THE CONNECTED AND CONTINUITY IN BITOPOLOGICAL SPACES

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ABSTRACT :- In this paper we introduce and study several properties of the connected and continuity in bitopological spaces by using β-open set.

ملخص البحث :- سندرس في هذا البحث الترابط والاستمرارية في الفضاءات التبولوجية الثنائية باستخدام المجموعة المفتوحة (β-open set)

1- INTRODUCTION :-

In 1963 , J.C. Kelly [5] initiated the study of bitopological spaces . A set X equipped with two topologies τ and $\tau^{\prime}$ is called a bitopological space denoted by (X, τ, $\tau^{\prime}$) . The notion of β-open sets due to Mashhour et al .[1]or semi – preopen sets due to Andrijevic' [2]. Plays a significant role in general topology. In [1] the concept of β-continuous function is introduced and further Popa and Noiri [4] studied the concept of weakly β-continuous functions. In 1992,Khedr et al [3] introduced and studied semi- precontinuity or β-continuous in bitopological spaces. In this paper we introduce and study the definition of β-open sets to define the connected and continuous in bitopological spaces .

Throughout the present paper ,(X, τ, τ_{β}) denotes a bitopological space . Let (X, τ) be a topological space and A be a subset of X the closure and interior of A are denoted by cl(A) and Int(A) respectively .

Let ,(X, τ, τ_{β}) be bitopological space and let A be a subset of X . The closure and interior of A with respect to τ are denoted by τ-cl(A) and τ-Int(A) . The closure and interior of A with respect to τ_{β} are denoted by τ_{β}-cl(A) and τ_{β}-Int(A). A separation of space X denoted by $X = C\backslash D$, where C and D are two nonempty disjoint sets .

2-BASIC DEFINITION:-

Definition (2.1) [1] :- Let (X, τ) be a topological space , a subset A of space X will be called β-open if A$\subseteq$ cl(Int(cl(A))).

The complement of a β-open set is said to be β-closed sets containing A the subset of X is known as β $-$closure of A and it is denoted by τ_{β}-cl(A)

(i.e) A$\subseteq$ τ_{β}-cl(A) .

PROPOSITION (2.2) :- Let U be open set then U is β-open .

Proof :- since U is open set .so we have U = Int(U) .
Since Int(U)$\subseteq$cl(U) . then Int(Int(U))$\subseteq$ Int(cl(U))
and cl(Int(Int(U)))$\subseteq$ cl(Int(cl(U))) .
Therefore U$\subseteq$ cl(u)$\subseteq$ cl(Int(cl(U))) .
So we have U$\subseteq$ cl(Int(cl(U))). Hence U is β-open.

Definition (2.3) [1]:- Bitopological space is any set X with two topological spaces τ and τ_β

Example (2.4) :- let $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$ and $\tau_\beta = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b\}\}$ so ,(X, τ, τ_β) is bitopological spaces

3- CONNECTED SPACES

Definition (3.1) :- Abitopological space ,(X, τ, τ_β) is connected if X cannot be expressed as the union of two nonempty disjoint sets U and V such that
$[U \cap \tau - cl(V)] \cup [\tau_\beta - cl(U) \cap V] = \emptyset$
Suppose X can be expressed then X is called disconnected and we write $X = U \backslash V$ and call this separation of X .

Example (3.2) :- :- let $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$ and $\tau_\beta = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b\}\}$, then $\{a, c\}$ is connected

PROPOSITION (3.3) :- if X contains no nonempty proper subset which is both τ- open and τ_β-closed , then X is connected .

Proof :- Let X contains no nonempty proper subset which is both τ- open and τ_β-closed. Suppose that X is disconnected .
Then X can be expressed as the union of two nonempty disjoint sets U and V such that $[U \cap \tau - cl(V)] \cup [\tau_\beta - cl(U) \cap V] = \emptyset$
Since $U \cap V = \emptyset$ and $U \cup V = X$, we have $U = V^c$ and $V = U^c$
Since $\tau_\beta - cl(U) \cap V = \emptyset$,so $\tau_\beta - cl(U) \subseteq V^c$.
Hence $\tau_\beta - cl(U) \subseteq U$. therefore U is τ_β-closed set .
Similarly V is τ –closed set .

Since $U = V^c, U$ is τ- open.Therefore there exists a nonempty proper subset which is both τ- open and τ_β-closed . this is contradiction to our assumption Therefore X is connected .

PROPOSITION (3.4):- if U is connected subset of bitopological space (X, τ, τ_β) which has separation $X = C \backslash D$, then $U \subseteq C$,or $U \subseteq D$.

Proof :- suppose that(X, τ, τ_β) has separation $X = C \backslash D$.

Then $X = C \cup D$, where C and D are two nonempty disjoint sets such that $[C \cap \tau - cl(D)] \cup [\tau_\beta - cl(C) \cap D] = \emptyset$.

Since $C \cap D = \emptyset$, we have $C = D^c$and $D = C^c$.

Now ,

$$[(C \cap U) \cap \tau - cl(D \cap U)] \cup [\tau_\beta - cl(C \cap U) \cap (D \cap U)] \subseteq$$
$$\big[[C \cap \tau - cl(D)] \cup [\tau_\beta - cl(C) \cap D]\big] = \emptyset \ .$$

Hence $U = (C \cap U) \backslash (D \cap U)$ is separation of U.

Since U is connected ,So we have either $C \cap U = \emptyset$ or $D \cap U = \emptyset$.

Consequently $U \subseteq C^c$ or $U \subseteq D^c$. Therefore $U \subseteq C$ or $U \subseteq D$.

PROPOSITION (3.5):- if $U = \cup U_i$ be any family of connected sets in bitopological space (X, τ, τ_β) with $\cap U_i \neq \emptyset$,then U is connected set in (X, τ, τ_β) .

Proof :- Let $U = \cup U_i$ be any family of connected sets in bitopological space (X, τ, τ_β) for each $i \in I$,where I be index set with $\cap U_i \neq \emptyset$.

Suppose that U is disconnected .

Then $U = C \cup D$, where C and D are two nonempty disjoint sets such that

$$\big[[C \cap \tau - cl(D)] \cup [\tau_\beta - cl(C) \cap D]\big] = \emptyset$$

Since U_i is connected and $U_i \subseteq U$,we have $U_i \subseteq C$ or $U_i \subseteq D$.

Therefore $\cup U_i \subseteq C$ or $\cup U_i \subseteq D$, hence $U \subseteq C$ or $U \subseteq D$.

Since $\cap U_i \neq \emptyset$, we have $x \in (\cap U_i)$. therefore $x \in U_i$ for all i .

Consequently , $x \in U$. So either $x \in C$ or $x \in D$.

Suppose $x \in C$. since $C \cap D = \emptyset$, so we have $x \notin D$. therefore $U \not\subset D$ and $U \subseteq C$

This is contradiction with the assumption of $U = C \cup D$.

So U is connected .

4- CONTINUOUS FUNCTION

Definition (4.1):- A function $f:(X,\tau,\tau_\beta) \to (Y,\omega,\omega_\beta)$ is said to be continuous if $f^{-1}(U)$ is (τ,τ_β) - β-open in X for each ω-open set U of Y.

PROPOSITION (4.2):- A function $f:(X,\tau,\tau_\beta) \to (Y,\omega,\omega_\beta)$ is continuous iff $f^{-1}(U)$ is (τ,τ_β) - β-closed for each ω-closed set in Y

Proof :- Suppose that f is continuous and let U be ω-closed set in Y .
Then U^c is ω-open set in Y. since f is continuous ,
So we have $f^{-1}(U^c)$ is (τ,τ_β) - β-open in X.
Consequently , $f^{-1}(U)$ is (τ,τ_β) - β-closed in X .
Now , let $f^{-1}(U)$ is (τ,τ_β) - β-closed in X for each ω-closed set U in Y.
Let V be ω-open set in Y. Then V^c be ω-closed set in Y .
Therefore by our assumption , $f^{-1}(V^c)$ is (τ,τ_β) - β-closed in X .
Hence $f^{-1}(V)$ is (τ,τ_β) - β-open in X .this complete the proof .

PROPOSITION (4.3):- A function $f:(X,\tau) \to (Y,\omega)$ is continuous iff $f:(X,\tau,\tau_\beta) \to (Y,\omega,\omega_\beta)$ is continuous .

Proof :- Suppose that $f:(X,\tau) \to (Y,\omega)$ is continuous.
So we have $f^{-1}(U)$ is τ- open set . since every τ- open set is (τ,τ_β) - β-open set
Then $f^{-1}(U)$ is (τ,τ_β) - β-open set .
Therefore $f:(X,\tau,\tau_\beta) \to (Y,\omega,\omega_\beta)$ is continuous.
Conversely , let $f:(X,\tau,\tau_\beta) \to (Y,\omega,\omega_\beta)$ is continuous function .
Let U be any ω-open set in (Y,V) .
Since $f:(X,\tau,\tau_\beta) \to (Y,\omega,\omega_\beta)$ is continuous .
So we have $f^{-1}(U)$ is (τ,τ_β) - β-open set (X,τ,τ_β).
Therefore $f^{-1}(U)$ is τ- open. This completes the proof .

PROPOSITION (4.4):- if $f:(X,\tau,\tau_\beta) \to (Y,\omega,\omega_\beta)$ is continuous and $g:(Y,\omega,\omega_\beta) \to (Z,\varphi,\varphi_\beta)$ is continuous then $g o f$ is continuous .

Proof :- let U be any ω-open set in $(Z,\varphi,\varphi_\beta)$.
Since g is continuous function, then $g^{-1}(U)$ is (ω,ω_β) - β-open in (Y,ω,ω_β) .
So $g^{-1}(U)$ is ω-open in (Y,ω,ω_β) .
Since f is continuous function .
So $f^{-1}(g^{-1}(U)) = (g o f)^{-1}(U)$ is (τ,τ_β) - β-open set (X,τ,τ_β) .

PROPOSITION (4.4):- let $f:(X,\tau,\tau_\beta) \rightarrow (Y,\omega,\omega_\beta)$ be continuous function then the image of connected space under f is connected.

Proof :- let $f:(X,\tau,\tau_\beta) \rightarrow (Y,\omega,\omega_\beta)$ be continuous function and let X be connected space .

Suppose that Y is disconnected .

Then $Y = C \cup D$ where C is ω-open set in Y , D is τ_β-open in Y .

Since f is continuous , so we have $f^{-1}(C)$ is (τ,τ_β) - β-open set and $f^{-1}(D)$ is (τ,τ_β) - β-open set in X .

Also $X = f^{-1}(C) \cup f^{-1}(D)$,where $f^{-1}(C)$ and $f^{-1}(D)$ are two nonempty disjoint sets .

Then X is disconnected . this is contradiction to fact that X is connected . therefore Y is connected .

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