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The Adomian Decomposition Method for Solving Fractional Integro-Differential Equations

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Abstract

In this paper , numerical solutions of fractional integro differential equations of a deferent order by using Adomian decomposition method (ADM) and compare our result with exact solution numerical result show that (ADM) is more efficient and powerful method .

Key words:Fractional differential equations,Integral equations,Adoman decomposition method.()

الخلاصة :

تناولنا في هذا البحث حل المعادلات التفاضلية التكاملية الكسرية ذات الأسس الكسرية المختلفة باستعمال طريقة (Adomian decomposition method) ، (ADM) حيث تم مقارنة النتائج التي تم الحصول عليها باستعمال هذه الطريقة مع الحل الدقيق لهذه المعادلة واثبتت النتائج ان هذه الطريقة متقاربة وأكثر كفاءة .

الكلمات المفتاحية: المعادلات التفاضليه –التكامليه الكسريه,طريقه ادومن.

1.Introduction and Basic Concepts:

Let us consider a linear fractional integro-differential equation

$D^q u(t)=f(t)+J^B u(t)$ with the intial condition $u(0)=0,\ 0<t<a$ and $0<q , B\leq 1$

Where D^q refers to the capnto derivative , f is a continuous function on (t,u) for u Є R , a >0

There is som of the moste important nations and definitions and theorem of fractional integro-defferential equations.

Definition 1.1: The Gama Function .

The complete gamma function Γ (t) , is $\Gamma(t) = \int X^t\ e^{-x}\,dx$, t> 0

, $\Gamma(t+1) = t\,\Gamma t$

$\Gamma\, n+1 = n\,!$

Definition 1.2: The fractional Derivative .

The fractional derivative , given in

$$D^q u(t) = 1/\Gamma\, m-q\ \ d^m/dx^m \int (t-s)^{m-q-1} u(s)\,ds \qquad (1.2.1)$$

The caputo definition of fractional derivative is given by :

$$D^q u(t) = 1/\Gamma\, m-q \int (t-s)^{m-q-1} u^m(s)\, ds \qquad (1.2.2)$$

Definition 1.3 : the fractional integral .

$$J^q u(t) = 1\backslash\Gamma\, q \int (t-s)^{q-1} u(s)\, ds ,\ q> 0 \qquad (1.3.1)$$

The properties of the operator J^q can be found in [Rawashdeh , 2005] , for $q \geq 0$,$\alpha > 0$, we have :

$$J^q J^\alpha u(t) = J^{q+\alpha} u(t)$$

$$J^q t^\gamma = \Gamma\gamma+1/\Gamma\gamma+1+q \quad t^{q+\gamma} ,\ t> 0 ,\ q \geq 0 ,\ \gamma = -1 \qquad (1.3.2)$$

$$D^q J^q u(t) = u(t) \qquad (1.3.3)$$

Definition 1.4 ,(Samah,2010; Samko,1993).
Fractional integro – Differential equations:

Consider the linear fractional integro – differential equation :

$$D^q u(t)=f(t)+J^\beta u(t), \text{ with initial condition } u(0) = u_0 ,0 < q , \beta < 1 \qquad (1.4.1)$$

where D^q refers to the caputo derivative operator of order $0< q < 1$

wich is defined

$$D^q u(t)=1/\Gamma m-q\int(t-s)^{m-q-1}u^m(s)ds ,-1< q \leq m , m \in N ,t \in [0,T]$$

and J^β , denotes the fractional integro operator of β , $0< \beta < 1$ where

$$J^\beta u(t) = 1/\Gamma\beta \int (t-s)^{\beta-1} u(s)\, ds ,\quad Au(t) = f(t) \qquad (1.4.2)$$

where

$$Au(t) = 1/\Gamma m-q \int (t-s)^{m-q-1} u^m(s)\, ds - 1/\Gamma\, \beta \int (t-s)^{\beta-1} u(s)\, ds \qquad (1.4.3)$$

Defintion 1.5 ,(Mittal and Ruchi,2008; Loverro,2004):

The Adomain Decompsion method (ADM) :

Let $D^q u(t) = f(t) + J^\beta u(t)$, where $t \in [0 , T]$, $0< q , \beta < 1$

by tack the J^q in the tow said we have

$u(t) = J^q \; f(t) + J^q \; [\, 1/\Gamma\beta \int (t-s)^{\beta-1} u(s) \, ds \,]$

where

$u_0(t) = u(0) + J^q f(t)$

$u(t) = J^q \; f(t) + J^q \; [\, 1/\Gamma\beta \int (t-s)^{\beta-1} u(s) \, ds \,]$

.
:
:

$u_{n+1} = J^q \; [\, 1/\Gamma\beta \int (t-s)^{\beta-1} u_n(s) \, ds \,]$

So $u(t) = \sum_{i=0}^{\infty} u_n(t)$

That is the Approximiat solution of the fractional – integro differential equ.

2.Some Theorems:

Theorem 2.1 ,(Al-husseiny,2006,Momani ,2007):(The uniqueness theorem)

The uniqueness of the solution of the fractional integro – differential equations

Consider the initial value problem,which consists of the fractional integro-differential equations ,

$$D^q u(t) = f(t) + J^\beta u(t) \qquad 0 < q , \beta < 1, \; u(0)=u_0 \qquad (2.1.1)$$

Where D^q refers to the caputo derivative operator of order $0< q < 1$, and $u(0)=u_0$

the initial condition , f is a continuous function on t for $u \in R$, $t \in [0, T]$, u_0 is

areal positive constant

We shall use Biharis inequality to to obtain the uniqueness to equations given by (2.1.1) can be transformed in the next lemma .

Now , some additional properties are given for completeness purposes,

Lemma 2.2:

The solution of the initial value problem given by eqs (2.1.1) has the form :

$U(t) = u_0 + 1/\Gamma q \int (t-s)^{q-1} f(s)\, ds + 1/\Gamma q \int (t-s)^{q-1} [\, 1/\Gamma\beta \int (s-\sigma)^{\beta-1} u(\sigma) \, d\sigma] \; ds$

Proof :

From (2.1.1)

$D^q u(t) = f(t) + J^\beta u(t)$, $0 < q,\beta < 1$ with initial condition $u(0) = u_0$ (2.2.1)

Applying the integral :

$J^q D^q u(t) = J^q f(t) + J^q [\ 1/\Gamma\beta \int (s - \sigma)^{\beta-1} u(\sigma)\, d\sigma]$

$U(t) - u_0 = 1/\Gamma q \int (t - s)^{q-1} f(s)\, ds + 1/ \Gamma q \int (t - s)^{q-1} [\ 1/\Gamma\beta \int (s - \sigma)^{\beta-1} u(\sigma)\, d\sigma]\ ds.$

The initial value problem by eq (2.1.1) has a unique solution on the interval [0 , T] if u is continuous function in the region :

$D = \{ (t, u) / \ 0 < t < T , | u - u_0 | \leq b \}$, and satisfy the condition :

$\int | 1/ \Gamma q (s - \sigma)^{q-1} u(\sigma) - 1/\Gamma q (s - \sigma)^{q-1} y(\sigma) | d\sigma \leq M\varphi (| u-y |)$ (2.2.2)

Where M is a positive constant and φ is a nondecreasing continuous function and satisfy

$1/\alpha\ \varphi (x) \leq \varphi (x /\alpha)$, for $x \geq 0$, $\alpha > 0$ and the following integral :

$$\Phi (x) = \int dx / \varphi (x) \qquad (2.2.3)$$

Where $\varphi (x)$ is a primitive of the function $1/ \varphi (x)$, and φ^{-1} denotes the inverse of φ .

let that there exists two solutions u and y of eq (2.1.1) then

$U(t) = u_0 + 1/\Gamma q \int (t-s)^{q-1} f (s)\, ds + 1/\Gamma q \int (t - s)^{q-1} \{ 1/ \Gamma\beta \int (s-\sigma)^{\beta-1} u (\sigma)\, d\sigma \}\, ds$

$Y (t) = u_0 + 1/\Gamma q \int (t-s)^{q-1} f (s)\, ds + 1/\Gamma q \int (t - s)^{q-1} \{ 1/ \Gamma\beta \int (s-\sigma)^{\beta-1} y (\sigma)\, d\sigma \}\, ds$

This implies to :

$| u(t) - y (t) | \leq 1/\Gamma q \int (t - s)^{q-1} \{ 1/ \Gamma\beta \int (s-\sigma)^{\beta-1} | u (\sigma) - y(t) | d \sigma \}\, ds$

It follows from eq (2.2.2) that :

$$| u(t) - y(t) | \leq 1 / \Gamma q \int (t-s)^{q-1} M \varphi (| u - y |)\, ds .$$

Thus

$$| u(t) - y(t) | \leq \xi + M/\Gamma q \int (t-s)^{q-1} \varphi (| u - y |) ds .$$

For any $\xi > 0$, $0 < t < T$ by using theorem(Biharis inequality) , then :

$| u (t) - y (t) | \leq \varphi^{-1} [\varphi (\xi) + MT^q / \Gamma q+1]$, for any fixed $t \in [0, T]$ (2.2.3)

We shall proof that the riqht – hand side of eq (2.2.3)

Tend towards zero as $\xi \to 0$.

Since $| u(t) - y(t) |$ is independent of ξ , it follows that $u (t) = y (t)$, which

we need .

Let us remark that condition (2.2.3) implies that φ , $\xi \to -\infty$ as $\xi \to 0$, no matter how

we choose the primitive of $1/ \varphi (x)$.

Thus $\varphi^{-1}(x) \to 0$ as $x \to -\infty$. consequently , when $\xi \to 0$ in ineq (2.2.3)

the right – hand side tends towardeds zero (for all finite t) therefore , $u(t) = y (t)$,

for $t \in [0 , T]$.

Theorem 2.3, (Momani,2001) : (The existence theorem)

Let u and u^m be a real non negative function in c [0 ,T] , and that $t \in [0 , T]$, $0 < q < 1$, then equ (2.1.1) has a solution u .

Proof :

In order to discuss the condition for the existence for the solution of eqs (2.1.1) , let

us define B = c[0, T] , to be the banach space with the supremum norm , let us define

the set : $u = \{u \in c [0 , T]: \| u \| \leq c_1 , \| u^{(km)} \| \leq c_2 , \quad c_1, c_2 > 0 , k \in N \}$

Now , since our proof depends on the schander fixed point theorem , then it is sufficient
to prove that u is a nonempty , close , bounded and convex subset of the banach space B

and then the operal or $A : U \rightarrow U$ is compact operator .

It is easy to see that the set U in nonempty since from the properties of the norm we

have $0 \in U$ and also bounded and closed (from the definition of U) to prove U is

convex subset of B .

Let $u_1 , u_2 \in U$, $\| u_1 \| \leq c_1$, $\| u_1^{(km)} \| \leq c_2$, $\| u_2 \| \leq c_1$, $\| u_2^{(km)} \| \leq c_2$, such that

$$u(t) = \lambda u_1 + (1-\lambda) u_2(t) \quad , \quad \lambda \in [0,1]$$

To prove $u \in U$, $\| u \| \leq c_1$, $\| u^{(km)} \| \leq c_2$,

$$\| u \| = \| \lambda u_1 + (1-\lambda) u_2 \| \leq |\lambda| \, \| u_1 \| + |(1-\lambda)| \, \| u_2 \| \leq \lambda c_1 + (1-\lambda) c_1 = c_1$$

$$\| u^{(km)} \| = \| [\lambda u_1 + (1-\lambda) u_2]^{(km)} \| = \| \lambda u_1^{(km)} \| + \| (1-\lambda) u_2^{(km)} \|$$

$$\leq |\lambda| \, \| u_1^{(km)} \| + |(1-\lambda)| \, \| u_2^{(km)} \|$$

$$\leq \lambda c_2 + (1-\lambda) c_2$$

$$= c_2 \quad .$$

Hence , $u \in U$, U is convex set .

Now , in order to show that eqs (3.1.1) , (3.1.2) , has a solution , we have to show that the operator A in eq

$$Au(t) = 1/\Gamma(m-q) \int (t-s)^{m-q-1} u^{(m)}(s)\, ds - 1/\Gamma\beta \int (t-s)^{\beta-1} u(s)\, ds .$$

Is completely continuous .

Let $v(t) = Au(t)$, to prove that $v(t) \in U$,

$$\| v \| = \| 1/\Gamma(m-q) \int (t-s)^{m-q-1} u^{(m)}(s)\, ds - 1/\Gamma\beta \int (t-s)^{\beta-1} u(s)\, ds \|$$

$$\leq \| u^{(m)} \| / \Gamma(m-q) \int (t-s)^{m-q-1} ds + \| u \| / \Gamma\beta \int (t-s)^{\beta-1} ds$$

$$\leq c_2 T^{m-q} / \Gamma\, m-q+1 + c_1 T^{\beta} / \Gamma\beta+1$$

$$\leq C ,$$

That is $v(t)$ is bounded .

$$\| v^{(km)} \| = \| 1/\Gamma(m-q) \int (t-s)^{m-q-1} u^{((k+1)m)}(s)\, ds - 1/\Gamma\beta \int (t-s)^{\beta-1} u^{(km)}(s)\, ds \|$$

$\leq \| u^{((k+1)m)} \| \setminus \Gamma(m-q+1) \; T^{m-q} + \| u^{(km)} \| T^{\beta} \setminus \Gamma\beta+1$

$\leq c_2 T^{m-q} / \Gamma m-q+1 + c_2 T^{\beta} / \Gamma\beta+1$

$\leq c^*$.

That is $V^{(km)}(t)$ is $V^{(km)}(t)$ is bounded , $v(t) \in U$. then the operator A maps U into it self.
Since for all $u \in U$ we have $A(u) \leq c$, then A(u) is bounded operator .
To prove that A is continuous operator .
Let $u , v \in U$, then we have

$\| Au - Av \| = \| 1/\Gamma(m-q) \int (t-s)^{m-q-1} u^{(m)}(s)\, ds - 1/\Gamma\beta \int (t-s)^{\beta-1} u(s)\, ds - [1/\Gamma(m-q) \int (t-s)^{m-q-1} v^{(m)}(s)\, ds - 1/\Gamma\beta \int (t-s)^{\beta-1} v(s)\, ds] \|$

$= \| 1/\Gamma(m-q) \int (t-s)^{m-q-1} (u^{(m)}(s) - v^{(m)}) - 1/\Gamma\beta \int (t-s)^{\beta-1} (u(s) - v(s))ds \|$

$\leq \| u^{(m)} - v^{(m)} \| / \Gamma(m-q+1)\, T^{m-q} + \| u - v \| / \Gamma\beta+1 \; T^{\beta}$

$\leq \| (u-v)^{m} \| / \Gamma(m-q+1)\, T^{m-q} + \| u - v \| / \Gamma\beta+1 \; T^{\beta}$

Let $w = u - v$

$\leq \| w^{(m)} \| / \Gamma(m-q+1) \; T^{m-q} + \| w \| / \Gamma\beta+1 \; T^{\beta}$

$\leq c$

That is Au is bounded operator , Au is continuous

operator .
Now , we shall prove that A is equicontinuous operator .

Let $u \in U$ and $t_1 , t_2 \in [0 , T]$, then :

$\| Au(t_1) - Av(t_2) \| = \| [1/\Gamma(m-q) \int (t_1-s)^{m-q-1} u^{(m)}(s)\, ds - 1/\Gamma\beta \int (t_1-s)^{\beta-1} u(s)\, ds \|] - [1/\Gamma(m-q) \int (t_2-s)^{m-q-1} u^{(m)}(s)\, ds -$

$1/\Gamma\beta \int (t_2-s)^{\beta-1} u(s)\, ds] \|$

$\leq \| u^{(m)} \| / \Gamma(m-q) \,| \int (t_1 - s)^{m-q-1} ds - \int (t_2 - s)^{m-q-1} ds | + \| u \| / \Gamma\beta \,| \int (t_1 - s)^{\beta-1} ds - \int (t_2 - s)^{\beta-1} ds |$

$\leq c_2 / \Gamma(m-q+1) \; | (t_1^{m-q} - t_2^{m-q}) | + c_1/\Gamma\beta+1 \; | (t_1^{\beta} - t_2^{\beta}) |$

$\leq 2c_2 / \Gamma(m-q+1) \; T^{m-q} + 2c_1 / \Gamma(\beta+1)\, T^{\beta}$

$\leq c$, where $\beta , q > 0$

Au is equicontinuous operator .
A is relatively compact , now from Arzela - Ascoli theorem , A is completely continuous operator then A is compact .

Then schander fixed point , which corresponds to the solution of eq .

3.Some Examples

In this chapter we display the adomen decompsion method for solve fractional –integro defferential equations .

Example 3.1:

Consider the fractional integlro – Differential equation

$$D^{0.5} u(t) = f(t) + J^{0.3} u(t) \quad , u(0) = 0 \ , \ t \in [0 , T]$$

Where

$$f(t) = 6 / \Gamma 3.5 \ t^{2.5} - 6 / \Gamma 4.5 \ t^{3.5}$$

and the exact solution is given by

$$u(t) = t^{3}$$

According to the Adomian Decomposition method , the approximate solution :

$$U(t) = u(0) + J^{0.5} f(t) + J^{0.5} [J^{0.3} u(t)]$$

$$U(t) = u(0) + J^{0.5} f(t) + J^{0.5} [1/\Gamma 0.3 \int (t-s)^{0.3-1} u(s) ds]$$

$$= u(0) + 6/\Gamma 3.5 \ J^{0.5} \ t^{2.5} - 6/\Gamma 4.5 \ J^{0.5} \ t^{3.5} + J^{0.5} [1/ \Gamma 0.5 \int (t-s)^{-0.7} u(s) ds]$$

therefor,

$$U_0(t) = u(0) + J^{0.5} f(t)$$

$$= 0 + 6/ \Gamma 3.5 \ J^{0.5} \ t^{2.5} - 6 / \Gamma 4.5 \ J^{0.5} t^{3.5}$$

$$= 0 + (6 / \Gamma 3.5) (\Gamma 3.5 / \Gamma 4) \ t^{3} - (6/ \Gamma 4.5) (\Gamma 4.5 / \Gamma 5) \ t^{4}$$

$$= t^{3} - 0.25 \ t^{4}$$

$$U_1(t) = J^{0.5} [1/ \Gamma 0.3 \int (t-s)^{-0.7} u_0(s) ds]$$

$$= J^{0.5} [1/ \Gamma 0.3 \int (t- s)^{-0.7} [t^{3} - 0.25 t^{4}] ds]$$

$$= J^{0.5} [t^{3.3} / 0.3 + 0.25 / 0.3 \ t^{4.3}]$$

$$= J^{0.5} \ t^{3.3} / 0.3 + 0.25 / 0.3 \ J^{0.5} t^{4.3}$$

Since

$$J^{q} t^{\gamma} = \Gamma 1+\gamma / \Gamma \gamma+1+q \quad t^{\gamma+q}$$

$$= (- 0.1) \ t^{3.8} - (0.0 6) t^{4.8}$$

$$U_2(t) = J^{0.5} [1/ \Gamma 0.3 \int (t-s)^{-0.7} u_1(s) ds]$$

$$= J^{0.5} [J^{0.3} [(- 0.1) \ t^{3.8} - (0.0 6) t^{4.8}]]$$

$= (-0.05) t^{4.1} - 0.03 t^{5.1}$

So that

$U(t) = u_0(t) + u_1(t) + u_2(t) + \ldots\ldots..$

$= t^3 – 0.25 t^4 + (-0.1) t^{3.8} – 0.06 t^{4.8} – (0.05) t^{4.1} - 0.03 t^{5.1} + \ldots$

That is the approximat solution

Now to find the error of this solution we have

$Q_3(t) = \sum_{i=0}^{n-1} u_i$

So the error

$| u(t) – Q_3(t) |$

Where u(t) is the exact solutio

Now we have atable of the solution

Exact and approximate results

T	Exact solution	ADM_{Q3}	E_3
0	0	0	0
0.1	1×10^{-3}	9.546×10^{-4}	4.6×10^{-5}
0.2	8×10^{-3}	7.283×10^{-3}	7.17×10^{-4}
0.3	2.7×10^{-2}	2.3×10^{-2}	4×10^{-3}
0.4	6.4×10^{-2}	6.100×10^{-2}	0.005
0.5	0.125	0.110	0.015
0.6	0.216	0.145	0.071
0.7	0.343	0.230	0.113
0.8	0.512	0.500	0.012
0.9	0.729	0.6999	0.0291
1	1	0.999	0.001

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