Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

Binomial Transform Technique For Solving Linear Difference Equations

(1) Dr. Eman A. Hussain (2) Asmaa S. Jasim

(1) Asst. Prof. in Department of Mathematics -College of Science-Mustansiriyah University, Baghdad –IRAQ. Email : dreman@uomustansiriyah.edu.iq
(2) Graduate student in Department of Mathematics -College of Science-Mustansiriyah University, Baghdad –IRAQ. Email : assma93salih@gmial.com.

ARTICLE INFO

Article history:
Received: 27 /05/2021
Rrevised form: 05 /06/2021
Accepted : 24 /06/2021
Available online: 30 /06/2021

Keywords: difference equations, linear difference equations, Binomial transform.

ABSTRACT

Our goal in this paper is to find a new transformation technique for solving linear difference equations, as in the case of Z-transformation. And we were able to find the binomial transform, and this transform is one of the most common transformations. The linear difference equations can be solved by the binomial transformation.

MSC. 41A25; 41A35; 41A36

DOI : https://doi.org/10.29304/jqcm.2021.13.2.815

1. Introduction and main result.

The binomial transformation is useful in many applications, whether in applied or pure mathematics.The generalization of a binomial transformation was first introduced by Prodinger (1994). The binomial transformation is a discrete transformation of one series to another with many interesting applications incombinations and analysis. This transformation is useful for researchers interested in numerical aggregation, special numbers, and classical analysis. The binomial transform is closely related to the Euler transform. The binomial transformation is usually used to speed up the series or, in

∗Corresponding author: Asmaa S. Jasim.

Email addresses:: assma93salih@gmial.com..

Communicated by: *Dr.Rana Jumaa Surayh aljanabi.*

the opposite direction, to simplify the structure of the hypergeometric terms of a series. This conversion has the elegant property that it is self-inverse. This conversion is one of the most common transfers. For more information about this conversion, see [8,9,10,11,12,13,14,15,16,17].

Difference equations in the form of recurring relations/series appeared in 1718 at the latest (before that time they were usually shown in an indirect or descriptive manner). de Moivre's methods were further organized and studied by Euler. Later in 1759 Lagrange studied the "integration" (i.e. solvability) of linear difference equations by modifying methods that had been used in the study of differential equations and essentially laid the foundation for further investigations. and are the discrete equivalent of differential equations and arises whenever an independent variable can have only discrete values. The Difference equations are used in situations of real life, in various sciences (population models, genetics, psychology, economics, sociology, stochastic time series, combinatorial analysis, queuing problems, number theory, geometry, radiation quanta and electrical networks).see [2,3,4,5,6,7].

Linear difference equations play an important role in various fields of science and engineering. Significant progress has been made in recent years in the theory of linear difference equations. Previously, the development of this topic was far behind the related field of linear differential, as a return to some fundamental difficulties that called for the introduction of new ideas and methods. The first of this new idea was the representation of affinity, which Poincaré developed and applied in 1885 to study distinction quotations [1]. About 1910 effective methods of attack were devised almost simultaneously by Norlund in Denmark, Carmichael and Berkoff in that country, and Dalbrun in France. These four mathematicians all succeeded in proving in different ways the existence of analytical solutions of linear homogeneous difference equations and studying their properties. In this paper we use the Binomial transformation to solve linear difference equations. This paper consists of several sections, the first section includes the introduction, the second section contains Binomial transform , the third section includes technic the solution of difference equation by Binomial Transform and the four section includes "applications".

2. Binomial transform

The binomial transform takes the sequence $a_0, a_1, a_2, \ldots$ to the sequence $b_0, b_1, b_2, \ldots$via the transformation

$$b_n = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} a_n$$

The inverse transform is

$$a_n = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} b_n$$

Where $\binom{n}{k} = 0 \; if \; k < 0.$

Now we define the binomial transform of difference equations

Given a sequence $\{y_k\}$, $k = 0,1,2, \ldots$, its binomial transform is the new sequence $\{B(y_n)\}$, $n = 0,1,2 \ldots$

Generated by the formula

$$B(y_n) \equiv b(y_n) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} y_k$$

But the inverse binomial transform we define

$$y_n = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} B(y_n)$$

A Table of Properties of Binomial Transform

	y_k	$B(y_n)$
1.	$y_k = 1$	$B(y_n) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} 1 = \binom{n}{1}(-1)^0 + \binom{n}{2}(-1)^1 + \binom{n}{3}(-1)^2 + \cdots = n - \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \cdots$
2.	$y_k = a^k$	$B(y_n) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} a^k = \binom{n}{1}(-1)^0 a^1 + \binom{n}{2}(-1)^1 a^2 + \binom{n}{3}(-1)^2 a^3 + \cdots = na^1 - a^2\frac{n(n-1)}{2!} + a^3\frac{n(n-1)(n-2)}{3!} + \cdots$
3.	$y_k = k$	$B(y_n) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} k = \binom{n}{1}(-1)^0 1 + \binom{n}{2}(-1)^1 2 + \binom{n}{3}(-1)^2 3 + \cdots = n - 2\frac{n(n-1)}{2!} + 3\frac{n(n-1)(n-2)}{3!} + \cdots = n - n(n-1) + \cdots$

4.	y_{k+1}	$B(y_{n+1}) = \sum_{k=1}^{n} \binom{n}{k+1} (-1)^k y_{k+1} = \binom{n}{2} (-1)^1 y_2 + \binom{n}{3} (-1)^2 y_3 + \cdots$
4.	y_{k+2}	$B(y_{n+2}) = \sum_{k=1}^{n} \binom{n}{k+2} (-1)^{k+1} y_{k+2} = \binom{n}{3} (-1)^2 y_3 + \cdots$
5.	y_{k-1}	$B(y_{n-1}) = \sum_{k=1}^{n} \binom{n}{k-1} (-1)^{k-2} y_{k-1} = \binom{n}{0} (-1)^{-1} y_0 + \binom{n}{1} (-1)^0 y_1 + \cdots$
6.	y_{k-2}	$B(y_{n-2}) = \sum_{k=1}^{n} \binom{n}{k-2} (-1)^{k-3} y_{k-2} = \binom{n}{-1} (-1)^{-2} y_{-1} + \binom{n}{0} (-1)^{-1} y_0 + \cdots$

3. Technic The Solution Of Difference Equation By Binomial Transform

Let

$$y_{k+m} + a_1 y_{k+m-1} + \cdots + a_m y_k = 0 \qquad (1)$$

Be an mth-order linear homogeneous difference equations with given constant coefficients $a_1, a_2, \dots, a_m$ and having $a_m \neq 0$.

To solve difference equations by binomial transformation
We will take the binomial transformation of both sides of the equation (1).
According to homogeneous boundary conditions

$$B(y_{n+m}) + a_1 B(y_{n+m-1}) + \cdots + a_m B(y_n) = 0$$

And the solution method is as follows:

$$B(y_{n+m}) = \sum_{k=1}^{n} \binom{n}{k+m} (-1)^{k-1+m} y_{k+m}$$
$$= \binom{n}{1+m} (-1)^m y_{1+m} + \binom{n}{2+m} (-1)^{1+m} y_{2+m} + \binom{n}{3+m} (-1)^{2+m} y_{3+m} + \cdots$$

$$B(y_{n+m-1}) = \sum_{k=1}^{n} \binom{n}{k+m-1} (-1)^{k-2+m} y_{k+m-1}$$
$$= \binom{n}{1+m-1} (-1)^{-1+m} y_{1+m-1} + \binom{n}{2+m-1} (-1)^{m} y_{2+m-1} + \binom{n}{3+m-1} (-1)^{1+m} y_{3+m-1} + \cdots$$

$$B(y_n) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} y_k = \binom{n}{1}(-1)^0 y_1 + \binom{n}{2}(-1)^1 y_2 + \binom{n}{3}(-1)^2 y_3 + \cdots$$

the following examples illustrate the application of the above technique.

4. Applications

Example (1) solve the first order difference equation by The binomial transform

$y_{k+1} - y_k = 0 \qquad (1),$

with initial boundary

$y(0) = 1, \quad y(1) = 1, \quad y(2) = 1$

Now, we will take the binomial transform of both sides of the equation (1)

$$b(y_{n+1}) - b(y_n) = 0$$

Where $B(y_n) \equiv b(y_{n+1}) - b(y_n)$

$$b(y_{n+1}) = \sum_{k=0}^{n} \binom{n}{k+1} (-1)^k y_{k+1} = \binom{n}{1}(-1)^0 y_1 + \binom{n}{2}(-1)^1 y_2 + \binom{n}{3}(-1)^2 y_3 + \cdots$$

$$= n - \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \cdots$$

$$= n - \frac{n^2}{2!} + \frac{n}{2!} + \frac{n^3}{3!} - \frac{n^2}{2} + \frac{2n}{3!} + \cdots$$

$$= \frac{11}{6} n - n^2 + \frac{n^3}{3!} + \cdots$$

$$b(y_n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} y_k$$

$$= \binom{n}{0}(-1)^{-1} y_0 + \binom{n}{1}(-1)^0 y_1 + \binom{n}{2}(-1)^1 y_2 + \binom{n}{3}(-1)^2 y_3 + \cdots$$

$$= -1 + n - \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \cdots$$

$$= -1 + \frac{11}{6} n - n^2 + \frac{n^3}{3!} + \cdots$$

$$B(y_n) \equiv b(y_{n+1}) - b(y_n) = 0$$

$$\left[\frac{11}{6}n - n^2 + \frac{n^3}{3!} + \cdots\right] - \left[-1 + \frac{11}{6}n - n^2 + \frac{n^3}{3!} + \cdots\right] = 1$$

$$\therefore\ B(y_n) = 1 \qquad (2)$$

Now, we will taking the inverse (2)

$$y_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} B(y_n)$$

$$y_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} 1 = \binom{n}{0}(-1)^{-1} + \binom{n}{1}(-1)^{0} + \binom{n}{2}(-1)^{1} + \binom{n}{3}(-1)^{2} + \cdots$$

$$= -1 + n - \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \cdots$$

$$= -1 + \frac{11}{6}n - n^2 + \frac{n^3}{3!} + \cdots$$

$$\therefore y_n = 1$$

Example (2) solve the second order difference equation by The binomial transform

$y_{k+2} - y_k = 0 \qquad (1),$

with initial boundary

$y(0) = 2, \quad y(1) = 0,$

$y(2) = 2\,, y(3) = 0, \quad y(4) = 2$

Now, we will take the binomial transform of both sides of the equation (1)

$$B(y_n) \equiv b(y_{n+2}) - b(y_n) = 0$$

$$b(y_{n+2}) = \sum_{k=1}^{n} \binom{n}{k+2} (-1)^{k+1} y_{k+2} = \binom{n}{3}(-1)^2 y_3 + \binom{n}{4}(-1)^3 y_4 + \cdots$$

$$= 0 - \frac{2n(n-1)(n-2)(n-3)}{4!} + \cdots$$

$$= -\frac{n^4}{12} + \frac{n^3}{2} - 11\frac{n^2}{12} + \frac{n}{2} + \cdots$$

$$b(y_n) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} y_k = \binom{n}{1}(-1)^0 y_1 + \binom{n}{2}(-1)^1 y_2 + \cdots$$

$$= 0 - n(n-1) + \cdots$$

$$= -n^2 + n + \cdots$$

$$b(y_{n+2}) - b(y_n) = 0$$

$$B(y_n) = \left[-\frac{n^4}{12} + \frac{n^3}{2} - 11\frac{n^2}{12} + \frac{n}{2} + \cdots\right] - [-n^2 + n + \cdots]$$

$$\therefore B(y_n) = \frac{-1}{2}n + \frac{1}{12}n^2 - \frac{1}{12}n^4 + \frac{1}{2}n^3 + \cdots \qquad (2)$$

Now, we will taking the inverse (2)

$$y_n = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} B(y_n)$$

$$y_n = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \left[\frac{-1}{2}n + \frac{1}{12}n^2 - \frac{1}{12}n^4 + \frac{1}{2}n^3 + \cdots\right]$$

$$= \binom{n}{1}(-1)^0\left[\frac{-1}{2} + \frac{1}{12} - \frac{1}{12} + \frac{1}{2} + \cdots\right] + \binom{n}{2}(-1)^1\left[-1 + \frac{1}{3} - \frac{4}{3} + 4 + \cdots\right]$$

$$= -n^2 + n + \cdots$$

$$\therefore y_n = 1 + (-1)^n.$$

Since the binomial transformation of $[1 + (-1)^n]$ is

$$= \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1}[1 + (-1)^n] \quad = \binom{n}{1}(-1)^0 0 + \binom{n}{2}(-1)^1 2 + \cdots$$

$$= -\frac{2}{2!}n(n-1) + \cdots$$

$$= -n^2 + n + \cdots$$

Example (3) solve the first order difference equation by The binomial transform

$y_k - 3y_{k-1} = 0 \qquad (1),$

with initial boundary

$y(0) = 1, \quad y(1) = 3,$

$y(2) = 9\,, y(3) = 27,$

Now, we will take the binomial transform of both sides of the equation (1)

$$B(y_n) \equiv b(y_n) - 3b(y_{n-1}) = 0$$

$$b(y_n) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} y_k = \binom{n}{1}(-1)^0 y_1 + \binom{n}{2}(-1)^1 y_2 + \cdots$$

$$= 3n - \frac{9}{2}n(n-1) + \cdots$$

$$b(y_{n-1}) = \sum_{k=1}^{n} \binom{n}{k-1} (-1)^{k-2} y_{k-1} = \binom{n}{0}(-1)^{-1} y_0 + \binom{n}{1}(-1)^0 y_1 + \cdots$$

$$= -1 + 3n$$

$$B(y_n) \equiv b(y_n) - 3b(y_{n-1}) \qquad (2)$$

$$= 3n - 3[-1 + 3n]$$

$$= -6n + 3$$

Now, we will taking the inverse of the equation (2)

$$y_n = \sum_{k=1}^{n} \binom{n}{k} (-1)^k B(y_n)$$

$$y_n = \sum_{k=1}^{n} \binom{n}{k} (-1)^k [-6k + 3] = \binom{n}{1} (-1)^1 (-3) + \binom{n}{2} (-1)^2 (-9) + \cdots$$

$$= 3n - \frac{9}{2} n(n-1) + \cdots$$

As compared to the binomial transformation of a^k

$$a^n = B(a^n) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} a^k \quad = \binom{n}{1} (-1)^0 a^1 + \binom{n}{2} (-1)^1 a^2 + \cdots$$

$$= an - \frac{a^2}{2!} n(n-1) + \cdots$$

Since $a = 3$ from equation (2) then we have

$$\therefore y_n = 3^n$$

Example (4) solve the first order difference equation by The binomial transform
$y_{k+1} - 2y_k = 0 \qquad (1),$
with initial boundary
$y(0) = 1, \quad y(1) = 2, \quad y(2) = 4$
Now, we will taking the binomial transform of y_k of the equation (1)

$$B(y_n) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} y_k \quad = \binom{n}{1} (-1)^0 y_1 + \binom{n}{2} (-1)^1 y_2 + \cdots$$

$$= n - \frac{4}{2!} n(n-1) \qquad (2)$$

As compared to the binomial transformation of a^k

$$a^n = B(a^n) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} a^k \quad = \binom{n}{1} (-1)^0 a^1 + \binom{n}{2} (-1)^1 a^2 + \cdots$$

$$= an - \frac{a^2}{2!} n(n-1) + \cdots$$

Since $a = 2$ from equation (2) then we have

$$\therefore y_n = 2^k$$

Conclusion

In this paper, a new method has been proposed to solve linear difference equations of first and second order homogeneous with initial conditions, and this method is a binomial transformation technique through which we were able to solve linear difference equations.

Acknowledgements

I would like to express my thanks and deep gratitude to my supervisor Asst. Prof. Dr. Eman A. Hussein for her support and valuable guidance through this work .

References

1. H. . Poincaré, Sur les equations linéaires aux différentielles ordinaires et aux differences finies, Amer. J. Math. 7 (1885), 203-258.
2. Jerri, Abdul J.," Linear Difference Equations With Discrete Transform Methods", Kluwer Academic Publishers, 1996.
3. Miclcnes,R.E."Difference Equations. Theory Applications and advanced Topics", 3^{ed} ed. Champam & Hall , 2015.
4. Elaydi, S.N. "An introduction to difference equations" , Springer, 2^{ed} ed . 1999.
5. Kelley W.G,& Pcterson A.C." Difference Equations. An Introduction with Applications", 2^{ed} ed .Academic Press , 2001.
6. Ladas, G., Open problems and conjecture. Journal of Difference Equations and Applications, 5, 317–321, 1995.
7. S. Goldberg, Introduction to Difference Equations: With Illustrative Examples from Economics, Psychology and Sociology, New York: Dover Publications, Inc., 2010.
8. S. Weinzierl, Expansions around half-integer values, binomial sums and inverse binomial sums, J. Math. Phys. 45 (2004), 2656–2673.
9. Derrick Henry Lehmer, Interesting series involving the central binomial coefficient, Amer. Math. Monthly 92 (1985), 449–457.
10. Khristo N. Boyadzhiev, Binomial transform and the backward difference. Adv. Appl. Discrete Math., 13(1)(2014),43-63.
11. Khristo N. Boyadzhiev, Binomial transform of products, Ars Comb., 126 (2016), 415 434.
12. K. W. Chen, Identities from the binomial transform, J. Number Theory 124(2007), 142-150.
13. Yan-Ping Mu, Symmetric recurrence relations and binomial transforms, J. Number Theory, 133 (2013) 3127-3137.
14. Mogens E. Larsen and Peter J. Larcombe, Some binomial coefficient identities of specific and general type, Util. Math., 74 (2007), 33-53.
15. N. YÄ´slmaz, N. Taskara, Binomial transforms of the Padovan and Perrinnumbers, Journal of Abstract and Applied Mathematics, (2013) ArticleID941673.
16. H. Prodinger, Some information about the binomial transform, The Fibonacci Quarterly, 32(5), 1994, 412-415.
17. Boyadzhiev, K. (2018). Notes on the Binomial Transform: Theory and Table with Appendix on Stirling Transform, World Scientific, Singapore.
18. A. Taha, B., & J. Kadhim, A. (2018). Non-polynomial spline finite difference method for solving second order boundary value problem. *Journal of Al-Qadisiyah for Computer Science and Mathematics*, *10*(2), Math Page 92 - 101. https://doi.org/10.29304/jqcm.2018.10.2.384
19. AbdulAmeer AbdulJabbar, I., & Ali Yakoob, Z. (2018). Hybrid techniques to improve face recognition based on features extraction methods and Haar discrete wavelet transformation.

Journal of Al-Qadisiyah for Computer Science and Mathematics, *10*(2), Comp Page 36 - 46. https://doi.org/10.29304/jqcm.2018.10.2.372

20. Ali Mohamad, H., & Falih Jaddoa, A. (2019). Property of oscillation of first order impulsive neutral differential equations with positive and negative coefficients. *Journal of Al-Qadisiyah for Computer Science and Mathematics*, *11*(1), Math Page 45 - 51. https://doi.org/10.29304/jqcm.2019.11.1.457
21. S. Rahma, A. M., & Ayad Jabbar, D. (2019). Development cryptography protocol based on Magic Square and Linear Algebra System. *Journal of Al-Qadisiyah for Computer Science and Mathematics*, *11*(1), Comp Page 72 - 75. https://doi.org/10.29304/jqcm.2019.11.1.470
22. Hadi, A. M., & Abdulrahman, A. A. (2020). Multi Discrete Laguerre Wavelets Transforms with The Mathematical aspects. *Journal of Al-Qadisiyah for Computer Science and Mathematics*, *12*(1), Comp Page 26-37. https://doi.org/10.29304/jqcm.2020.12.1.672.