = R(b_1 b_2 \dots b_l)\bar{m} = R(b_2 \dots b_l)\bar{m}$. This implies that a_i will remain an inessential divisor for $(b_2 \dots b_l)\bar{m}$.

Using the above we can rearrange the given factorization to be a U-factorization.

Corollary 9. A (α,β)-atomic module is also (α,β)-U-atomic.

The rearrangement is not unique. For example in Z_6 considered as a Z-module, we have $\bar{2} = 10.2.\bar{1}$. Either 10 nor 2 can be an essential divisor, while the other is the inessential divisor.

Next we will deal with the issue of comparing U-factorizations.

Two factorizations $m = (a_1 a_2 \dots a_k)[b_1 b_2 \dots b_l]\bar{m}$ and $m = (a'_1 a'_2 \dots a'_{k'})[b'_1 b'_2 \dots b'_{l'}]\bar{n}$ will be called isomorphic if

$\bar{m} \sim \bar{n}$ and $l = l^{-}$ and for some permutation σ we have $b_i \sim b'_{\sigma(i)}$.

Here $\sim$ means that the two module elements generate the same cyclic submodule and for ring elements it means that the two nonunits generate the same principal ideal. We could have used other notions of equivalent of elements to get more restrictive notions of isomorphism of U-factorizations. With a notion of isomorphism in place, we can define the following.

U-(α,β)-UFM: A nonzero R-module M is called a U-(α,β)-unique factorization module if M is (α,β) atomic and any two (α,β) factorizations are isomorphic.

If we write just U-UFM, we intend U-(irreducible, primitive)-UFM.

U-(α,β)-HFM: A nonzero R-module M is called a U-(α,β)-half factorial module if M is (α,β) atomic and any two (α,β) factorizations have the same number of essential divisors.

U-BFM: A nonzero R-module M is called a U-bounded factorization module if for each $m \neq 0$, there exists a positive integer N depending on m, such that for any U-factorization of m the number of essential divisors is less than N.

In the above terms, the ring R is implicitly understood. We will not write that M is a U-UFM as a R-module, but just M is a U-UFM.

Theorem 10. If R is a PID, then a finitely generated torsion R-module is a U-UFM.

Proof. Let M be the finitely generated R-module. Since R is a PID, any nonunit in R factors in to primes. Also since M is finitely generated, it is a Noetherian R-module, and hence satisfies ACCM (ascending chain condition on cyclic submodule). From this we conclude that primitive elements exist in M and every element of M is a multiple of some primitive elements. This establishes the atomicity of the module. A finitely generated torsion module over a PID has a decomposition as follows.

$M \cong [R/(p_1)^{n11} \oplus \dots \oplus R/(p_1)^{n1s1}] \oplus \dots \oplus [R/(p_k)^{nk1} \oplus \dots \oplus R/(p_k)^{nksk}]$.

For nonassociate primes $p_1, p_2, \dots, p_k$ and positive integer n_{ij}. Here s_i gives the number of summands which are annihilated by some power or p_i.

Denote this decomposition as $M \cong M_1 \oplus M_2 \dots \oplus M_k$ with the submodule $M_i \cong M_{i1} \oplus M_{i2} \dots \oplus M_{isi}$. So here M_{ij} is $R/(p_i)^{nij}$ and M_i is the direct summand of M annihilated by some power of p_i. Note that

$p_j : M_i \to M_i$ gives by $m_i \to p_j m_i$ is an automorphism of M_i as long as $I \neq j$. Also M_{ij} is a cyclic submodule. M_{ij} has a finite number of distinct submodules given by $(p_i)^{\alpha} M_{ij}$ for $\alpha = 0, 1, \dots, n_{ij}$. Of course we could have $(p_i)^{\alpha} M_{ij} = (p_i) M_{ij}$ for different integers α, β, but if we restrict ourselves to the range$(0, \dots, n_{ij})$ the exponentα is uniquely specified by the submodule of M_{ij} under consideration

An element $m_i \in M_i$ can be written as $m_i = p_i \alpha_i \bar{m}_i$ with $\bar{m}_i$ primitive in M_i. Within M_i, $\bar{m}_i = (\bar{m}_1, \bar{m}_2, \dots, \bar{m}_k)$ is primitive if and only if some $\bar{m}_{ij}$ is primitive in M_{ij}. The reason is that if all the $\bar{m}_{ij}$ were non-primitive, we would be able to write $\bar{m}_i = p_i m'_i$. By taking the projection of Rm_i to M_{ij} we conclude that α_I is uniquely specified, by m_i and does depend on the primitive element element $\bar{m}_i$.

In M, $\bar{m} = (\bar{m}_1, \bar{m}_2, \dots, \bar{m}_k)$ is primitive iff each $\bar{m}_i$ is primitive in M_i. The reason is that if some $\bar{m}_i$ were not primitive we would be able to write $\bar{m} = p_i m'$, and the submodule $R\bar{m}_i$ would be properly contained in Rm'.

Now let $m \in M$ and let $m = (a_1 \dots a_N)\bar{m}$ be a factorization with a_i prime and $\bar{m}$ primitive. Let ζ denote the list $(a_1, a_2, \dots, a_N)$.

Let $\mathcal{E}$ denote the list of non-negative integers $(\alpha_1, \alpha_2, \dots, \alpha_k)$, i.e. the list of exponents that we obtain by projecting Rm onto the direct summands M_i. The primes p_i appear in the list ζ at least α_i times. The first α_I appearances of p_i constitute the essential factors, the rest are inessential. The uniqueness of the essential factors follows from the uniqueness of α_i.

Note that Z_6 as a Z-module is not a UFM as a Z-module. The equation $\bar{3} = 3^n.\bar{3}$ holds for any non-negative integer n. However, Z_6 is a U-UFM as a Z-module. The theorem shows that the class of U-UFM modules is larger than the class of UFM modules.

Unfortunately, the above result cannot be extended even to UFD's, as the following example shows.

Example .Let R be the UFD Z[x], the ring of polynomials over Z. Let M be the Z[x]-module Z[x] / J where J is the ideal (x^2+5).

Now $6+J = 3.2.(1+J)$ and $6+J = (1+x)(1-x).(1+J)$.

The factors 3,2, (1+x) and (1-x) are all prime, essential and nonassociate.

The module element 6+J admits at least two non-comparable atomic factorizations and hence M is not a U-UFM .We will now define a notion of finite factorization modules in the framework provided by U-factorizations.

Surpisingly, the choice whether to include atomicity in the definition of a finite factorization module makes a diffence .Definition 11. A module M is called a U-FF (a) module if it is (irreducible, primitive)-atomic and given $0 \neq m \in M$, there are only a finite number of U-(irreducible, primitive) factorizations of m up to associates and order on the essential irreducible divisors and the primitive element of the factorization .

A module M is called a U-FF(b) module if for every given $0 \neq m \in M$, there are only a finite number of U-factorizations up to associates and order on the essential divisors and the module element appearing in the factorization . The first proposition we have is that a U-FF (b) module is a primitive –atomic module . Proposition 12.A U-FF(b) module satisifies the ascending chain condition on cyclic submodule (ACCM) and hence is primitive-atomic .

Proof. Suppose $Rm_1 \subset Rm_2 \subset \ldots$ is a strictly chain of cyclic submodules. Then $m_1 = \alpha_1 m_1 = \alpha_2 m_2 = \ldots$ and so m_1 has an infinite number of distinct factorizations .

If R is assumed to be an atomic ring then a U-FF(b) module is also a U-FF(a) module. The converse however in not true. The Z-module Z_6 is a U-FF(a) module, as it is a U-UFM by Theorem 10 . However, $\bar{2} = 2.\bar{1} = 8.\bar{1} = (6n+2).\bar{1}$ for any integer n. The ring elements are all essential in the factorizations and so $\bar{2}$ has infinitely many distinct U-factorizations .

In [7], M. Axtell gives a definition of a U-FF module over a domain D. We reproduce that definition here and show that Axtell's definition is equivalent to our U-FF(b) .

Definition 13.(Axtell) Let D be an integral domain and M a D-module. Let $Dm = \{dm \mid d \in D\}$, the cyclic submodule generated by m. Given a unitary D-module M, we say that $Dd_1 d_2 \ldots d_n m$ is a reduced submodule factorization if $d_j \notin D(U)$, $m \neq 0$ and for no cancelling and reordering of the d_j 's is it the case that $Dd_1 d_2 \ldots d_n m = Dd_1 d_2 \ldots d_t m$ where $t < n$.

The D-module M is said to be a U-FF module if for every $0 \neq m \in M$, there exist only finitely many reduced submodule factorizations $Dm = Dd_1 d_2 \ldots d_n m_j$, up to orders and associates on the d_i , as well as up to cyclic submodules on the m_j, i.e. $Dm_i \neq Dm_j$ for $i \neq j$.

We will refer to the U-FF modules in definition 13 above as U-FF(ax) modules .

Proposition 14. Let D be an integral domain. A D-module M, is a U-FF(ax) module if and only if is a U-FF(b) module .

Proof. A ssume M is a U-FF (b) module .

Let $0 \neq m \in M$. Let $Dm = D(r_1 r_2 \ldots r_n)\bar{m}$ be a reduced submodule factorization for Dm. Then $m = r' (r_1 r_2 \ldots r_n)\bar{m}$ for some $r' \in D$. Clearly, r' is an inessential divisor in this factorization of m .

Also $Dr_i (r_1 r_2 \ldots \hat{r_i} \ldots r_n)\bar{m} \neq D(r_1 r_2 \ldots \hat{r_i} \ldots r_n)\bar{m}$ as $D(r_1 r_2 \ldots r_n)\bar{m}$ is a reduced submodule factorization . So $m = R' [r_1 r_2 \ldots r_n]\bar{m}$. This implies that every reduced submodule factorization gives to a U-factorization with the r_i being essential factors. Since there are only a finite number of U-factorizations of m, m can have at most a finite number of reduced submodule factorizations .

The argument above can be reversed to give the converse .

References

[1]-A.G. Agargun, D.D. Anderson, and S. Valdes-Leon. Factorization commutative rings with zero divisors, III, Rocky Mountain Journal of Mathematics, 31(2001) (1) :1-20 .

[2]-D. D. Anderson. Some finiteness conditions on a commutative ring. Houston Journal of Mathematics, 4(1978) : 289-299.

[3]- D. D. Anderson. D.F. Anderson and M. Zafrullah. Factorization in integral domains . Journal of Pure and Applied Algebra, 69(1990) (1): 1-19.

[4]- D. D. Anderson and S. Valdes-Leon . Factorization in commutative rings with zero divisors , Rocky Mountain Journal of Mathematics, 26(1996) (1):439-480.

[5]- D. D. Anderson and S. Valdes-Leon. Factorization I commutative rings with zero divisors, II, In D. D. Anderson, editor, Factorization in integral domains, (1997) pages 197-219. Marcel Dekker, New York .

[6]-M. F. Atiyah and I.G. MacDonald. Introduction to Commutative Algebra. Addison-Wesley,(1969), New York .

[7]-M. Axtell U-factorizations in commutative rings with zero divisors. Communications in Algebra, 3(2002) (30): 1241-1255.

بعض الملاحظات حول مقاسات التحليل -U

سنان عمر ابراهيم

قسم الرياضيات ، كلية التربية للبنات ، جامعة تكريت ، تكريت ، العراق

(تاريخ الاستلام: 26 / 10 / 2013 ---- تاريخ القبول: 3 / 12 / 2013)

الملخص

يسعى هذا البحث الى توسع في مفهوم مقاسات التحليل – U كما هو الحال في حالة الحلقات . نقوم بتعريف المقاسات الوحيدة للتحليل – U والمقاسات المحددة للتحليل – U والمقاسات المنتهية للتحليل – U . ويقدم البحث مثالا للمقاسات التحليل الوحيد – U ويختتم البحث باعادة تعريف للمقاسات U-FF على الساحة والمعطى من قبل M. Axtell .