On Second –Order Differential Subordinations for Multivalent Functions Associated with Komatu Operator

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Abstract . In this paper , we obtain some results for second - order differential subordinations , for multivalent functions in the open unit disk associated with the komatu operator .

Keywords : Univalent function , Multivalent function , Differential subordination , Komatu operator .

2000 AMS Classification : 30C45 , 30A20 , 34A30 .

1. Introduction and preliminaries

Let be the open unit disk in the complex plane and let denote the class of analytic functions defined in U , for positive integer and a ℂ . Let = { f : f(z) = , with , .

Let f and g be members of . The function f is said to be subordinate to g , written f g or f(z) g(z) , if there exists a schwarz function w(z) analytic in U, with w(0)= 0 and |w(z)|<1 such that f(z)= g (w(z)) , (z U) .

In particular , if the function g is univalent in U , then f g if and only if f(0) = g(0) and f(U) ⊂ g(U) .

Let ψ : U and let be univalent in If f is analytic in and satisfies the (second –order) differential subordination

$$\psi(f(z), zf'(z), z^2 f''(z); z) \prec h(z) , \quad (1.1)$$

then is called a solution of the differential subordination . The univalent function is called a dominant of the solutions of the differential subordination , or more simply dominant if for all satisfying (1.1) A dominant that satisfies for all dominants of (1.1) is said to be the best dominant of (1.1).

Let L(p) denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (z \in U, p \in \mathbb{N} = \{1,2,3,\ldots\}) , \quad (1.2)$$

which are analytic and p-valent in

For f L(p) , let the komatu operator [4] be denote by

$$K_{c,p}^{\delta} f(z) = \frac{(c+p)^{\delta}}{\Gamma(\delta) z^c} \int_0^z t^{c-1} \left(log \frac{z}{t}\right)^{\delta-1} f(t)\, dt$$

$$= z^p + \sum_{n=1}^{\infty} \left(\frac{c+p}{c+p+n}\right)^{\delta} a_{n+p} z^{n+p} \quad (c > -p, \delta > 0) . \quad (1.3)$$

In order to prove the results , we shall use the following definitions and theorem.

Definition 1.1[2] . Denote by the set of all functions that are analytic and injective on , where

$$E(q) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \right\} \quad (1.4)$$

and are such that 0 for . Further let the subclass of for which a be denoted by .

Definition 1.2 [2]. Let Ω be a set in ℂ , q Q and let be positive integer. The class of admissible functions consists of those functions : that satisfy the admissibility condition ψ , whenever $= q($, $s =$ $q'($, and

$$\operatorname{Re}\left\{\frac{t}{s}+1\right\} \geq m \operatorname{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\} \qquad (1.5)$$

. Let .

Theorem 1.1[2]. Let ψ with . If the analytic function satisfies

$$\psi\,(F(z), zF'(z), z^2F''(z); z) \in \Omega\,, \qquad (1.6)$$

then
$F(z) \prec q(z)$.

2. Main Results

Definition 2.1. Let Ω be a set in and .The class of admissible functions consists of those functions : that satisfy the admissibility condition :

$$\Omega\,, \qquad (2.1)$$

whenever

$$u = q(\zeta)\,, v = \frac{m\zeta q'(\zeta) + cq(\zeta)}{c+p} \quad (p \in \mathbb{N}\,, c > -p)\,,$$

and

$$\operatorname{Re}\left\{\frac{(c+p)^2 w - c^2 u}{(c+p)v - cu} - 2c\right\} \geq m \operatorname{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\}, \qquad (2.2)$$

.

Theorem 2.1. Let

$$\{\phi(\mathrm{K}_{c,p}^{\delta+2} f(z), \mathrm{K}_{c,p}^{\delta+1} f(z), \mathrm{K}_{c,p}^{\delta} f(z); z) : z \in U\} \subset \Omega\,, \qquad (2.3)$$

then

$$\mathrm{K}_{c,p}^{\delta+2} f(z) \prec q(z)\,.$$

Proof. We note from (1.3)that , we have

$$z\left(\mathrm{K}_{c,p}^{\delta+1} f(z)\right)' = (c+p)\mathrm{K}_{c,p}^{\delta} f(z) - c\,\mathrm{K}_{c,p}^{\delta+1} f(z)\,, \qquad (2.4)$$

is equivalent to

$$K_{c,p}^{\delta}f(z)=\frac{z\left(K_{c,p}^{\delta+1}f(z)\right)'+\ cK_{c,p}^{\delta+1}f(z)}{(c+p)}\ , \qquad (2.5)$$

and

$$K_{c,p}^{\delta+1}f(z)=\frac{z\left(K_{c,p}^{\delta+2}f(z)\right)'+\ cK_{c,p}^{\delta+2}f(z)}{(c+p)}\ . \qquad (2.6)$$

Let the analytic function F in U defined by

$$F(z)=K_{c,p}^{\delta+2}f(z)\ . \qquad (2.7)$$

Then we have

$$K_{c,p}^{\delta+1}f(z)=\frac{zF'(z)+cF(z)}{c+p}\ ,$$

$$K_{c,p}^{\delta}f(z)=\frac{z^2F''(z)+(1+2c)zF'(z)+c^2F(z)}{(c+p)^2}\ . \qquad (2.8)$$

Further , let us define the transformations from by

$$u=r\ ,\qquad v=\frac{s+cr}{c+p}\ ,\qquad w=\frac{t+(1+2c)s+c^2r}{(c+p)^2}.$$

Let

$$\psi\,(r,s,t;z)=\phi(u,v,w;z)=\phi\left(r,\frac{s+cr}{c+p},\frac{t+(1+2c)s+c^2r}{(c+p)^2};z\right). \qquad (2.9)$$

The proof will make use of Theorem 1.1. Using (2.7) and (2.8) , from (2.9) , we obtain

$$\psi\,(F(z),zF'(z),z^2F''(z);z)=\phi\left(K_{c,p}^{\delta+2}f(z),K_{c,p}^{\delta+1}f(z),K_{c,p}^{\delta}f(z);z\right). \qquad (2.10)$$

Therefore (2.3) becomes

$$\psi\,(F(z),zF'(z),z^2F''(z);z)\in\Omega\ . \qquad (2.11)$$

Note that

$$\frac{t}{s}+1=\frac{(c+p)^2w-c^2u}{(c+p)v-cu}-2c\ , \qquad (2.12)$$

and since the admissibility condition for is equivalent to the admissibility condition for ψ as given in Definition 1.2 , hence ψ , and by Theorem 1.1, *F(z)*

By (2.7), we get

$$\mathrm{K}_{c,p}^{\delta+2} f(z) \prec q(z) .$$

In the case , we have the following example .

Example 2.1. Let the class of admissible functions consist of those functions that satisfy the admissibility condition :

$$v = \frac{m\zeta q'(\zeta) + cq(\zeta)}{c+p} \notin \Omega ,$$

. , then

We consider the special situation when is a simply connected domain. In this case , where is a conformal mapping of U onto and the class is written as . The following result follows immediately from Theorem 2.1.

Theorem 2.2. Let

$$\phi\left(\mathrm{K}_{c,p}^{\delta+2} f(z), \mathrm{K}_{c,p}^{\delta+1} f(z), \mathrm{K}_{c,p}^{\delta} f(z); z\right) \prec h(z), \tag{2.13}$$

then

$$\mathrm{K}_{c,p}^{\delta+2} f(z) \prec q(z) .$$

The next results occurs when the behavior of on is not known .

Corollary 2.1. Let , q be univalent in U and $q(0)$. Let for some (0,1) , where .

$$\phi\left(\mathrm{K}_{c,p}^{\delta+2} f(z), \mathrm{K}_{c,p}^{\delta+1} f(z), \mathrm{K}_{c,p}^{\delta} f(z); z\right) \in \Omega , \tag{2.14}$$

then
$K_{c,p}^{\delta+2} f(z) \prec q(z)$.

Proof. From Theorem 2.1,we have and the proof is complete .

Theorem 2.3. Let and be univalent in , with and set Let satisfy one of the following conditions :

(1) , for some (0,1) , or

(2) there exists (0,1) such that , for all (0,1) .

(2.13) ,then
$K_{c,p}^{\delta+2} f(z) \prec q(z)$.

Proof .
case (1): By applying Theorem 2.1 ,we obtain , since we deduce

$K_{c,p}^{\delta+2} f(z) \prec q(z)$.

case (2): If we let *F(z)* f(z) and let , then

$$\phi\left(F_\rho(z), zF_\rho{}'(z), z^2F_\rho{}''(z); \rho z\right) = \phi(F(\rho z), \rho zF'(\rho z), \rho^2z^2F''(\rho z); \rho z) \in h_\rho(U) .$$

By using Theorem 2.1 and the comment associated with Ω , Where w is any function mapping U into U , with , we obtain for (,1) . By letting , we get .
Therefore
$K_{c,p}^{\delta+2} f(z) \prec q(z)$.

The next result give the best dominant of the differential subordination (2.13)

Theorem 2.4. Let be univalent in U and let : . Suppose that the differential equation

$$\phi\left(q(z), zq'(z), z^2q''(z); z\right) = h(z) \qquad (2.15)$$

has a solution with and satisfy one of the following conditions :

(1) q and ,

(2) q is univalent in U and , for some (0,1) , or

(3) q is univalent in U and there exists (0,1) such that ,for all (0,1).

(2.13) ,then and q is the best dominant .

Proof . By applying Theorem 2.2 and Theorem 2.3 , we deduce that q is a dominant of (2.13) . Since q satisfies (2.15) , it is also a solution of (2.13) and therefore q will be dominated by all dominants of (2.13) . Hence q is the best dominant of (2.13) .

Definition 2.2. Let Ω be a set in and q . The class of admissible functions consists of those functions : that satisfy the admissibility condition :

Ω ,

whenever

$$u = q(\zeta)\,, \quad v = \frac{m\zeta q'(\zeta) + (c+p-1)q(\zeta)}{c+p} \quad (p \in \mathbb{N}, c > -p)\,,$$

and

$$\operatorname{Re}\left\{\frac{(c+p)^2 w - (c+p-1)^2 u}{(c+p)v - (c+p-1)u} - 2(c+p-1)\right\} \geq m \operatorname{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\}, \qquad (2.16)$$

.

Theorem 2.5. Let

$$\left\{\phi\left(\frac{\mathrm{K}_{c,p}^{\delta+2} f(z)}{z^{p-1}}, \frac{\mathrm{K}_{c,p}^{\delta+1} f(z)}{z^{p-1}}, \frac{\mathrm{K}_{c,p}^{\delta} f(z)}{z^{p-1}}\,;\, z\right) : z \in U\right\} \subset \Omega\,, \qquad (2.17)$$

then

$$\frac{\mathrm{K}_{c,p}^{\delta+2} f(z)}{z^{p-1}} \prec q(z)\,.$$

Proof . Let the analytic function F in U defined by

$$F(\mathrm{z}) = \frac{\mathrm{K}_{c,p}^{\delta+2} f(z)}{z^{p-1}}\,. \qquad (2.18)$$

By using the relations (2.4) and (2.18) , we get

$$\frac{\mathrm{K}_{c,p}^{\delta+1} f(z)}{z^{p-1}} = \frac{zF'(z) + (c+p-1)F(z)}{c+p}\,,$$

$$\frac{\mathrm{K}_{c,p}^{\delta} f(z)}{z^{p-1}} = \frac{z^2 F''(z) + [2(c+p)-1]zF'(z) + (c+p-1)^2 F(z)}{(c+p)^2}\,. \qquad (2.19)$$

Further ,let us define the transformations from by

$$u = r\ , v = \frac{s + (c+p-1)r}{c+p}\ , w = \frac{t + [2(c+p)-1]s + (c+p-1)^2 r}{(c+p)^2}\ .$$

Let

$$\psi(r,s,t;z) = \phi(u,v,w;z) = \phi\left(r, \frac{s+(c+p-1)r}{c+p}, \frac{t+[2(c+p)-1]s+(c+p-1)^2 r}{(c+p)^2}; z\right). \tag{2.20}$$

The proof will make use of Theorem 1.1. Using (2.18) and (2.19) , from (2.20) , we obtain

$$\psi\ (F(z), zF'(z), z^2F''(z); z) = \phi\left(\frac{\mathrm{K}_{c,p}^{\delta+2} f(z)}{z^{p-1}}, \frac{\mathrm{K}_{c,p}^{\delta+1} f(z)}{z^{p-1}}, \frac{\mathrm{K}_{c,p}^{\delta} f(z)}{z^{p-1}};\ z\right). \tag{2.21}$$

Therefore (2.17) becomes

$$\psi\ (F(z), zF'(z), z^2F''(z); z) \in \Omega\ . \tag{2.22}$$

Note that

$$\frac{t}{s} + 1 = \frac{(c+p)^2 w - (c+p-1)^2 u}{(c+p)v - (c+p-1)u} - 2(c+p-1)\ , \tag{2.23}$$

and since the admissibility condition for is equivalent to the admissibility condition for ψ as given in Definition 1.2 , hence ψ , and by Theorem 1.1, F(z) .

By (2.18) , we get

$$\frac{\mathrm{K}_{c,p}^{\delta+2} f(z)}{z^{p-1}} \prec q(z)\ .$$

In case , we have the following example .

Example 2.2. Let the class of admissible functions consist of those functions that satisfy the admissibility condition :

$$v - u\ = \frac{m\zeta q'(\zeta)\ -\ q(\zeta)}{c+p} \notin\ \Omega\ ,$$

.

$$\frac{\mathrm{K}_{c,p}^{\delta+1} f(z)}{z^{p-1}} - \frac{\mathrm{K}_{c,p}^{\delta+2} f(z)}{z^{p-1}} \subset \Omega\ ,$$

then

$$\frac{\mathrm{K}_{c,p}^{\delta+2}f(z)}{z^{p-1}} \prec q(z)\,.$$

We consider the special situation when is a simply connected domain. In this case , where is a conformal mapping of onto and the class is written as .
The following result follows immediately from Theorem 2.5.

Theorem 2.6. Let

$$\phi\left(\frac{\mathrm{K}_{c,p}^{\delta+2}f(z)}{z^{p-1}},\frac{\mathrm{K}_{c,p}^{\delta+1}f(z)}{z^{p-1}},\frac{\mathrm{K}_{c,p}^{\delta}f(z)}{z^{p-1}};z\right) \prec h(z), \qquad (2.24)$$

then

$$\frac{\mathrm{K}_{c,p}^{\delta+2}f(z)}{z^{p-1}} \prec q(z)\,.$$

The next results occurs when the behavior of q on is not known .

Corollary 2.1. Let , be univalent in U and $q(0)$. Let for some (0,1),where .

$$\phi\left(\frac{\mathrm{K}_{c,p}^{\delta+2}f(z)}{z^{p-1}},\frac{\mathrm{K}_{c,p}^{\delta+1}f(z)}{z^{p-1}},\frac{\mathrm{K}_{c,p}^{\delta}f(z)}{z^{p-1}};z\right) \in \Omega\,, \qquad (2.25)$$

then

$$\frac{\mathrm{K}_{c,p}^{\delta+2}f(z)}{z^{p-1}} \prec q(z)\,.$$

Proof. From Theorem 2.5,we have

$$\frac{\mathrm{K}_{c,p}^{\delta+2}f(z)}{z^{p-1}} \prec q_{\rho}(z)$$

and the proof is complete .

Theorem 2.7 Let h and q be univalent in U , with $q(0)$ and set Let : satisfy one of the following conditions :

(1) , for some (0,1) , or

(2) there exists (0,1) such that , for all (0,1) .

(2.24) ,then

$$\frac{\mathrm{K}_{c,p}^{\delta+2}f(z)}{z^{p-1}} \prec q(z)$$

Proof .

case (1): By applying Theorem 2.5, we obtain , since we deduce

$$\frac{K_{c,p}^{\delta+2} f(z)}{z^{p-1}} \prec q(z).$$

case (2): If we let *F(z)* and let , then

$$\phi\,(F_\rho(z), zF_\rho{}'(z), z^2 F_\rho{}''(z); \rho z) = \phi(F(\rho z), \rho z F'(\rho z), \rho^2 z^2 F''(\rho.$$

By using Theorem 2.5 and the comment associated with Ω , where is any function mapping U into U , with , we obtain for (,1) . By letting , we get .

Therefore

$$\frac{K_{c,p}^{\delta+2} f(z)}{z^{p-1}} \prec q(z).$$

The next result give the best dominant of the differential subordination (2.24)

Theorem 2.8. Let be univalent in U and let : .Suppose that the differential equation

$$\phi\,(q(z), zq'(z), z^2 q''(z); z) = h(z) \qquad (2.26)$$

has a solution q with $q(0)$ and satisfy one of the following conditions :

(1) q and ,

(2) q is univalent in U and , for some (0,1) , or

(3) q is univalent in U and there exists (0,1) such that ,for all (0,1).

(2.24) ,then and q is the best dominant .

Proof . By applying Theorem 2.6 and Theorem 2.7 , we deduce that q is a dominant of (2.24) . Since q satisfies (2.26) , it is also a solution of (2.24) and therefore q will be dominated by all dominants of (2.24) . Hence q is the best dominant of (2.24) .

Definition 2.3. Let Ω be a set in and q . The class of admissible functions consists of those functions : that satisfy the admissibility condition :

Ω ,

whenever

$$u = q(\zeta)\,, \quad v = \frac{m\zeta q'(\zeta) + (c+p)\big(q(\zeta)\big)^2}{(c+p)q(\zeta)} \quad (p \in \mathbb{N}\,, c > -p)\,,$$

and

$$Re\left\{\frac{(w-u)(c+p)u}{v-u} - (c+p)(w-3u)\right\} \geq m\, Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\}, \qquad (2.27)$$

.

Theorem 2.9. Let

$$\left\{\phi\left(\frac{\mathrm{K}_{c,p}^{\delta+2} f(z)}{\mathrm{K}_{c,p}^{\delta+3} f(z)}, \frac{\mathrm{K}_{c,p}^{\delta+1} f(z)}{\mathrm{K}_{c,p}^{\delta+2} f(z)}, \frac{\mathrm{K}_{c,p}^{\delta} f(z)}{\mathrm{K}_{c,p}^{\delta+1} f(z)}\,;\, z\right) : z \in \mathrm{U}\right\} \subset \Omega\,, \qquad (2.28)$$

then

$$\frac{\mathrm{K}_{c,p}^{\delta+2} f(z)}{\mathrm{K}_{c,p}^{\delta+3} f(z)} \prec q(z)\,.$$

Proof . Let the analytic function F in U defined by

$$F(z) = \frac{\mathrm{K}_{c,p}^{\delta+2} f(z)}{\mathrm{K}_{c,p}^{\delta+3} f(z)}\,. \qquad (2.29)$$

Differentiating (2.29) yields

$$\frac{zF'(z)}{F(z)} = \frac{z\left(\mathrm{K}_{c,p}^{\delta+2} f(z)\right)'}{\mathrm{K}_{c,p}^{\delta+2} f(z)} + \frac{z\left(\mathrm{K}_{c,p}^{\delta+3} f(z)\right)'}{\mathrm{K}_{c,p}^{\delta+3} f(z)}\,. \qquad (2.30)$$

By using the relation (2.4) , we get

$$\frac{z\left(\mathrm{K}_{c,p}^{\delta+2} f(z)\right)'}{\mathrm{K}_{c,p}^{\delta+2} f(z)} = \frac{zF'(z)}{F(z)} + (c+p)F(z) - c\,. \qquad (2.31)$$

Therefore

$$\frac{\mathrm{K}_{c,p}^{\delta+1} f(z)}{\mathrm{K}_{c,p}^{\delta+2} f(z)} = \frac{zF'(z) + (c+p)(F(z))^2}{(c+p)F(z)}\,. \qquad (2.32)$$

Further computations show that

$$\frac{K_{c,p}^{\delta+1}f(z)}{K_{c,p}^{\delta+2}f(z)} = \frac{z^2F''(z) + [1+3(c+p)F(z)]zF'(z) + (c+p)^2(F(z))^3}{(c+p)zF'(z) + (c+p)^2(F(z))^2} . \qquad (2.33)$$

Further , let us define the transformations from by

$$u = r\,, v = \frac{s+(c+p)r^2}{(c+p)r}\,, w = \frac{t+[1+3(c+p)r]s+(c+p)^2r^3}{(c+p)s+(c+p)^2r^2} .$$

Let

$$\psi(r,s,t;z) = \phi(u,v,w;z) = \phi\left(r, \frac{s+(c+p)r^2}{(c+p)r}, \frac{t+[1+3(c+p)r]s+(c+p)^2r^3}{(c+p)s+(c+p)^2r^2}; z\right) . \qquad (2.34)$$

The proof will make use of Theorem 1.1.Using (2.29), (2.32) and (2.33), from (2.34), we obtain

$$\psi\,(F(z), zF'(z), z^2F''(z); z) = \phi\left(\frac{K_{c,p}^{\delta+2}f(z)}{K_{c,p}^{\delta+3}f(z)}, \frac{K_{c,p}^{\delta+1}f(z)}{K_{c,p}^{\delta+2}f(z)}, \frac{K_{c,p}^{\delta}f(z)}{K_{c,p}^{\delta+1}f(z)}\,;\, z\right). \qquad (2.35)$$

Therefore (2.28) becomes

$$\psi\,(F(z), zF'(z), z^2F''(z); z) \in \Omega\,. \qquad (2.36)$$

Note that

$$\frac{t}{s} + 1 = \frac{(w-u)(c+p)u}{v-u} - (c+p)(w-3u)\,, \qquad (2.37)$$

and since the admissibility condition for is equivalent to the admissibility condition for ψ as given in Definition 1.2 , hence ψ , and by Theorem 1.1,

,

By (2.29), we get

$$\frac{K_{c,p}^{\delta+2}f(z)}{K_{c,p}^{\delta+3}f(z)} \prec q(z)\,.$$

We consider the special situation when is a simply connected domain . In this case = , where is a conformal mapping of U onto and the class is written as .
The following result follows immediately from Theorem 2.9.

Theorem 2.10. Let

$$\phi\left(\frac{K_{c,p}^{\delta+2}f(z)}{K_{c,p}^{\delta+3}f(z)}, \frac{K_{c,p}^{\delta+1}f(z)}{K_{c,p}^{\delta+2}f(z)}, \frac{K_{c,p}^{\delta}f(z)}{K_{c,p}^{\delta+1}f(z)}\,;\, z\right) \prec h(z)\,, \qquad (2.38)$$

then

$$\frac{K_{c,p}^{\delta+2}f(z)}{K_{c,p}^{\delta+3}f(z)} \prec q(z) .$$

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