Weakly quasi-prime radical of submodules

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Abstract:

Let R be acommutative ring with identity, and M be a unitary R-module. In this paper we introduce the concept weakly quasi-prime radical of submodules N as a generalization of a prime radical of submodule N (for short, $Wqprad_M(N)$) is define as the intersection of all weakly quasi-prime submodules of M which contain N. Also, we introduce the concept weakly quasi-prime radical submodule, where a proper submodule N of an R-module M which satisfies the property $Wqprad_M(N)=N$ is called a weakly quasi-prime radical submodule of M. Many properties of these concepts are given.

Introduction:

Throughout this paper R will be denoted a commutative ring with identity, and M be a unitary R-module. A prime radical of an R-submodule N of M, denoted by $rad_M(N)$was defined in [7] as the intersection of all prime submodule of M containing N, then $rad_M(N) = M$. Where a proper submodule K of M is called a prime if r $m\in K$ for $r\in R$, $m\in M$ implies that either $m\in N$ or $r\in[K:M]$.

Weakly quasi-prime submodules are generalization of a prime submodules are introduce in [3], where a proper submodule N of an R-module M is called weakly quasi-prime, if wherever $0\neq r_1r_2m\in N$, for each non-Zero elements $r_1r_2\in R$ and for each $0\neq m\in M$, then either $r_1m\in N$ or $r_2m\in N$.

Equivalently, a proper submodule N of an R-module M is weakly quasi-prime if and only if $[N:_R(m)]$ is weakly prime ideal of R for each $0\neq m\in M$.We give the concept of weakly quasi-prime radical of a submodule, and the concept of weakly quasi-prime radical submodules as a generalization of prime radical of N, and prime radical submodules respectively.

S1:Basic properties of Weakly quasi-prime radical:

In this section, we introduce, the concept of weakly quasi-prime radical of a submodule N of an R-module M as a generalization of prime radical of a submodule N, and gives some basic properties of it.

Definition (1.1):

A weakly quasi-prime radical of a submodule N of an R-module M, denoted by $Wqprad_M(N)$ is defined as the intersection of all weakly quasi-prime submodule of M which contain N. If there exists no weakly quasi-prime submodule of M containing N, we put $Wqprad_M(N)=M$. If M=R, and I is an ideal of R, then $Wqprad_M(I)$ is the intersection of all weakly quasi-prime ideals of R containing I.

In the following proposition we give some fundamental properties of weakly quasi-prime radical.

Proposition (1.2):

Let $F:M\to M'$ be an epimorphis from an R-module M in to an R-module M', and K be a submodule of M with $Kerf\subseteq K$, then:

1. $f(Wqprad_M(K)) = Wqprad_{M'}(f(K))$
2. $f^{-1}(Wqprad_{M'}(K')) = Wqprad_M(f^{-1}(K'))$.Where K' is a submodule of M'.

Proof:

1. Since $Wqprad_M(K) = \cap N$ where the intersection is over all weakly quasi-prime submodule N of M with $K\subseteq N$, then $f(Wqprad_M(K)) = f(\cap N)$. Since $Kerf\subseteq K\subseteq N$ then $f(Wqprad_M(K))=\cap f(N)$ [5] where the intersection is over all weakly quasi-prime submodule $f(N)$ of M' with $f(K)\subseteq f(N)$ thus $f(Wqprad_M(f(K))$.
2. Let K' be a submodule of M'. Then $Wqprad_{M'}(K') = \cap N'$ where the intersection is over all weakly quasi-prime submodule N' of M' with $K'\subseteq M'$, then $f^{-1}(Wqprad_M(K'))=f^{-1}(\cap N') = \cap f^{-1}(N')$ [5]. Where the intersection is over all weakly quasi-prime submodule $f^{-1}(N')$ of M with $f^{-1}(K')\subseteq f^{-1}(N')$ therefore $f^{-1}(Wqprad_M(K'))=Wqprad_M(f^{-1}(K'))$.

Proposition(1.3):

Let M be an R-module, and N,L are two submodule of M then:

1. $N\subseteq Wqprad_M(N)$.
2. If $N\subseteq L$ then $Wqprad_M(N)\subseteq Wqprad_M(L)$.
3. $Wqprad_M(Wqprad_M(N)) = Wqprad_M(N)$.
4. $Wqprad_M(N\cap L)\subseteq Wqprad_M(N)\cap Wqprad_M(L)$.
5. $Wqprad_M(N+L) = Wqprad_M(Wqprad_M(N)+ Wqprad_M(L))$.

Proof:

1. Since $Wqprad_M(N)=\cap L$, where the intersection is over all weakly quasi-prime submodule L, with $N\subseteq L$, So $N\subseteq Wqprad_M(N)$.
2. Suppose that $N\subseteq L$, and let K be a weakly quasi-prime submodule of M with $L\subseteq K$. But $N\subseteq L\subseteq K$, implies that $N\subseteq K$. Hence $Wqprad_M(N)\subseteq Wqprad_M(L)$.
3. By part one, we have $Wqprad_M(N)\subseteq Wqprad_M(Wqprad_M(N))$. Now, $Wqprad_M(Wqprad_M(N)) = \cap L$, where the intersection is over all weakly quasi-prime submodule L with $Wqprad_M(N)\subseteq L$. But by (1) we have $N\subseteq Wqprad_M(N)$, and consequently, $Wqprad_M(Wqprad_M(N)\subseteq Wqprad_M(N)$. Hence $Wqprad_M(Wqprad_M(N)) = Wqprad_M(N)$.
4. Sine $N\cap L\subseteq L$ and $N\cap L\subseteq N$, then by (2) we have $Wqprad_M(N\cap L)\subseteq Wqprad_M(L)$ and $Wqprad_M(N\cap L)\subseteq Wqprad_M(N)$. Thus $Wqprad_M(N\cap L)\subseteq Wqprad_M(N)\cap Wqprad_M(L)$.
5. Since $N+L\subseteq Wqprad_M(N)+ Wqprad_M(L)$ then by (2) we have $Wqprad_M(N+L)\subseteq Wqprad_M(Wqprad_M(N)+ Wqprad_M(L))$.

Now let K be a weakly quasi-prime submodule of M such that $N+L\subseteq K$, we are going to prove that $Wqprad_M(N) + Wqprad_M(L) \subseteq K$. Since $N+L \subseteq K$, $N \subseteq K$ and $L \subseteq K$. Thus $Wqprad_M(N) \subseteq K$ and $Wqprad_M(L) \subseteq K$. Hence $Wqprad_M(N)+ Wqprad_M(L) \subseteq K$, and $Wqprad_M(Wqprad_M(N)+ Wqprad_M(L)) \subseteq K$. Therefore $Wqp\ rad_M(Wqprad_M(N)+ Wqprad_M(L)) \subseteq Wqprad_M(N+L)$ and we have $Wqprad_M(Wqp\ rad_M(N)+ Wqprad_M(L))= Wqprad_M(N+L)$.

As a consequence of proposition (1.3) we get the following corollaries.

Corollary (1.4): Let N be a submodule of $Wqprad_M(N(S))$, where S is a multiplicative set of R.

Corollary(1.5): Let N be a submodule of an R-module M then $Wqprad_M(N)\subseteq Wqprad_M(cl(N))$.

Corollary (1.6): Let N be a submodule of an R-module M then $Wqprad_M(N) \subseteq Wqp\ rad_M([N:I])$ for every ideal I of R.

Recall that a submodule N of an R-module M is completely irreducible if for any submodule L_1,L_2 of $M, L_1\cap L_2\subseteq N$, implies that either $L_1=N$ or $L_2=N$[6].

The following proposition gives equality of prop. (1.3(4)) holds under certain condition.

Proposition (1.7): Let M be an R-module, and N, L are two submodules of M. If every weakly quasi-prime submodule of M which contain $N\cap L$ is completely irreducible submodule, then $Wqprad_M(N\cap L) = Wqprad_M(N)\cap W\ qprad_M(L)$.

Proof:

Since $Wqprad_M(N\cap L) \subseteq Wqprad_M(N)\cap W\ qprad_M(L)$ holds by prop.(1.3(4)). If $Wqp\ rad_M(N\cap L)= M$, then $Wqprad_M(N)= Wqprad_M(L)=M$.

If $Wqprad_M(N\cap L)\neq M$, then there exists a weakly quasi-prime submodule K of M such that $N\cap L\subseteq K$, then by hypothesis either $N\subseteq K$ or $L\subseteq K$ so that either $Wqprad_M(N) \subseteq K$ or $Wqprad_M(L) \subseteq K$, since every weakly quasi-prime submodule containing $N\cap L$ is completely irreducible, then we have either $Wqprad_M(N) \subseteq Wqprad_M(N\cap L)$ or $Wqprad_M(L) \subseteq Wqprad_M(N\cap L)$.

Therefore $Wqprad_M(L) \cap Wqprad_M(N) \subseteq Wqprad_M(N\cap L)$. Hence $Wqprad_M(N\cap L) = Wqp\ rad_M(N) \cap Wqprad_M(L)$.

Proposition (1.8):

Let N be a submodule of an R-module M. Then $Wqprad_M([N:M])\ M \subseteq Wqp\ rad_M(N)$.

Proof:

Let $Wqprad_M(N)=M$, then $Wqprad_M([N:M])M \subseteq Wqprad_M(N)$. Let L be any weakly quasi-prime submodule of M containing N, $[N:M] \subseteq [L:M]$. But L is a weakly quasi-prime, then by (3, coro.3.1.4,ch.3). [L:M] is a weakly prime ideal of R, and hence [L:M] is a weakly quasi-prime ideal of R .

Thus $Wqprad_M([N:M])\ M \subseteq [L:M]M\subseteq L$. Therefore $Wqprad_M([N:M]M) \subseteq Wqprad_M(N)$.

Proposition (1.9):

Let M be an R-module, and N,L are submodule of M such that [N:M]+[K:M]= R for each weakly quasi-prime submodule K of M containing $N\cap L$. Then $Wqp\ rad_M(N\cap L)=Wqprad_M(N)\cap Wqprad_M(L)$.

Proof:

Since $N\cap L\subseteq K$, and K is weakly quasi-prime submodule of M, then by (3,coro. 3.1.4)ch.3) we have $L\subseteq K$, thus K is completely irreducible and hence by prop.(1.7) we have $Wqprad_M(N\cap L)=Wqprad_M(N)\cap Wqprad_M(L)$. We can generalize proposition (1.7).

Proposition (1.10):

Let $N_1,N_2,\ldots,N_n$ be a submodule of an R-module M such that wherever $N_1\cap N_2\cap\ldots\cap N_n \subseteq K$, for some i=1,2,….n for any quasi-prime submodule H of M. Then $Wqprad_M(\bigcap_{i=1}^{n} Ni)=\bigcap_{i=1}^{n} Wqprad_M(Ni)$.

Proposition (1.11):

Let N be a submodule of an R-module M. If M satisfies the ascending chain condition on submodules, then $Wqprad_M(N) = M$ if and only if N=M.

Proof:

Suppose that N=M, then $Wqprad_M(N)= Wqprad_M(M)=M$. Now, suppose that M satisfies the ascending chain condition on submodule, then every proper submodule of M is contained in a prime submodule. Hence every proper submodule is contained in a weakly quasi-prime submodule. Thus if N is a proper, then $Wqprad_M(N)\neq M$. Hence if $Wqprad_M(N)= M$ then N=M.

Corollary (1.12):

If an R-module M satisfies the asecending chain condition on a submodules, and N,L are submodules of M. Then $Wqprad_M(N)+Wqprad_M(L)=M$ if and only if N+L=M.

Proof:

Assume that $Wqprad_M(N)+ Wqprad_M(L)=M$. Thus $Wqprad_M(Wqprad_M(N)+ Wqp\ rad_M(L))= Wqprad_M(M) =M$. If N+L=M, then $Wqprad_M(N+L)= Wqprad_M(M)=M$. Mean $Wqprad_M(Wqprad_M(N)+ Wqprad_M(L)=M$, implies that $Wqprad_M(N)+ Wqprad_M(L)=M$. Mean $Wqprad_M(N+L)=M$ by prop.(1.3(5)). But M satisfies ascending chain condition then N+L=M.

Proposition (1.13):

Let M be an R-module. If M is regular, then $Wqprad_M(K)=K$ for all submodule K of M.

Proof:

Suppose that M is regular R-module, and let K be a proper submodule of M. Then by prop.(1.3) we have $K\subseteq Wqprad_M(K)$. To prove first that K is the intersection of prime submodules, we must prove that K is semi-prime submodule of $rad_M(N)=M$.

Let $r^2x\in K$ for $r\in R$, $x\in M$. Then since M is regular by [2] we have $rx\in(r)\ M\cap(rx) =(r)(rx)$. Thus $rx\in K$ and K is a semi-prime submodule. Hence by [4] K is the intersection of a prime submodules. Hence $K= \bigcap_{\alpha\in\Lambda} P_\alpha$ where P_α is a prime submodule of M for each $\alpha\in\Lambda$, therefor $\bigcap_{\alpha\in\Lambda} K_\alpha\subseteq K$, where K_α is a prime submodule of M is a weakly quasi-prime then $Wqprad_M(K) \subseteq \bigcap_{\alpha\in\Lambda} K_\alpha$, implies that $Wqprad_M(K) \subseteq K$. Hence $Wqprad_M(K)=K$.

S2: Weakly quasi-prime radical submodules:

In this section, we introduce the definition of weakly quasi-prime radical submodule as a generalization of prime radical submodule, and study some properties of this concept.

Definition (2.1):

A proper submodule of an R-module M is called weakly quasi-prime radical, if Wqp rad $_M$(N)=N.

Proposition (2.2):

If N is a submodule of an R-module M then Wqprad$_M$ (N) is a weakly quasi-prime radical submodule.

Proof:-

from prop.(1.3(3),we have Wqprad$_M$(Wqprad$_M$(N)= Wqprad$_M$(N).hence Wqp rad $_M$(N)is aweakly quasi-prime radical Submoaule of M.

Proposition (2.3):

If N is a weakly quasi-prime submodule of M, then Wqprad $_M$(N)=N.

Proof:-

from prop (1.3(1), we have N$\subseteq$ Wqprad$_M$ (N).And from definition of Wqprad $_M$(N),we have Wqprad$_M$ (N) $\subseteq$,hene Wqprad$_M$ (N)=N.

Now, we are going to consider the relationship among the following three statements for any R-module.

1. M satisfies the ascending chain condition for weakly quasi-prime radical submodules. 2. Each weakly quasi-prime radical submodule is an intersection of a finite number of weakly quasi-prime submodule.

3. Every weakly quasi-prime radical submodule is the weakly quasi-prime radical of a finitely generated submodule of it.

Proposition (2.4):

Let M be an R-module. If M satisfies the ascending chain condition for weakly quasi-prime radical submodules, then every weakly quasi-prime radical submodule of M is an intersection of a finite number of weakly quasi-prime submodules.

Proof:

Let N be a weakly quasi-prime radical submodule of M. Put N=$\bigcap_{i\in I}$ Ni, where Ni is a weakly quasi-prime radical submodule of M for each i$\in$I, and the expression is reduced. Assume that I is an infinite index set. without loss of generality we may assume that I is countable.

Then N= $\bigcap_{i=1}^{\infty}$ Ni$\subseteq\bigcap_{i=2}^{\infty}$ Ni$\subseteq\bigcap_{i=3}^{\infty}$ Ni$\subseteq$...., is ascending chain of weakly quasi-prime radical submodules. Then by prop.(1.3(1)),we have $\bigcap_{i\in I}$ Ni $\subseteq$ Wqp rad$_M$ ($\bigcap_{i\in I}$ Ni)$\subseteq\cap$Wqprad$_M$(Ni)= $\bigcap_{i\in I}$ Ni.

By hypothesis this ascending chain must terminate, so there exists j$\in$I such that $\bigcap_{i=j}^{\infty}$ Ni=$\bigcap_{i=j+1}^{\infty}$ Ni, therefore $\bigcap_{i=j+1}^{\infty}$ Ni$\subseteq$Nj, which contradicts that the expression N= $\bigcap_{i=1}^{\infty}$ Ni is reduced. Therefore, I must be finite and hence N=$\bigcap_{i=1}^{\infty}$ Ni.

Proposition (2.5)

Let M be an R-module. If M satisfies the ascending chain condition for weakly quasi-prime radical submodules then every proper submodule of M is a weakly quasi-prime radical of a finitely generated submodule.

Proof:

Assume that there exists aproper submodule N of M which is not the weakly quasi-prime radical of finitely generated submodule of it.

Let $m_1\in$N and N_1=Wqprad$_M$(Rm_1), so $N_1\subset$N. Thus there exists $m_2\in$N-N_1. Let N_2=Wqp rad$_M$(Rm_1+Rm_2), then $N_1\subset$N, hence there exists $m_3\in$N-N_3 . This implies an ascending chain of weakly quasi-prime radical submodules $N_1\subseteq N_2\subseteq N_3\subseteq$....,which does not terminate and this contradicts with the hypothesis.

Proposition (2.6):

Let M be a finitely generated R-module. If every weakly quasi-prime submodule of M is weakly quasi-prime radical of a finitely generated submodule of it, then M satisfies the ascending chain condition for weakly quasi-prime submodules.

Proof:

Let $N_1\subseteq N_2\subseteq N_3\subseteq$...., be ascending chain of weakly quasi-prime submodules of M. Since M is finitely generated, then N=$\cup$ Ni is weakly quasi-prime submodule of M. Thus by hypothesis, N is the weakly quasi-prime radical for some finitely generated submodule L=Rm_1+Rm_2+Rm_3+...++Rm_n=$\sum_{i=1}^{n}$ Rmi, where mi$\in$N for all i=1,2,...,n. Hence L $\subseteq$ Wqprad$_M$ (L) = N =$\cup$ Ni. Then there exists j$\in$J such that $\cup$ Ni = Nj. Thus the chain of weakly quasi-prime submodules Ni terminates.

The following proposition shows that weakly quasi-prime radical submodule and prime-radical sumodule are equivalent under acertian condition.

Proposition (2.7):

Let M be an R-module such that every submodule of M is irreducible. Then N is prime-radical submodule iff N is aweakly quasi-prime radical submodule

Proof:

Suppose that N is a prime-radical submodule, that is N= rad$_M$(N)=$\cap${Li:where Li is a prime submodule of M such that N$\subseteq$ Li} since every submodule is irreducible, then by [1,prop 2.1.3, ch2] every prime submodule of M is weakly quasi-prime. Hence rad $_M$(N) = $\cap$ {Li:where Li is a prime submodule of M such that N$\subseteq$Li}=Wqprad$_M$(N).

Conversely: Similary.

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المقاسات الجزئية الضعيفة الاولية الشبه جذرية

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الملخص

لتكن R حلقة ابدالية بمحايد وMمقاساً أحادياً على R في هذا البحث قدمنا مفهوم المقاس الجزئي الضعيف الاولي الشبه جذري N كأعمام لجذر الاولي للمقاس الجزئي الاولي N. حيث أن المقاس الجزئي الضعيف الاولي الشبه جذري Nويرمز له بالرمز $Wqprad_M(N)$ويعرف بأنه تقاطع كل المقاسات الجزئية الضعيفة الشبه الاولية للمقاس M والتي تحتوي على N .

المقاس الجزئي كذلك قدمنا مفهوم المقاس الجزئي الضعيف الاولي الشبه جذري، حيث ان المقاس الجزئي الفعلي N من M الذي يحقق الخاصة $Wqprad_M(N)=N$ يدعى مقاس جزئي أولي شبه جذري من M . العديد من الصفات لهذين المفهومين أُعطيت.