On θ-Centralizers of Prime and Semiprime Rings

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Abstract : The purpose of this paper is to prove the following result : Let R be a 2-torsion free ring and T : R→R an additive mapping such that $2T(x^2) = T(x)\theta(x) + \theta(x)T(x)$ holds for all $x \in R$. . In this case T is left and right θ-centralizer , if one of the following statements hold (i) R semiprime ring has a commutator which is not a zero divisor . (ii) R is a non commutative prime ring . (iii) R is a commutative semiprime ring , where θ be surjective endomorphism of R

Keywords: prime ring, semiprime ring, derivation, Jordan derivation, left (right) centralizer, left (right) θ-centralizer, centralizer, θ-centralizer, Jordan centralizer, Jordan θ-centralizer .

1- Introduction : This research has been motivated by the work of Brešar [2] , Zalar [6] and we [7] . Throughout, R will represent an associative ring with the center Z(R). As usual we write [x, y] for xy - yx and use basic commutator [xy, z] = [x, z]y + x[y, z], $x, y, z \in R$. Recall that R is prime if aRb = (0) implies a = 0 or b = 0, and is semiprime if aRa = (0) implies a = 0. An additive mapping $D : R \to R$ is called a derivation if D(xy) = D(x)y + xD(y) holds for all pairs $x, y \in R$ and is called Jordan derivation in case $D(x^2) = D(x)x+xD(x)$ is fulfilled for all $x \in R$. A derivation D is inner if there exists $a \in R$ such that D(x) = [a,x] holds for all $x \in R$. A classical result of Herstein [4] states that in case R is a prime ring of characteristic different from two, then every Jordan derivation is a derivation. A brief proof of Herstein's result can be found in [1]. Cusak [3] has generalized Herstein's result on 2-torsion free semiprime rings (see also [2] for an alternative proof). An additive mapping $T : R \to R$ is called a left (right) centralizer in case T(xy) = T(x)y (T(xy) = xT (y)) holds for all pairs $x, y \in R$. An additive mapping $T : R \to R$ is called a left (right) θ-centralizer in case T(xy) = T(x)θ(y) (T(xy) = θ(x)T(y)) holds for all pairs $x, y \in R$.

In case R has an identity element $T : R \to R$ is a left (right) centralizer iff T is of the form T(x) = ax (T(x) = xa) for some fixed element $a \in R$. An additive mapping

$T : R \to R$ is called a left (right) Jordan centralizer in case $T(x^2) = T(x)x$ $(T(x^2) = xT(x))$ is fulfilled for all $x \in R$. Following ideas from [2], Zalar has proved in [6] that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer. Some results concerning centralizers in prime and semiprime rings can be found in [5] and [8].

An additive mapping $T : R \to R$ is a left (right) θ-centralizer iff T is of the form $T(x) = a\theta(x))$ $(T(x) = \theta(x)a)$ for some fixed element $a \in R$. An additive mapping $T: R\to R$ is called a left (right) Jordan θ-centralizer in case $T(x^2) = T(x)\theta(x)$ $(T(x^2) = \theta(x)T(x))$ is fulfilled for all $x \in R$. In [7] that any left (right) Jordan θ-centralizer on a 2-torsion free semiprime ring is a left (right) θ-centralizer.

If the mapping $T : R \to R$, where R is an arbitrary ring, is both left and right Jordan centralizer, then obviously T satisfies the relation $2T(x^2) = T(x)x + xT(x)$, $x \in R$, in [9] J. Vukman prove the converse when R is 2- torsion free semiprime ring . In this paper we generalize this result to θ-centralizer .

The Results

Theorem 1

Let R be a 2-torsion free semiprime ring and let $T : R \to R$ be such an additive mapping that $2T(x^2) = T(x)\theta(x) + \theta(x)T(x)$ holds for all $x \in R$. then T is left and right Jordan θ-centralizer, where θ be a surjective endomorphism of R

Proof :

$$2T(x^2) = T(x)\theta(x) + \theta(x)T(x), \quad x \in R, \tag{1}$$

We intend to prove that T is commuting on R. In other words, it is our aim to prove that

$$[T(x), \theta(x)] = 0 \tag{2}$$

holds for all $x \in R$. In order to achieve this goal we shall first prove a weaker result that T satisfies the relation

$$[T(x), \theta(x^2)] = 0, \ x \in R \tag{3}$$

Since the above relation can be written in the form $[T(x),\theta(x)]\theta(x)+\theta(x)[T(x),\theta(x)]= 0$, it is obvious that T satisfies the relation (3) in case T is commuting on R.

Putting in the relation (1) x + y for x one obtains

$$2T(xy + yx) = T(x)\theta(y) + \theta(x)T(y) + T(y)\theta(x) + \theta(y)T(x), \ x, y \in R \tag{4}$$

Our next step is to prove the relation

$$8T(xyx) = T(x)(\theta(xy) + 3\theta(yx)) + (\theta(yx) + 3\theta(xy))T(x) + 2\theta(x)T(y)\,\theta(x) - \theta(x^2)T(y) - T(y)\,\theta(x^2), \ x, y \in R \tag{5}$$

For this purpose we put in the relation (4) 2(xy + yx) for y. Then using (4) we obtain

$$4T(x(xy+yx)+(xy+yx)x) = 2T(x)\,\theta(xy+yx)+2\theta(x)T(xy+yx)+2T(xy+yx)\theta(x)$$

$$+ 2\theta(xy + yx)T(x) = 2T(x)\theta(xy + yx) + \theta(x)T(x)\theta(y) + \theta(x^2)T(y) + \theta(x)T(y)\theta(x)$$

$$+ \theta(xy)T(x) + T(x)\theta(yx) + \theta(x)T(y)x + T(y)\,\theta(x^2) + \theta(y)T(x)\,\theta(x) + 2\theta(xy + yx)T(x):$$

Thus we have

$4T(x(xy+yx) + (xy+yx)x) = T(x)\theta(2xy+3yx) + \theta(3xy+2yx)T(x) + \theta(x)T(x)\theta(y) + \theta(y)T(x)\theta(x) + 2\theta(x)T(y)\theta(x) + \theta(x^2)T(y) + T(y)\,\theta(x^2), \; x, y \in R$ (6)

On the other hand, using (4) and (1) we obtain

$$4T(x(xy+yx) + (xy+yx)x) = 4T(x^2y+yx^2) + 8T(xyx) = 2T(x^2)\theta(y) + 2\theta(x^2)T(y) + 2T(y)\theta(x^2) + 2\theta(y)T(x^2) + 8T(xyx) = T(x)\theta(xy) + \theta(x)T(x)\theta(y) + 2\theta(x^2)T(y) + 2T(y)\theta(x^2) + \theta(y)T(x)\theta(x) + \theta(yx)T(x) + 8T(xyx)$$

We have therefore

$$4T(x(xy+yx)+(xy+yx)x) = T(x)\theta(xy)+\theta(yx)T(x)+\theta(x)T(x)\theta(y)+ \theta(y)T(x)\theta(x)+ 2\theta(x^2)T(y) + 2T(y)\theta(x^2) + 8T(xyx), \; x, y \in R \quad (7)$$

By comparing (6) with (7) we arrive at (5). Let us prove the relation

$$T(x)\theta(xyx - 2yx^2 - 2x^2y) + \theta(xyx - 2x^2y - 2yx^2)T(x) + \theta(x)T(x)\theta(xy + yx) + \theta(xy + yx)T(x)\theta(x) + \theta(x^2)T(x)\theta(y) + \theta(y)T(x)\theta(x^2) = 0 \quad , x, y \in R \quad (8)$$

Putting in (4) 8xyx for y and using (5) we obtain

$$16T(x^2yx+xyx^2)=8T(x)\theta(xyx)+8\theta(x)T(xyx)+8T(xyx)\theta(x)+8\theta(xyx)T(x)=8T(x)\theta(xyx) +\theta(x)T(x)\theta(xy+3yx)+\theta(xyx+3x^2y)T(x)+2\theta(x^2)T(y)\theta(x)-\theta(x^3)T(y)-\theta(x)T(y)\theta(x^2)+ T(x)\theta(xyx+3yx^2) + \theta(yx+3xy)T(x)\theta(x) + 2\theta(x)T(y)\theta(x^2) - \theta(x^2)T(y)\theta(x) - T(y)\theta(x^3) + 8\theta(xyx)T(x)$$

We have therefore

$$16T(x^2yx + xyx^2) =T(x)\theta(9xyx + 3yx^2) +\theta(9xyx + 3x^2y)T(x)+ \theta(x)T(x)\theta(xy +3yx) + \theta(yx + 3xy)T(x)\theta(x) + \theta(x^2)T(y)\theta(x) + \theta(x)T(y)\theta(x^2) - T(y)\theta(x^3) - \theta(x^3)T(y) \quad x, y \in R \quad (9)$$

On the other hand, we obtain first using(5) and then after collecting some terms using(4)

$$16T(x^2yx+xyx^2) = 16T(x(xy)x) + 16T(x(yx)x) = 2T(x)\theta(x^2y+3xyx) + 2\theta(xyx+3x^2y)T(x) + 4\theta(x)T(xy)\theta(x) - 2\theta(x^2)T(xy) - 2T(xy)\theta(x^2) + 2T(x)\theta(xyx +3yx^2) + 2\theta(yx^2+3xyx)T(x) + 4\theta(x)T(yx)\theta(x) - 2\theta(x^2)T(yx) - 2T(yx)\theta(x^2) = T(x)\theta(2x^2y+6yx^2+8xyx) + \theta(8xyx+2yx^2+6x^2y)T(x) + 4\theta(x)T(xy+yx)\theta(x) - 2\theta(x^2)T(xy+yx) - 2T(xy+yx)\theta(x^2) = T(x)\theta(2x^2y+6yx^2+8xyx) +\theta(8xyx+2yx^2+ 6x^2y)T(x) + 2\theta(x)T(x)\theta(yx) + 2\theta(x^2)T(y)\theta(x) + 2\theta(x)T(y)\theta(x^2) + 2\theta(xy)T(x)\theta(x) - \theta(x2)T(x)\theta(y) - \theta(x3)T(y) - \theta(x^2)T(y)\theta(x) - \theta(x^2y)T(x) - T(x)\theta(yx^2) - \theta(x)T(y)\theta(x^2) - T(y)\theta(x^3) -\theta(y)T(x)\theta(x^2)$$

We have therefore

$16T(x^2yx + xyx^2) = T(x)\ \theta(2x^2y + 5yx^2 + 8xyx) + \theta(2yx^2+5x^2y+8xyx)T(x) + 2\theta(x)T(x)\theta(yx) + 2\theta(xy)T(x)\theta(x) + \theta(x^2)T(y)\theta(x) + \theta(x)T(y)\theta(x^2) - \theta(x^2)T(x)\theta(y) - \theta(y)T(x)\theta(x^2) - \theta(x^3)T(y) - T(y)\theta(x^3) \quad x, y \in R \quad (10)$

By comparing (9) with (10) we obtain (8).

Replacing in (8) y by yx we obtain

$T(x)\ \theta(xyx^2 - 2yx^3 - 2x^2yx) + \theta(xyx^2 - 2x^2yx - 2yx^3)T(x) + \theta(x)T(x)\theta(xyx+yx^2) + \theta(xyx+yx^2)\ T(x)\ \theta(x) + \theta(x^2)\ T(x)\ \theta(yx) + \theta(yx)\ T(x)\ \theta(x^2) = 0 \quad x, y \in R \quad (11)$

Right multiplication of (11) by θ(x) gives

$T(x)\theta(xyx^2-2yx^3-2x^2yx) + \theta(xyx-2x^2y-2yx^2)T(x)\theta(x) + \theta(x)T(x)\theta(xyx+yx^2) + \theta(xy+yx)T(x)\theta(x^2) + \theta(x^2)T(x)\theta(yx) + \theta(y)T(x)\theta(x^3) = 0 \quad x, y \in R \quad (12)$

Subtracting (12) from (11) we obtain

$\theta(xyx)[\theta(x),T(x)] + 2\theta(x^2y)[T(x),\theta(x)] + 2\theta(yx^2)[T(x),\theta(x)] + \theta(xy)[\theta(x),T(x)] + \theta(yx)[\theta(x),T(x)]\theta(x) + \theta(y)[\theta(x),T(x)]\theta(x^2) = 0, \quad x, y \in R$

Which reduces after collecting the first and the fourth term together to

$\theta(xy)[\theta(x^2),T(x)] + 2\theta(x^2y)[T(x),\theta(x)] + 2\theta(yx^2)[T(x),\theta(x)] + \theta(yx)[\theta(x),T(x)] + \theta(y)[\theta(x),T(x)]\theta(x^2) = 0, \quad x, y \in R \quad (13)$

Substituting T(x) θ(y) for θ(y) in the above relation gives

$\theta(x)T(x)\theta(y)[\theta(x^2),T(x)] + 2\theta(x^2)T(x)\theta(y)[T(x),\theta(x)] + 2T(x)\theta(yx^2)[T(x),\theta(x)] + T(x)\theta(yx)[\theta(x),T(x)]\theta(x) + T(x)\theta(y)[\theta(x),T(x)]\theta(x^2) = 0 \quad x, y \in R \quad (14)$

Left multiplication of (13) by T(x) leads to

$T(x)\quad \theta(xy)[\theta(x^2),T(x)] + 2T(x)\theta(x^2y)[T(x),\theta(x)] + 2T(x)\theta(yx^2)[T(x),\theta(x)] + T(x)\theta(yx)[\theta(x),T(x)]\theta(x) + T(x)\theta(y)[\theta(x),T(x)]\theta(x^2) = 0 \quad x, y \in R \quad (15)$

Subtracting (15) from (14) we arrive at

$[T(x),\theta(x)]\theta(y)[T(x),\theta(x^2)] - 2[T(x),\theta(x^2)]\theta(y)[T(x),\theta(x)] = 0, \quad x, y \in R$

We set

$$a = [T(x),\theta(x)], \qquad b = [T(x),\theta(x^2)], \qquad c = -2[T(x),\theta(x^2)]$$

Then the above relation becomes

$$a\theta(y)b + c\theta(y)a = 0,\ y \in R: \quad (16)$$

Putting in (19) θ(y)aθ(z) for θ(y) we obtain

$$a\theta(y)a\theta(z)b + c\theta(y)a\theta(z)a = 0,\ z, y \in R \quad (17)$$

Left multiplication of (16) by ay gives

$$a\theta(y)a\theta(z)b + a\theta(y)c\theta(z)a = 0,\ z, y \in R \quad (18)$$

Subtracting (18) from (17) we obtain

$$(a\theta(y)c - c\theta(y)a)\ \theta(z)a = 0,\ z, y \in R \tag{19}$$

Let in (19) θ(z) be θ(z)cθ(y) we obtain

$$(a\theta(y)c - c\theta(y)a)\ \theta(z)c\theta(y)a = 0,\ z, y \in R \tag{20}$$

Right multiplication of (19) by θ(y)c gives

$$(a\theta(y)c - c\theta(y)a)\ \theta(z)a\theta(y) = 0,\ z, y \in R \tag{21}$$

Subtracting(20) from(21) we obtain $(a\theta(y)c-c\theta(y)a)\theta(z)(a\theta(y)c-c\theta(y)a) = 0$, $z,y\in R$, whence it follows

$$a\theta(y)c = c\theta(y)a \quad , \quad y \in R \tag{22}$$

Combining (16) with (22) we arrive at

$$a\theta(y)(b + c) = 0\ , \qquad y \in R$$

In other words

$$[T(x),\theta(x)]\theta(y)[T(x),\theta(x^2)] = 0\ , \qquad x, y \in R \tag{23}$$

From the above relation one obtains easily

$$([T(x),\theta(x)]\ \theta(x) + \theta(x)\ [T(x),\theta(x)])\ \theta(y)[T(x), \theta(x^2)] = 0,\ x, y \in R$$

We have therefore

$$[T(x),\theta(x^2)]\theta(y)[T(x),\theta(x^2)] = 0\ , \quad x, y \in R$$

which implies

$$[T(x),\theta(x^2)] = 0, \qquad x \in R \tag{24}$$

Substitution x + y for x in (24) gives

$$[T(x),\theta(y^2)] + [T(y),\theta(x^2)] + [T(x),\theta(xy+yx)] + [T(y),\theta(xy+yx)] = 0,\ x, y \in R$$

Putting in the above relation -x for x and comparing the relation so obtained with the above relation we obtain, since we have assumed that R is 2-torsion free

$$[T(x),\theta(xy + yx)] + [T(y),\theta(x^2)] = 0,\ \ x, y \in R \tag{25}$$

Putting in the above relation 2(xy+yx) for y we obtain according to (4) and (24)

$$0 = 2[T(x)\ ,\ \theta(x^2y + yx^2 + 2xyx)] + [T(x)\theta(y) + \theta(x)T(y) + T(y)\theta(x) + \theta(y)T(x)\ ,\ \theta(x^2)]$$

$$= 2\ \theta(x^2)\ [T(x)\ ,\ \theta(y)] + 2\ [T(x)\ ,\ \theta(y)]\ \theta(x^2) + 4\ [T(x)\ ,\ \theta(xyx)] + T(x)\ [\theta(y),\theta(x^2)] + \theta(x)[T(y),\theta(x^2)] + [T(y),\theta(x^2)]\theta(x) + [\theta(y),\theta(x^2)]T(x)$$

Thus we have

$$2\theta(x^2)[T(x),\theta(y)] + 2[T(x),\theta(y)]\theta(x^2) + 4[T(x),\theta(xyx)] + T(x)[\theta(y),\theta(x^2)] + [\theta(y),\theta(x^2)]T(x) + \theta(x)[T(y),\theta(x^2)] + [T(y),\theta(x^2)]\theta(x) = o\ , \quad x, y \in R \tag{26}$$

For y = x the above relation reduces to

$$\theta(x^2)[T(x),\theta(x)] + [T(x),\theta(x)]\theta(x^2) + 2[T(x),\theta(xx^2)] = 0,$$

which gives

$$\theta(x^2)[T(x),\theta(x)] + 3[T(x),\theta(x)]\theta(x^2) = 0, \quad x \in R$$

According to the relation $[T(x), \theta(x)]\,\theta(x) + \theta(x)\,[T(x), \theta(x)] = 0$ (see (24)) one can replace in the above relation $\theta(x^2)\,[T(x), \theta(x)]$ by $[T(x), \theta(x)]\,\theta(x^2)$, which gives

$$[T(x), \theta(x)]\,\theta(x^2) = 0, \; x \in R \tag{27}$$

and

$$\theta(x^2)\,[T(x), \theta(x)] = 0, \; x \in R \tag{28}$$

We have also

$$\theta(x)\,[T(x), \theta(x)]\,\theta(x) = 0, \; x \in R \tag{29}$$

Because of (25) one can replace in (26) $[T(y),\theta(x^2)]$ by $-[T(x),\theta(xy+yx)]$, which gives

$$0 = 2\theta(x^2)[T(x),\theta(y)] + 2[T(x),\theta(y)]\theta(x^2) + 4[T(x),\theta(xyx)] + T(x)[\theta(y),\theta(x^2)] + [\theta(y),\theta(x^2)]T(x) - \theta(x)[T(x),\theta(xy+yx)] - [T(x),\theta(xy+yx)]\theta(x) = 2\theta(x^2)[T(x),\theta(y)] + 2[T(x),\theta(y)]\theta(x^2) + 4[T(x),\theta(x)]\theta(yx) + 4\theta(x)[T(x),\theta(y)]\theta(x) + 4\theta(xy)[T(x),\theta(x)] + T(x)[\theta(y),\theta(x^2)] + [\theta(y),\theta(x^2)]T(x) - \theta(x)[T(x),\theta(x)]\theta(y) - \theta(x^2)[T(x),\theta(y)] - \theta(x)[T(x),\theta(y)]\theta(x) - \theta(xy)[T(x),\theta(x)] - [T(x),\theta(x)]\theta(yx) - \theta(x)[T(x),\theta(y)]\theta(x) - [T(x),\theta(y)]\theta(x^2) - \theta(y)[T(x),\theta(x)]\theta(x)$$

We have therefore

$$\theta(x^2)[T(x),\theta(y)] + [T(x),\theta(y)]\theta(x^2) + 3[T(x),\theta(x)]\theta(yx) + \theta(3xy)[T(x),\theta(x)] + 2\theta(x)[T(x),\theta(y)]\theta(x) + T(x)[\theta(y),\theta(x^2)] + [\theta(y),\theta(x^2)]T(x) + \theta(x)[T(x),\theta(x)]\theta(y) - \theta(y)[T(x),\theta(x)]\theta(x) = 0, \quad x, y \in R \tag{30}$$

The substitution yx for y gives

$$0 = \theta(x^2)[T(x),\theta(yx)] + [T(x),\theta(yx)]\theta(x^2) + 3[T(x),\theta(x)]\theta(yx^2) + 3\theta(xyx)[T(x),\theta(x)] + 2\theta(x)[T(x),\theta(yx)]\theta(x) + T(x)[\theta(yx),\theta(x^2)] + [\theta(yx),\theta(x^2)]T(x) + \theta(x)[T(x),\theta(x)]\theta(yx) - \theta(yx)[T(x),\theta(x)]\,\theta(x) = \theta(x^2)[T(x),\theta(y)]\theta(x) + \theta(x^2y)[T(x),\theta(x)] + [T(x),\theta(y)]\theta(x^3) + \theta(y)[T(x),\theta(x)]\theta(x^2) + 3[T(x),\theta(x)]\theta(yx^2) + 3\theta(xyx)[T(x),\theta(x)] + 2\theta(x)[T(x),\theta(y)]\theta(x^2) + 2\theta(xy)\,[T(x), \theta(x)]\,\theta(x) + T(x)\,[\theta(y), \theta(x^2)]\,\theta(x) + [\theta(y), \theta(x^2)]\,\theta(x)\,T(x) + \theta(x)\,[T(x), \theta(x)]\,\theta(yx) - \theta(yx)\,[T(x), \theta(x)]\,\theta(x)$$

which reduces because of (27) and (29) to

$$\theta(x^2)[T(x),\theta(y)]\theta(x) + \theta(x^2y)[T(x),\theta(x)] + [T(x),\theta(y)]\theta(x^3) + 3[T(x),\theta(x)]\theta(yx^2) + 3\theta(xyx)\,[T(x), \theta(x)] + 2\theta(x)\,[T(x), \theta(y)]\,\theta(x^2) + 2\theta(xy)\,[T(x), \theta(x)]\,\theta(x) + T(x)[\theta(y),\theta(x^2)]\theta(x) + [\theta(y),\theta(x^2)]\theta(x)T(x) + \theta(x)[T(x),\theta(x)]\theta(yx) = 0, \quad x,y \in R \tag{31}$$

Right multiplication of (30) by $\theta(x)$ gives

$\theta(x^2)\,[T(x)\,,\,\theta(y)]\,\theta(x) + [T(x)\,,\,\theta(y)]\,\theta(x^3) + 3\,[T(x)\,,\,\theta(x)]\,\theta(yx^2) + \theta(3xy)\,[T(x)\,,\,\theta(x)]\,\theta(x) + 2\theta(x)\,[T(x)\,,\,\theta(y)]\,\theta(x^2) + T(x)\,[\theta(y)\,,\,\theta(x^2)]\,\theta(x) + [\theta(y),\theta(x^2)]\,T(x)\,\theta(x) + \theta(x)\,[T(x)\,,\theta(x)]\,\theta(yx) = 0, \qquad x, y\in R \qquad (32)$

Subtracting (32) from (31) we obtain

$$\theta(x^2y)[T(x),\theta(x)] + 3\theta(xy)[\theta(x),[T(x),\theta(x)]] + 2\theta(xy)[T(x),\theta(x)]\theta(x) + [\theta(y),\theta(x^2)][\theta(x),T(x)] = 0,$$

which reduces because of (28) to

$$2\theta(x^2y)[T(x),\theta(x)] + 3\theta(xyx)[T(x),\theta(x)] - \theta(xy)[T(x),\theta(x)]\theta(x) = 0, \; x, y \in R$$

Replacing in the above relation $-[T(x),\theta(x)]\theta(x)$ by $\theta(x)[T(x),\theta(x)]$ we obtain

$$\theta(x^2y)[T(x),\theta(x)] + 2\theta(xyx)[T(x),x] = 0, \qquad x, y \in R$$

Because of (24), (27), (28) and (29) the relation (13) reduces to $\theta(x^2y)[T(x),\theta(x)] = 0$, $x, y \in R$, which gives together with the relation above $\theta(xyx)[T(x),\theta(x)] = 0$, $x,y \in R$, whence it follows

$$\theta(x)[T(x),\theta(x)]\theta(yx)[T(x),\theta(x)] = 0, \; x, y \in R$$

Thus we have

$$\theta(x)\,[T(x)\,,\,\theta(x)] = 0, \; x \in R \qquad (33)$$

Of course we have also

$$[T(x)\,,\,\theta(x)]\,\theta(x) = 0, \; x \in R \qquad (34)$$

From (33) one obtains (see the proof of (25))

$$\theta(y)\,[T(x)\,,\,\theta(x)] + \theta(x)\,[T(x)\,,\,\theta(y)] + \theta(x)\,[T(y)\,,\,\theta(x)] = 0, \; x, y \in R:$$

Left multiplication of the above relation by $[T(x)\,,\,\theta(x)]$ gives because of (34)

$$[T(x)\,,\,\theta(x)]\,\theta(y)\,[T(x)\,,\,\theta(x)] = 0, \; x, y \in R,$$

whence it follows

$$[T(x)\,,\,\theta(x)] = 0, \; x \in R \qquad (35)$$

Combining (35) with (1) we obtain

$$T(x^2) = T(x)\,\theta(x), \; x \in R$$

and also

$$T(x^2) = \theta(x)\,T(x), \; x \in R,$$

which means that T is left and also right Jordan θ-centralizer.
The proof of the theorem is complete.

Corollary 1

Let R be a 2-torsion free ring and let T : R → R be such an additive mapping that $2T(x^2) = T(x)\theta(x) + \theta(x)T(x)$ holds for all $x \in R$. In this case T is left and right θ-centralizer , if one of the following statements hold

(i) R semiprime ring has a commutator which is not a zero divisor .
(ii) R is a non commutative prime ring .
(iii) R is a commutative semiprime ring .

Where θ be surjective endomorphism of R

Proof :

By Theorem 1 we get T is left and also right Jordan θ-centralizer.
By Theorem (1.3) in [7] we get T is both left and also right θ-centralizer.
The proof of the Corollary is complete.

Corollary 2

Let R be a 2-torsion free prime ring and let T : R → R be such an additive mapping that $2T(x^2) = T(x)\theta(x) + \theta(x)T(x)$ holds for all $x \in R$. In this case T is left and right θ-centralizer , where θ be surjective endomorphism of R .

Corollary 3

Let R be a 2-torsion free ring and let T : R → R be such an additive mapping that $2T(x^2) = T(x)x + xT(x)$ holds for all $x \in R$. In this case T is left and right centralizer , if one of the following statements hold

(i) R semiprime ring has a commutator which is not a zero divisor .
(ii) R is a non commutative prime ring .
(iii) R is a commutative semiprime ring .

Corollary 4

Let R be a 2-torsion free prime ring and let T : R → R be such an additive mapping that $2T(x^2) = T(x)x + xT(x)$ holds for all $x \in R$. In this case T is left and right θ-centralizer

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