An easier method for finding all types of (k,n;{1,2})-arcs in PG(2,q) , q odd

Makbola J. Mohammed
Civil Engg. Department , College of Engineering

Abstract:

In this work, we study the (k,n;f)-arcs in a projective plane PG(2,q), $q=p^h$, p is any positive prime number greater than 2 , and h is any positive integer number.

This research presents a new and fast method to determine these arcs, by using this method we can prove that the number of (k,n;{1,2})-arcs in PG(2,q), $q=p^h$ are (q+2). Then finding for example all (k,n;{1,2})-arcs in PG(2,17) whose numbers are (19). Only two types of these arcs are known in [6], also finding all (k,n;{1,2})-arcs in the two projective planes PG(2,23) and PG(2,25) which has been not researched until now, to enable finding in general all arcs in PG(2,q), $q=p^h$, by using an easier equations and in a short time.

Bearing in mind that the previous approach was able only in limited manner to specify a limited number of those arcs in the projective plane PG (2,19), no other study exists for those arcs of q >19 values.

1. Introduction

The idea of (k,n;f)-arcs was originally proposed by Tallini- Scafati[9], in the papers of Barnabei[1] and d'Agostini[2], and [3],…,[8],[10] an account has been given of some result concerning (k,n;f)-arcs in finite and particularly Galois planes. In this paper we are concerned with extending the work in any projective plane of order q , q odd.

A (k,n)-arc K in a finite projective plane of order q is defined as a set of k points such that some n, but no n+1, of them are collinear .

The definition of a weighted arc given in previous papers . Thus we are concerned with a set K of k>0 points in PG (2,q) to each point p of which is assigned a natural number f(p) called its weight and such that the total weight of the points on any line does not exceed a given natural no. n, line having total weight i is called an i-secant of K . Points not included in K are assigned the weight zero , and use L_j to denote number of points of weight j for j=0,1,…,w . Let t_i denote the number of i- secants of K for which i=0,1,…,n . If the values of i for which t_i is non-zero are $m_1 < .... < n$ then K is said to be of type $(m_1, m_2, ..., n)$.

If K is a (k,n;f)- arc , we call the type of K the set Imf.

If K is a (k,n;f)- arc of type (0,n) then Imf={0,w} and we get the *ordinary* case , i.e , a (k,n;f)- arc is a (k,n)-arc .

A (k,n;f)- arcs which have only one point of weight greater than one are called *monoidal* arcs, i.e a (k,n;f)- arc is called monoidal if Imf={0,1,w} and $L_w=1$.

(1-1) Lemma

The weight W of a (k,n;f)- arc of type (m,n) satisfies $m(q+1) \le W \le (n-w)(q+1)+w$

Proof: See [4] (p.11)

(1-2) Theorem

Let K be a (k,n;f)- arc of type (m,n), m>0 and let V_m^s and V_n^s be respectively the number of lines of weight m and number of lines of weight n passing through a point of weight s ,then

$$(n-m)V_m^s = (n-s)(q+1)-(W-s)$$

$$(n-m)V_n^s = (W-s)-(m-s)(q+1)$$

Proof: See [4] (p.12)

(1-3) Lemma

A necessary condition for the existence of a (k,n;f)-arc K of type (m,n),

m > 0 is that q ≡0 mod (n-m) .

Proof: See [4] (p.14)

Hint:

For a (k,n;f)- arc of type (1,n), it is necessary that Imf={0,1,w} where w≥2 and q ≡ 0 mod (n-1) .

From Theorem (1-2) , we obtain the following:

$$V_1^0 = \frac{qw}{n-1} \qquad V_n^0 = q - \frac{qw}{(n-1)} + 1$$

$$V_1^1 = \frac{q(w-1)}{n-1} \qquad V_n^1 = \frac{q(n-w)}{(n-1)} + 1$$

$$V_1^w = 0 \qquad V_n^w = q+1$$

$$t_n = \frac{q}{(n-1)}[(n-w-1)(q+1)+n]+1$$

$$t_1 = \frac{q}{(n-1)}(qw+w-n)$$

(1-4) Theorem

If K is a (k,n;f)- arc of type (m,n) then its necessary that $w \le n-m$

Proof: See [4] (p.14)

2. The numbers of (k,n;f)-arcs of type (n-q,n) with $L_j>0$ for j=0,1,2 , $L_j=0$,j=3,4,…,q in PG(2,q)

In this article we prove the following Theorem:

(2-1)Theorem: The numbers of (k,n;f)-arcs of type (n-q ,n) with $L_j>0$ for j=0,1,2 , $L_j=0$,j=3,4,…,q in PG(2,q) are exactly (q+2) .

Proof:

Let t_{n-q} be the number of lines of weight $(n-q)$ and t_n the number of the lines of weight n; then

$$t_{n-q} + t_n = q^2+q+1 \quad .... (2\text{-}1)$$

$$(n-q)t_{n-q} + nt_n = W(q+1) = (n-q)(q+1)^2 \quad .... (2\text{-}2)$$

Multiplying equation (2-1) by (n-q) and subtracting from equation (2-2), we get

$t_n = (n-q)$ ......(2-3)

substitute the value of t_n in equation (2-1), we have

$t_{n-q} = (q^2 + 2q + 1 - n)$ ...... (2-4)

Let N be an n-secant which has no points of weight 0 and suppose that on N there are α points of weight 1 and β points of weight 2 ,counting points of N gives $\alpha + \beta$ = q+1 , and counting weight of points on N, we get $\alpha + 2\beta$ = n

So α = 2(q+1) –n ....(2-5)

β = n–(q+1) ...(2-6)

Counting incidence between points of weight 1 and n-secant gives

$L_1 V_n^1 = t_n \alpha$ ...(2-7)

By using Theorem (1-2) and equations (2-3),(2-5) in equation(2-7), we get

$L_1 = (n-q)(2q+2-n)$ ...(2-8)

Similarly by using Theorem (1-2) and equations (2-3)&(2-6), we have

$L_2 = \frac{(n-q)(n-q-1)}{2}$ ....(2-9)

From equations (2-8) and (2-9) to get the values of L_1 and L_2 greater than zero, then the values of n must be $q < n \le 2(q+1)$.

Hence 2q +2 – q = q + 2.

i.e ,the number of (k,n;{1,2})-arcs in PG(2,q) , $q=p^h$ are exactly (q+2) . □

3-An easier method for finding a (k,n;{1,2})-arcs

All the previous research use the same method for finding a (k,n;f)-arcs which can be explained shortly as follows:

For example , from [6] since the points in the projective plane PG(2,17) are

$L_0 + L_1 + L_2 = q^2 + q + 1 = 307$

$L_0 = q^2 + q + 1 - L_1 - L_2$

then by using equations (2-8)&(2-9) , we obtained

$L_0 = (q^2 + q + 1) - (n-17)(2q+2-n) - \frac{(n-17)(n-q-1)}{2}$. (3-1)

Hence , we get

$2q^2 + (53-3n)q + n^2 - 20n + 53 - 2L_0 = 0$ .... (3-2)

For a solution of q in equation (3-2) ,we required

$(n-79)^2 - (3856 - 16L_0)$ is square ,

i.e $\delta^2 = (n-79)^2 - (3856 - 16L_0)$ ...(3-3)

To find the solution ,we need to find the value of L_0 which make equation (3-3) to be a square and then find from it n and δ ,that is ,we try to check L_0 from 1 to 307 and after large number of efforts we found only two types of a (k,n;f)-arcs in PG(2,17) see [6] .But in Theorem(2-1) we prove that the number of (k,n;{1,2})-arcs in PG(2,17) are exactly 19.

In this research we found an easier method to determine the number of (k,n;{1,2})-arcs in PG(2,q) which can be explain as follows:

We don't use equation (3-2) to check the values of L_0 from 1 to $q^2 + q + 1$. we determined the values of n from equations (2-3) and (2-4) , and from Theorem (2-1) we have $q < n \le 2(q+1)$, when q=17 implies $17 < n \le 36$.

Substitute these values of n in equation(3-2) we get directly the value of L_0

Case -1

If n=18 from equation (2-8) and equation (2-9) $L_1 = 18$, $L_2 = 0$ and from equation (3-1) $L_0 = 289$

Substitute the values of L_0 and n in equation (3-2) , we get $2q^2 - q - 561 = 0$

(2q+33) (q-17) = 0 $\Rightarrow$ q =17

$\delta^2 = (18-79)^2 - (3856 - 16*289)$

$\delta^2 = 4489 \Rightarrow \delta = 67$

∴ The type of (k,n;f)-arc is (1,18)

$m(q+1) \le W \le (n-w)(q+1) + w \Rightarrow 18 \le W \le 35$

Case -2

When n=19 from equation (2-8) and equation (2-9) $L_1 = 34$, $L_2 = 1$ and from equation (3-1) $L_0 = 272$, then from equation (3-2) we have $2q^2 - 4q - 510 = 0$

(2q+30) (q-17) = 0 $\Rightarrow$ q =17

And from equation (3-3) we see that δ is a square .

∴ The type of (k,n;f)-arc is (2,19) and is called ***monoidal*** and $36 \le W \le 53$

Case -3

When n=20 from equation (2-8) and equation (2-9) $L_1 = 48$, $L_2 = 3$ and from equation (3-1) $L_0 = 256$,from equation (3-2) we have $2q^2 - 7q - 459 = 0$

(2q+27) (q-17) = 0 $\Rightarrow$ q =17

And from equation (3-3) we see that δ is a square .

∴ The type of (k,n;f)-arc is (3,20) and $54 \le W \le 71$

Case -4

When n=21 from equation (2-8) and equation (2-9) $L_1 = 60$, $L_2 = 6$ and from equation (3-1) $L_0 = 241$,from equation (3-2) we have $2q^2 - 10q - 408 = 0$

(2q+24) (q-17) = 0 $\Rightarrow$ q =17

And from equation (3-3) we see that δ is a square .

∴ The type of (k,n;f)-arc is (4,21) and $72 \le W \le 89$

Case -5

When n=22 from equation (2-8) and equation (2-9) $L_1 = 70$, $L_2 = 10$ and from equation (3-1) $L_0 = 227$,from equation (3-2) we have $2q^2 - 13q - 357 = 0$

$(2q+21)(q-17)=0 \Rightarrow q=17$
And from equation (3-3) we see that δ is a square .
$\therefore$ The type of (k,n;f)-arc is (5,22) and $90 \le W \le 107$

Case -6
When n=23 from equation (2-8) and equation (2-9) $L_1 = 78$, $L_2 = 15$ and from equation (3-1) $L_0 = 214$,from equation (3-2) we have $2q^2 - 16q - 306 = 0$
$(2q+18)(q-17)=0 \Rightarrow q=17$
And from equation (3-3) we see that δ is a square .
$\therefore$ The type of (k,n;f)-arc is (6,23) and $108 \le W \le 125$

Case -7
When n=24 from equation (2-8) and equation (2-9) $L_1 = 84$, $L_2 = 21$ and from equation (3-1) $L_0 = 202$,from equation (3-2) we have $2q^2 - 19q - 255 = 0$
$(2q+15)(q-17)=0 \Rightarrow q=17$
And from equation (3-3) we see that δ is a square .
$\therefore$ The type of (k,n;f)-arc is (7,24) and $126 \le W \le 143$

Case -8
When n=25 from equation (2-8) and equation (2-9) $L_1 = 88$, $L_2 = 28$ and from equation (3-1) $L_0 = 191$,from equation (3-2) we have $2q^2 - 22q - 204 = 0$
$(2q+12)(q-17)=0 \Rightarrow q=17$
And from equation (3-3) we see that δ is a square .
$\therefore$ The type of (k,n;f)-arc is (8,25) and $144 \le W \le 161$

Case -9
When n=26 from equation (2-8) and equation (2-9) $L_1 = 90$, $L_2 = 36$ and from equation (3-1) $L_0 = 181$,from equation (3-2) we have $2q^2 - 25q - 153 = 0$
$(2q+9)(q-17)=0 = q=17$
And from equation (3-3) we see that δ is a square.
$\therefore$ The type of (k,n;f)-arc is (9,26) and $162 \le W \le 179$

Case -10
When n=27 from equation (2-8) and equation (2-9) $L_1 = 90$, $L_2 = 45$ and from equation (3-1) $L_0 = 172$,from equation (3-2) we have $2q^2 - 28q - 102 = 0$
$(2q+6)(q-17)=0 \Rightarrow q=17$
And from equation (3-3) we see that δ is a square .
$\therefore$ The type of (k,n;f)-arc is (10,27) and $180 \le W \le 197$

Case -11
When n=28 from equation (2-8) and equation (2-9) $L_1 = 88$, $L_2 = 55$ and from equation (3-1) $L_0 = 164$,from equation (3-2) we have $2q^2 - 31q - 51 = 0$
$(2q+3)(q-17)=0 \Rightarrow q=17$
And from equation (3-3) we see that δ is a square .
$\therefore$ The type of (k,n;f)-arc is (11,28) and $198 \le W \le 215$

Case -12
When n=29 from equation (2-8) and equation (2-9) $L_1 = 84$, $L_2 = 66$ and from equation (3-1) $L_0 = 157$,from equation (3-2) we have $2q^2 - 34q = 0$
$2q(q-17)=0 \Rightarrow q=17$
And from equation (3-3) we see that δ is a square .
$\therefore$ The type of (k,n;f)-arc is (12,29) and $216 \le W \le 233$

Case -13
When n=30 from equation (2-8) and equation (2-9) $L_1 = 78$, $L_2 = 78$ and from equation (3-1) $L_0 = 151$,from equation (3-2) we have $2q^2 - 37q + 51 = 0$
$(2q-3)(q-17)=0 \Rightarrow q=17$
And from equation (3-3) we see that δ is a square .
$\therefore$ The type of (k,n;f)-arc is (13,30) and $234 \le W \le 251$

Case -14
When n=31 from equation (2-8) and equation (2-9) $L_1 = 70$, $L_2 = 91$ and from equation (3-1) $L_0 = 146$,from equation (3-2) we have $2q^2 - 40q + 102 = 0$
$(2q-6)(q-17)=0 \Rightarrow q=17$
And from equation (3-3) we see that δ is a square .
$\therefore$ The type of (k,n;f)-arc is (14,31) and $252 \le W \le 269$

Case -15
When n=32 from equation (2-8) and equation (2-9) $L_1 = 60$, $L_2 = 105$ and from equation (3-1) $L_0 = 142$,from equation (3-2) we have $2q^2 - 43q + 153 = 0$
$(2q-9)(q-17)=0 \Rightarrow q=17$
And from equation (3-3) we see that δ is a square.
$\therefore$ The type of (k,n;f)-arc is (15,32) and $270 \le W \le 287$

Case -16

When n=33 from equation (2-8) and equation (2-9) $L_1 = 48$, $L_2 = 120$ and from equation (3-1) $L_0 = 139$,from equation (3-2) we have

$2q^2 - 46q + 204 = 0$

(2q-12) (q-17) = 0 $\Rightarrow$ q =17

And from equation (3-3) we see that δ is a square.

$\therefore$ The type of (k,n;f)-arc is (16,33) and $288 \le W \le 305$

Case -17

When n=34 from equation (2-8) and equation (2-9) $L_1 = 34$, $L_2 = 136$ and from equation (3-1) $L_0 = 137$, from equation (3-2) we have

$2q^2 - 49q + 255 = 0$

(2q-15) (q-17) = 0 $\Rightarrow$ q =17

And from equation (3-3) we see that δ is a square

$\therefore$ The type of (k,n;f)-arc is (17,34) and $306 \le W \le 323$

Case -18

When n=35 from equation (2-8) and equation (2-9) $L_1 = 18$, $L_2 = 153$ and from equation (3-1) $L_0 = 136$,from equation (3-2) we have

$2q^2 - 52q + 306 = 0$

(2q-18) (q-17) = 0 $\Rightarrow$ q =17

And from equation (3-3) we see that δ is a square .

$\therefore$ The type of (k,n;f)-arc is (18,35) and $324 \le W \le 341$

Case -19

When n=36 from equation (2-8) and equation (2-9) $L_1 = 0$, $L_2 = 171$ and from equation (3-1) $L_0 = 136$,from equation (3-2) we have $2q^2 - 55q + 357 = 0$

(2q-21) (q-17) = 0 $\Rightarrow$ q =17

And from equation (3-3) we see that δ is a square

$\therefore$ The type of (k,n;f)-arc is (19,36) and is called ***ordinary*** and $342 \le W \le 359$

Note : By the above method we found all (19) types (m,n) of a (k,n;{1,2})-arcs in PG(2,17) , but in [6] after large numbers of attempts found only two types of this arcs which are *case-2* and *case-12* ,see [6]

4. A (k,n;{1,2})-arcs of type (m,n) in PG(2,23):

The projective plane PG(2,23) contains (553) points and (553) lines each point lies on 24 lines and 24 points pass through each line.

In this section we found all (25) types of a (k,n;{1,2})-arcs in PG(2,23) which has been not researched until now .

From Theorem (2-1) we have $q < n \le 2(q+1)$.

When q=23 implies $23 < n \le 48$.

In PG (2,23) equation (3-2) becomes

$2q^2 + (71 - 3n)q + n^2 - 26n + 71 - 2L_0 = 0$ ....(4-1)

Similarly equations (2-8) and (2-9) in PG(2,23) becomes

$L_1 = (n-23)(48-n)$ .... (4-2)

$L_2 = \frac{(n-23)(n-24)}{2}$ ......(4-3)

And $L_0 = q^2 + q + 1 - L_1 - L_2$

Substitute the values of n and L_0 in equation (4-1) as in (article -3) we get all (25) types of a (k,n;f)-arcs in PG(2,23) ,see (table-1).

(Table -1)

n	L_1	L_2	L_0	Type (m,n)	W
24	24	0	529	(1,24)	$24 \le W \le 47$
25	46	1	506	(2,25) ***monoidal***	$48 \le W \le 71$
26	66	3	484	(3,26)	$72 \le W \le 95$
27	84	6	463	(4,27)	$96 \le W \le 119$
28	100	10	443	(5,28)	$120 \le W \le 143$
29	114	15	424	(6,29)	$144 \le W \le 167$
30	126	21	406	(7,30)	$168 \le W \le 191$
31	136	28	389	(8,31)	$192 \le W \le 215$
32	144	36	373	(9,32)	$216 \le W \le 239$
33	150	45	358	(10,33)	$240 \le W \le 263$
34	154	55	344	(11,34)	$264 \le W \le 287$
35	156	66	331	(12,35)	$288 \le W \le 311$
36	156	78	319	(13,36)	$312 \le W \le 335$
37	154	91	308	(14,37)	$336 \le W \le 359$
38	150	105	298	(15,38)	$360 \le W \le 383$
39	144	120	289	(16,39)	$384 \le W \le 407$
40	136	136	281	(17,40)	$408 \le W \le 431$
41	126	153	274	(18,41)	$432 \le W \le 455$
42	114	171	268	(19,42)	$456 \le W \le 479$
43	100	190	263	(20,43)	$480 \le W \le 503$
44	84	210	259	(21,44)	$504 \le W \le 527$
45	66	231	256	(22,45)	$528 \le W \le 551$
46	46	253	254	(23,46)	$552 \le W \le 575$
47	24	276	253	(24,47)	$576 \le W \le 599$
48	0	300	253	(25,48) ***ordinary***	$600 \le W \le 623$

5. A (k,n;{1,2})-arcs of type (m,n) in PG(2,25):

Now we found all (27) types of a (k,n;{1,2})- arcs in PG(2,25) which are also not researched until now ,this plane contains 651 points and 651 lines

From Theorem (2-1) we have $q < n \le 2(q+1)$.

When q=25 implies $25 < n \le 52$.

In PG(2,25) equation (3-2) becomes

$2q^2 + (77-3n)q + n^2 - 28n + 77 - 2L_0 = 0$ .....(5-1)

Similarly equations (2-8) and (2-9) becomes

$L_1 = (n-25)(52-n)$ ..... (5-2)

$L_2 = \frac{(n-25)(n-26)}{2}$ ....(5-3)

And $L_0 = q^2 + q + 1 - L_1 - L_2$

Substitute the values of n and L_0 in equation (5-1) as in (article -3) we get all (27) types of a (k,n;{1,2})- arcs in PG(2,25) , see (table-2) .

(Table -2)

n	L_1	L_2	L_0	Type (m,n)	W
26	26	0	625	(1,26)	$26 \le W \le 51$
27	50	1	600	(2,27) ***monoidal***	$52 \le W \le 77$
28	72	3	576	(3,28)	$78 \le W \le 103$
29	92	6	553	(4,29)	$104 \le W \le 129$
30	110	10	531	(5,30)	$130 \le W \le 155$
31	126	15	510	(6,31)	$156 \le W \le 181$
32	140	21	490	(7,32)	$182 \le W \le 207$
33	152	28	471	(8,33)	$208 \le W \le 233$
34	162	36	453	(9,34)	$234 \le W \le 259$
35	170	45	436	(10,35)	$260 \le W \le 285$
36	176	55	420	(11,36)	$286 \le W \le 311$
37	180	66	405	(12,37)	$312 \le W \le 337$
38	182	78	391	(13,38)	$338 \le W \le 363$
39	182	91	378	(14,39)	$364 \le W \le 389$
40	180	105	366	(15,40)	$390 \le W \le 415$
41	176	120	355	(16,41)	$416 \le W \le 441$
42	170	136	345	(17,42)	$442 \le W \le 467$
43	162	153	336	(18,43)	$468 \le W \le 493$
44	152	171	328	(19,44)	$494 \le W \le 519$
45	140	190	321	(20,45)	$520 \le W \le 545$
46	126	210	315	(21,46)	$546 \le W \le 571$
47	110	231	310	(22,47)	$572 \le W \le 597$
48	92	253	306	(23,48)	$598 \le W \le 623$
49	72	276	303	(24,49)	$624 \le W \le 649$
50	50	300	301	(25,50)	$650 \le W \le 675$
51	26	325	300	(26,51)	$676 \le W \le 701$
52	0	351	300	(27,52) ***ordinary***	$702 \le W \le 727$

Finally we can found by this method all types of a (k,n;{1,2})-arcs in a projective plane PG(2,q) , $q=p^h$,p is any positive prime number greater than 2 ,and h is any positive integer number.

References

1- Barnabei, M. , "On arcs with weighted points" ,Journal of Statistical Planning and Inference, 3 (1979) , 279-286

2- d` Agostini, E., "Sulla caratterizzazione celle (k,n; f)- calotte di tipo (n-2,n)" Atti Sem. Mat.Fis.University. Modena, XXIX, (1980) , 263-275

3- Hammed, F.K. "Weighted (k,n)-arcs in the projective plane of order nine" PH.D.Thesis University of London (1989).

4- Mahmood , R.D. " (k,n;f)-arcs of type (n-5,n) in PG(2,5)" M.Sc.Thesis ,college of science ,University of Mosul (1990).

5- Makbola J. & Ban A. "On Arcs with weighted points of type (n-13,n) in PG(2,13) ".J. Edu. Sci, vol.18 No.4, 83-93,2006.

6- Makbola J. & Ban A. "Weighted points and lines In projective plane of order 17" Raf.J.comp.& Math`s ,vol.4,No.2, 33-41,2007 .

7- Makbola J. & Ban A. & Layla Jasim "Classification of some arcs of type (m,n) in PG(2,19)".J.Edu. & Sci., vol.22, No.4, 83-93, 2009.

8- Mohammed, M, J. and Mahmood, R.D." (k,n;f)-arcs in Galois plane of order seven" Basrah J. science, vol. 13,No. 1,49-56 ,1995 .

9- Tallini Scafati , M. , "Graphic Curves on a Galois Plane", Atti del convegno di Geometria Combinatoria e sue Applicazioni Perugia (1971) , 413-419

10-Wilson B.J. "(k,n;f)-arc and caps in finite projective spaces " Annals of Discrete mathematics 30 , 355-362(1986).

الطريقة الاسهل لايجاد جميع انواع الاقواس (k,n;{1,2}) في المستوى (2,q) حيث q عدد فردي

مقبولة جاسم محمد المحترمة

قسم الهندسة المدنية ، كلية الهندسة ، جامعة الموصل ، الموصل ، العراق

الملخص

لدراسة الأقواس -(k,n;f) في المستوي ألإسقاطي PG(2,q) حيث $q=p^h$, لأي عدد أولي p أكبر من 2 و h عدد صحيح موجب . تم في هذا البحث إيجاد طريقة جديدة وسريعة لتحديد هذه الأقواس. وبالاعتماد على هذه الطريقة تمكنا من إثبات أن عدد الأقواس -(k,n;{1,2}) في المستوي ألإسقاطي PG(2,q) حيث $q=p^h$ هي (q+2) .

ثم وجدنا على سبيل المثال كل الأقواس -(k,n;{1,2}) في المستوي ألإسقاطي PG(2,17) وعددها (19) والمعروف منها سابقا نوعان فقط كما ورد في المصدر [6] . وكذلك وجدنا كل الأقواس -(k,n;{1,2}) في المستويين PG(2,23) و PG(2,25) والتي لم تكن معروفة إلى حد الآن على حد علمنا ، وبالإمكان الآن إيجاد الأقواس أعلاه لأي مستوي PG(2,q) حيث $q=p^h$ ، بمعادلات بسيطة وبفترة قصيرة .

علما أن الطريقة السابقة حددت عدد قليل من هذه الأقواس والى حد المستوي ألإسقاطي PG(2,19) ،ولا توجد دراسة لهذه الأقواس لقيم q>19 .