Positively Invariant Sets in Sliding Mode Control Theory with Application to Servo Actuator System with Friction*

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Abstract

In this paper two invariant sets are derived for a second order nonlinear affine system using a sliding mode controller. If the state started in these sets, it will not leave it for all future time. The invariant set is found function to the initial condition only, from which the state bound is estimated and used when determining the gain of the sliding mode controller. This step overcomes an arithmetic difficulty that consists of calculating suitable controller gain value that ensures the attractiveness of the switching manifold. Also, by using a differentiable form for the approximate signum function in sliding mode controller formula, the state will converge to a positively invariant set rather than the origin. The size of this set is found function to the parameters that can be chosen by the designer, thus, it enables us to control the size of the steady state error. The sliding mode controller is designed to the servo actuator system with friction where the derived invariant sets are used in the calculation of the sliding mode controller gain. The friction model is represented by the major friction components; Coulomb friction, the Stiction friction, and the viscous friction. The simulation results demonstrate the rightness of the derived sets and the ability of the differentiable sliding mode controller to attenuate the friction effect and regulate the state to the positively invariant set with a prescribed steady state error.

Key words: Positively Invariant Set, Sliding Mode Control, Servo Actuator System, Friction Model.

المجاميع الامتغيرة في نظرية المسيطر المنزلق مع تطبيقها على منظومة المحرك المؤازر بوجود الإحتكاك

الخلاصة

في هذا البحث تم إشتقاق مجموعتين من المجاميع الامتغيرة لمنظومة ديناميكية لاخطية من الدرجة الثانية مسيطر عليها بواسطة منظومة المسيطر المنزلق. في هذه المجاميع إذا نشأ متغير الحالة بداخلها فسوف لن يغادرها أبدا. وجدت المجموعة الامتغيرة دالة للشروط الإبتدائية فقط والتي يمكن منها تخمين المحدد لمتغير الحالة و إستخدامه في حساب معامل المسيطر المنزلق. هذه الخطوة تمكننا من تجاوز صعوبة حسابية متمثلة بقيمة هذا المعامل والذي يضمن جاذبية سطح التحول (switching manifold). أيظا إن إستخدام دالة تقريبية قابلة للإشتقاق لدالة الإشارة (signum function) سيجعل متغير الحالة يقترب ويبقى في مجموعة لامتغيرة بدلا من نقطة الأصل. إن حجم هذه المجموعة يمكن إختياره من قبل المصمم وهو ما يعني قابلية التحكم بمقدار الخطأ الدائمي. صمم المسيطر المنزلق لمنظومة الدافع المؤازر (servo actuator) بوجود الإحتكاك حيث تم إستخدم المجموعة الامتغيرة لحساب معامل المسيطر المنزلق. تم تمثيل الإحتكاك (النموذج الرياضي) بعناصر الإحتكاك الرئيسية: إحتكاك Coulomb , إحتكاك Stiction والإحتكاك اللزج. إن نتائج المحاكات بينت صحة المجاميع الامتغيرة التي تم إشتقاقها و قدرة المسيطر المنزلق في تقويض وإضعاف تأثير الإحتكاك وأيظا أظهرت قدريه على قيادة متغير الحالة الى المجموعة الامتغيرة والتي تم فيها تحديد مقدار الخطأ الدائمي مسبقا.

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1-Introduction

In this paper we are interested mainly to answer the following: *Consider the second order affine system with sliding mode controller*

$$\dot{x} = f(x) + g(x) * (-k * sgn(s)),\quad s = s(x)$$

Then, for a certain controller gain value k*, what is the area around the origin such that if the state initiated inside this region, it will not leave it and the origin is an attractive point. This area is known as the area of attraction.*

The area of attraction forms the so called positively invariant set. The set notion appears in control theory when we considered three aspects, which are crucial in control systems design, these are: constraints, uncertainties, and design specifications [1]. For the sliding mode controller

$$u = -k * sgn(s)\ , k > 0$$

the main design step is the calculation of an appropriate value for the controller gain$k$. This point is important since a large gain value may lead to the chattering problem. So a better estimate to gain value may help in reducing the amplitude of the chattering behavior (the chattering behavior frequently appears in sliding mode control system for many reasons such as the non ideality of the switching process [2]). In fact, this work is an issue in this direction. Furthermore, many methods are used to eliminate the chattering in sliding mode control system (see [2]&[3]), but the simplest method is introduced by Sloten J. J. [4], where the segnum function is replaced by a saturation function. This approximate sliding mode controller introduces a positively invariant set around the origin and its size is determined by the design parameters [5]. Khalil H.K. [5], derives the invariant set formed by the sliding mode controller that uses the saturation function as suggested by Sloten. The saturation function is a continuous but not differentiable function; and for this reason we are interested in replacing the segnum function by a continuous and differentiable function, and then derive the positively invariant set formed by the approximate sliding mode controller.

In recent applications of control theory, many dynamical systems have been modeled as interconnected systems where the state of the upper system is unaffected by the actual controller [6]. For this system type a virtual controller is used to control the upper system if the system be in a certain form to enable the application of the so called Backstepping approach. So, the presence of the disturbances in the upper system will lead to the non-matching property for the control system. The situation becomes more complicated if the disturbances are nonsmooth. This situation makes us use the arc tan function (the continuous and differentiable function), which may be used as a virtual controller for the interconnected system, and derive for it the positively invariant set. The servo actuator system is one of the interconnected system models, where the torque that actuates the mechanical system is not the actual input (for a D.C. motor the voltage is the actual servo actuator system input). Therefore, we select this system to design the sliding mode controller with the aid of the derived positively invariant sets.

2-Invariant Set

The terminologies of the invariant and positively invariant set are defined in this section, where we refer mainly to the excellent reference [5]. So, consider the second order autonomous system

$$\dot{x} = f(x) \tag{1}$$

where $x \in \mathcal{R}^2$and $f(x)$ is a locally Lipschitz map from a domain $D \subset \mathcal{R}^2$ into $\mathcal{R}^2$. Let $x(t)$ be a solution to the second order autonomous system in equation (1) and also let $x = 0$ be an equilibrium point; that is $f(0) = 0$. Now, the set M, with respect to the system in equation (1), is said to be invariant set if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall t \in \mathcal{R}$$

It means that: if $x(t)$ belongs to M at some time instant, then it belongs to $M$for all future and past time, i.e., it will never come from a region outside it or leave it for all future time. A set M is said to be a positively invariant setif

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall t \geq 0$$

In this case the state may come from outside the positively invariant set but will never leave for all future time. We also say that $x(t)$ approaches a set M as $t$approaches infinity, if for each $\varepsilon > 0$there is $T > 0$ such that

$$\text{dist}(x(t), M) < \varepsilon, \forall t > T$$

where $\text{dist}(x(t), M)$ denotes the distance from a point $x(t)$to a set M. The positive limit point is defined as the limit for the solution $x(t)$ when the time approaches infinity. The set of all positive limit points of $x(t)$is called the positive limit set of $x(t)$. Accordingly, the asymptotically stable equilibrium is the positive limit set of every solution starting sufficiently near the equilibrium point, while the stable limit cycle is the positive limit set of every solution starting sufficiently near the limit cycle. The solution approaches the limit cycle as $t \to \infty$. The equilibrium point and the limit cycle are invariant sets, since any solution starting in either set remains in the set for all $t \in \mathcal{R}$. Moreover, let the set of positively limit set for a point p denoted by the ω limit set of p, namely $\omega(p)$, then some properties of it are stated in the following fact [7]:

Let M be a compact, positively invariant set and $p \in M$, then $\omega(p)$ satisfies the following properties:

1. $\omega(p) \neq \emptyset$, that is, the ω limit set of a point is not empty.
2. $\omega(p)$ is closed.
3. $\omega(p)$ in a positively invariant set.
4. $\omega(p)$ is connected.

This fact, in later sections, will be helpful in determining the behavior of the state trajectory when it is initiated in a positively invariant set.

3-The First Positively Invariant Set

In the following analysis, the first invariant set for a second order system that uses a sliding mode controller is estimated. Consider the following second order affine system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = f(x) + g(x)u, \quad g(x) > 0 \qquad (2)$$

Let the controller in equation (2) be the sliding mode controller

$$u = -k\text{sgn}(s), s = x_2 + \lambda x_1, \quad \lambda > 0 \quad (3)$$

where s is the switching function which is selected such that the system at the switching manifold $(s = 0)$ is asymptotically stable. The main idea behind the selection of the sliding mode controller gain k is that the switching manifold will be attractive. To do that we use the following nonsmooth Lyapunov function

$$V = |s| \qquad (4)$$

The switching manifold is guaranteed to be attractive if the derivative of the Lyapunov function is negative. Consequently,

$$\dot{V} = \dot{s} * \text{sgn}(s)$$
$$= [f(x) - g(x)k * \text{sgn}(s) + \lambda x_2]\text{sgn}(s)$$

$$= -[g(x)k - (f(x) + \lambda x_2) * \mathrm{sgn}(s)] \quad (5)$$

Now if k is chosen such that $\dot{V} < 0$, then the switching manifold is attractive. Thus,

$$k > \max \left| \frac{f(x)+\lambda x_2}{g(x)} \right| = h \quad (6)$$

If k satisfies the inequality (6), then $s = 0$ is asymptotically stable. In fact satisfying inequality (6) is the main calculation problem during design process. Formally, we may use a large gain value to ensure satisfying (6), and consequently the area of attraction becomes large. But the gain cannot be chosen freely without limit due to the control saturation. Accordingly, the gain value determines directly the area of attraction size. In this work, we aim to find the invariant set for a second order system that use the sliding mode controller as given in (3), such that when the state initiated in it will never leave for all future time. Hence, the gain is calculated depending on the invariant set size and the region of attraction will include at least the invariant set. In literature, the existence of the invariant set is assumed (by assigning the maximum state value) and accordingly the sliding mode controller gain is calculated. In this case the sliding controller will be able to force the state toward the switching manifold at least when it initiated in this invariant set. However, the gain value may be large and again the saturation problem arises. Other designers, use a certain gain value in the design of sliding controller and, may be, by doing extensive simulations they prove that the area of attraction will include the nominal initial conditions for a certain application [8].

To find the invariant set, we need to derive its bounds. The first bound on the invariant set is derived by using the Lyapunov function given in equation (4). Suppose that we use a certain value for the gain k, then there is a certain basin of attraction such that the time rate of change of the Lyapunov function is less than zero, namely

$$\dot{V} < 0 \Rightarrow V(t) - V(t_o) < 0$$

or

$$|s(t)| - |s(t_o)| < 0$$

Therefore the switching function level is bounded by:

$$\therefore |s(t)| < |s(t_o)| \,, \forall t > t_o \quad (7)$$

Of course the inequality (7) holds due to the action of the sliding mode controller with gain k. However, the inequality (7) shows that the state will lie in a region bounded by

$$-s(t_o) < s(t) < s(t_o) \,, \forall t > t_o$$

but without assigning the equilibrium point with respect to the switching function. So we need to show that, as it is known, that the switching manifold is an asymptotically equilibrium manifold due to the sliding mode controller. To prove the stability of $s = 0$, the time derivative of the switching function is found first when k satisfies inequality (6), as follows:

$$\dot{s} = \dot{x}_2 + \lambda \dot{x}_1 = f(x) - g(x)ksgn(s) + \lambda x_2$$
$$\Rightarrow \dot{s} = -\beta(x)sgn(s) \,, \quad 0 \le \beta(x)$$

Now, we return to the Lyapunov function, equation (4), to find its derivative as:

$$\dot{V}(s) = \dot{s} * sgn(s)$$
$$\Rightarrow \dot{V}(s) = -\beta(x) < 0$$

Since $V(0) = 0$ and $\dot{V}(s) < 0$ in the set $\{x \in \mathcal{R}^2 : s \neq 0\}$, then $s = 0$ is asymptotically stable (theorem 4-1 in reference [5]). Moreover, we must note that the solution of the dynamical system in (7) at the switching manifold does not exist [9]. This is due to the discontinuity in sliding mode controller formula. Ideally the state will slide along the switching manifold to the origin, i.e., the state trajectory will identify the switching manifold until it reaches the origin. Therefore, the bound given in the inequality (7) becomes:

$$0 \le |s(t)| < |s(t_o)|$$
$$\Rightarrow 0 \le s(t) * sgn(s) < s(t_o) * sgn(s_o)$$

But in sliding mode control $sgn(s) = sgn(s_o)$, $\forall t > t_o$, thus,

$$0 \le s(t) * sgn(s) < s(t_o) * sgn(s) \quad (8)$$

Accordingly we have

$$0 \le s(t) < s(t_o) \text{ for } s > 0$$
$$0 \ge s(t) > s(t_o) \text{ for } s < 0 \quad (9)$$

In words, inequality (9) shows that if the state initiated in the positive side of the switching manifold, then the state will stay in an open region bounded by $s = s(t_o)$ and $s = 0, \forall t > t_o$. The same thing happens if the state was initiated with negative switching function level. Inequality (9) is the first bound; the second is derived here for x_1 as follows:

$$\dot{x}_1 + \lambda x_1 = s(t)$$
$$\Rightarrow d\{e^{\lambda t} x_1(t)\} = e^{\lambda t} s(t) dt$$
or
$$e^{\lambda t} x_1(t) - x_1(t_o) = \int_{t_o}^{t} s(\tau) e^{\lambda \tau} d\tau$$

By taking the absolute for both sides and considering the inequality (7), we obtain

$$|e^{\lambda t} x_1(t)| - |e^{\lambda t_o} x_1(t_o)| \le |e^{\lambda t} x_1(t) - e^{\lambda t_o} x_1(t_o)|$$
$$= \left|\int_{t_o}^{t} s(\tau) e^{\lambda \tau} d\tau\right| \le \int_{t_o}^{t} |s(\tau)| e^{\lambda \tau} d\tau$$
$$\le |s(t_o)| \int_{t_o}^{t} e^{\lambda \tau} d\tau$$
$$= \frac{|s(t_o)|}{\lambda}\left(e^{\lambda t} - e^{\lambda t_o}\right)$$
$$\Rightarrow |e^{\lambda t} x_1(t)| \le |e^{\lambda t_o} x_1(t_o)| + \frac{|s(t_o)|}{\lambda}\left(e^{\lambda t} - e^{\lambda t_o}\right)$$
$$\Rightarrow |x_1(t)| \le |x_1(t_o)| e^{-\lambda(t-t_o)} + \frac{|s(t_o)|}{\lambda}\left(1 - e^{-\lambda(t-t_o)}\right)$$
$$\therefore |x_1(t)| \le max\left\{|x_1(t_o)|, \frac{|s(t_o)|}{\lambda}\right\} \quad (10)$$

The result in the inequality (10) is a consequence of the convexity of the set

$$\Psi = \left\{x_1(t): x_1(t) = \mu|x_1(t_o)| + (1-\mu)\frac{|s(t_o)|}{\lambda}, 0 \le \mu \le 1\right\}$$

In this case the maximum element of the set is at $\mu = 0$ or at $\mu = 1$. Therefore the invariant set is bounded by the inequalities (9) and (10) in terms of the initial condition only and hence, the invariant set is given by:

$$\Theta = \left\{x \in \mathcal{R}^2: 0 \le s(t) sgn(s) < s(t_o) sgn(s),\ |x_1(t)| \le max\left(|x_1(t_o)|, \frac{|s(t_o)|}{\lambda}\right)\right\} \quad (11)$$

The figure below plot the invariant set in the phase plane and one can find geometrically that the bound for $x_2(t)$ inside Ψ is

$$|x_2(t)| \le max\{|x_2(t_o)|, |s(t_o)|\} \quad (12)$$

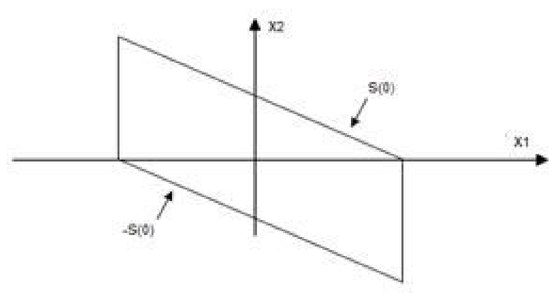


Figure (1):Positively Invariant Set.

4-The Second Positively Invariant Set

In classical sliding mode control theory, there exist a trivial invariant set. This set is the origin of the state space where the controller regulates the state to it and keeps the state there for all future time. The sliding mode control that does the above task is a discontinuous control and it may cause the chattering problem. Many solutions to the chattering problem exist in the literatures (see references [2], [3] and [10]). The simplest method to remove chattering is by replacing the segnum function, which it used in sliding mode controller, by an approximate form. This idea is first introduced by J.J. Sloten in [4] by using the saturation function instead of the signum function. Later, many other approximate segnum functions are used to remove chattering as found in reference [11]. However, when replacing the signum function the state will not be regulated to the origin, instead it will be regulated to a certain set around the origin known as the positively invariant set. The size of this set is determined by the design parameters and the approximation form. In the present work the signum function is replaced by the arc tan function, namely

$$sgn_{approx.}(s) = \frac{2}{\pi} tan^{-1}(\gamma s) \qquad (13)$$

Where $tan^{-1}(\gamma s)$ is a continuously differentiable, odd, monotonically increasing function with the properties:

$$tan^{-1}(0) = 0, \lim_{|s|\to\infty} tan^{-1}(\gamma s) = \lim_{\gamma\to\infty} tan^{-1}(\gamma s) = \frac{\pi}{2} sgn(s) \quad \text{and} \quad sgn(s) * tan^{-1}(\gamma s) = tan^{-1}(\gamma |s|) \geq 0$$

Accordingly, the sliding mode controller, given in equation (3), becomes

$$u_{approx.} = -\frac{2k}{\pi} tan^{-1}(\gamma s) \qquad (14)$$

Now, let us state the following, and then prove it:

When the sliding mode controller use the approximate signum function as given in equation (13), and the controller gain satisfied the inequality (6), then the state will be regulated to a positively invariant set defined by

$$\Delta_\delta = \left\{ x \in \mathcal{R}^2 : |x_1| < \frac{\delta}{\lambda}, |s| \leq \delta \right\} \qquad (15)$$

To prove that Δ_δ is a positively invariant set for a second order affine system (equation (3)), we return to use the Lyapunov function in equation (4) which has the time rate of change

$$\begin{aligned}\dot{V} &= \left\{ f(x) - g(x) \frac{2k}{\pi} tan^{-1}(\gamma s) + \lambda x_2 \right\} sgn(s) \\ &= -\left\{ g(x) \frac{2k}{\pi} tan^{-1}(\gamma |s|) - (f(x) + \lambda x_2) * \mathrm{sgn}(s) \right\}\end{aligned}$$

For the switching manifold to be attractive $\dot{V}$ must be less than zero, namely

$$-\left\{ g(x) \frac{2k}{\pi} tan^{-1}(\gamma |s|) - (f(x) + \lambda x_2) * \mathrm{sgn}(s) \right\} < 0$$

$$\Rightarrow \frac{2k}{\pi} tan^{-1}(\gamma |s|) > max \left| \frac{f(x) + \lambda x_2}{g(x)} \right| = h$$

or

$$k > \frac{\pi h}{2tan^{-1}(\gamma |s|)} \qquad (16)$$

Now, let $|s| = \delta$ be the chosen boundary layer, then inequality (16) reveals, for a certain γ, that: *for any δ there is k, such that the state will be regulated to an open region given by*

$$\Gamma = \{ x \in \mathcal{R}^2 : |s| < \delta \} \qquad (17)$$

Accordingly, the gain k will be

$$k = \frac{\alpha \pi h}{2tan^{-1}(\gamma \delta)} \quad , \alpha > 1 \qquad (18)$$

In addition, to determine γ, equation (18) may be written as:

$$k = \alpha h \beta, \quad \beta > 1 \qquad (19)$$

provided that;

$$\gamma\delta = tan\frac{\pi}{2\beta} \quad (20)$$

The next step in the determination of the invariant set Δ_δ is to found the boundary with respect to x_1 inside Γ. This is done by using the following Lyapunov function

$$V = \frac{1}{2}x_1^2 \quad (21)$$

with the x_1 dynamics, from equations (2) and (3):

$$\dot{x}_1 = -\lambda x_1 + s(t) \quad (22)$$

Therefore the time rate of change for the Lyapunov function is

$$\begin{aligned}\dot{V} &= x_1\dot{x}_1 = x_1\left(-\lambda x_1 + s(t)\right) \\ &= -\lambda|x_1|^2 + |x_1||s(t)| \\ &\leq -\lambda|x_1|^2 + |x_1|\delta \\ &= -|x_1|(\lambda|x_1| - \delta)\end{aligned}$$

Thus, $\dot{V} \leq 0$ for the following unbounded interval:

$$|x_1| > \frac{\delta}{\lambda} \quad (23)$$

Inequality (23) proves that the state x_1 will reach and stay within the interval $-\frac{\delta}{\lambda} \leq x_1 \leq \frac{\delta}{\lambda}$. This ends the proof that the set $\left\{x \in \mathcal{R}^2 : |x_1| < \frac{\delta}{\lambda}, |s| \leq \delta\right\}$ is positively invariant for the system in equation (2) that uses a sliding mode controller with the approximate signum function as given in equation (14).

Note that the state inside Δ_δ may or may not reaches an equilibrium point; the situation depends on system dynamics, i.e., the state, instead of that, will reach a limit cycle inside Δ_δ. Consequently, and for the design purpose, δ may be determined according to a desired permissible steady state deviation of the state x_1 and for a selected λ, as a design parameter, as follows:

$$\delta = \lambda * x_{1per.} \quad (24)$$

Thus, the set Δ_δ is now written as:

$$\Delta_\delta = \left\{x \in \mathcal{R}^2 : |x_1| < x_{1per.}, |s| \leq \delta\right\} \quad (25)$$

It is also noted that for arbitrary small $x_{1per.}$ the positively invariant set Δ_δ becomes arbitrary small and it may lead, again, to the state chattering. This situation may explain the chattering phenomena as the state oscillation with a very narrow width, i.e., the interval $|x_1| < x_{1per.}$ is very small.

5-Sliding Mode Controller Design for Servo Actuator with Friction

Consider the following model for the servo actuator with friction:

$$J\ddot{x} = u - F - T_L \quad (26)$$

Where x is the actuator position, J is the moment of inertia, u is the control signal, F is the friction torque, including the static and dynamic components, and T_L is the load torque. The friction model taken here is a combination of Coulomb friction F_c, Stiction friction F_s, and the viscous friction (for more details one can refer to the survey papers [12] &[13]), namely

$$F = F_s e^{-\left(\frac{\dot{x}}{\dot{x}_s}\right)^2} * sgn(\dot{x}) + F_c\left(1 - e^{-\left(\frac{\dot{x}}{\dot{x}_s}\right)^2}\right) * sgn(\dot{x}) + \sigma\dot{x}$$

or

$$F = \left\{F_s e^{-\left(\frac{\dot{x}}{\dot{x}_s}\right)^2} + F_c\left(1 - e^{-\left(\frac{\dot{x}}{\dot{x}_s}\right)^2}\right) + \sigma|\dot{x}|\right\} * sgn(\dot{x}) \quad (27)$$

Where $\dot{x}_s$ is called the Stribeck velocity and σ is the viscous friction coefficient. In addition, the servo actuator model in equation (26) is considered uncertain with a bounded load torque. The uncertainty in the model parameters reaches to 20% of their nominal values. Now, define $e_1 = x - x_d$ and $e_2 = \dot{x} - \dot{x}_d$, then the system model in equation (26) in state space form (in (e_1, e_2) plane)is written as:

$$\dot{e}_1 = e_2$$
$$\dot{e}_2 = \left(\frac{1}{J}\right)(u - F - T_L) - \ddot{x}_d \qquad (28)$$

In this work the desired position and velocity are taken as in reference [14]:

$$x_d = \frac{1}{16\pi}sin(8\pi t) - \frac{1}{24\pi}sin(12\pi t)$$
$$\Rightarrow |x_d| \le \frac{5}{48\pi}$$
$$\dot{x}_d = sin(10\pi t) * sin(2\pi t) \Rightarrow |\dot{x}_d| \le 1 \quad (29)$$

Also, the switching function and its derivative are

$$s = e_2 + \lambda e_1 \&$$
$$\dot{s} = \left(\frac{1}{J}\right)(u - F - T_L) - \ddot{x}_d + \lambda e_2 \qquad (30)$$

where $\ddot{x}_d = 10\pi * cos(10\pi t)sin(2\pi t) - 2\pi * sin(10\pi t)cos(2\pi t)$ and $|\ddot{x}_d| \le 12\pi$.

The sliding mode controller is designed for two initial conditions (the position and the velocity at time $t = 0$). The first initial condition lies in the 2nd positively invariant set (see (15)), while in the second case the 1st positively invariant set is taken according to the initial condition which lies outside the 2nd positively invariant set. The controller parameters are calculated for each case in appendices (A) and (B) for the following nominal parameters and external load values [14]

Table (1): Nominal Servo Actuator Parameters and the External Load values

Par.	Definition	Value	Units
J_o	Moment of inertia.	0.2	kgm^2
F_{so}	Stiction friction.	2.19	Nm
F_{co}	Coulomb friction.	16.69	Nm
$\dot{x}_{so}$	Stribeck velocity.	0.01	rad/sec
σ_o	viscous friction coefficient	0.65	$Nm \cdot sec/rad$
T_{Lo}	External Torque	2	Nm

The simulation results and discussions are presented in the following section.

6-Simulations Result and Discussions

For the first case the state is started from the rest, which means $e(0) = (0,0)$ (this is because $x_d(0) = \dot{x}_d(0) = 0$). In this case the state is initiated inside the 2nd positively invariant setΔ_δ, and accordingly the state will not leave it for $t \ge 0$.The state after that reaches an invariant set (it stills inside Δ_δ) , namely the ω limit set of the point $e(0)$. For the servo actuator with non-smooth disturbance (the friction), this set is a limit cycle lying inside the positively invariant set Δ_δ (the fact in section 2). Indeed, the state will reach the ω limit set if it is started at any point in Δ_δ. This situation will be demonstrated by the simulations result below.

The approximate sliding mode controller in this case is (the details of the calculations is found in Appendix (A))

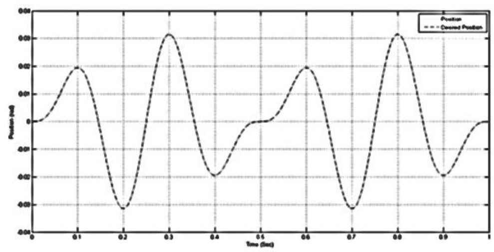
$$\left.\begin{aligned} u_{approx.} &= -(84/\pi) * tan^{-1}(141 * s) \\ s &= (\dot{x} - \dot{x}_d) + 25 * (x - x_d) \end{aligned}\right\} \quad (31)$$

This controller will be able to maintain the state in the following invariant set:

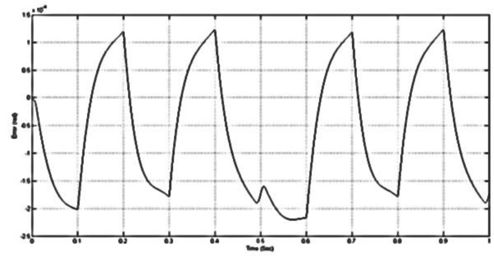
$$\Delta_\delta = \left\{x \in \mathcal{R}^2 : |x - x_d| < \frac{\pi}{3600}, |s| \le \frac{\pi}{144}\right\}, \qquad (32)$$

The response of the servo actuator system when started at the origin is shown in figure (2). In this figure the position response is plotted with time and it appears very close to the desired position. This result is demonstrated when plotting the error and the maximum error shown in the plot, where it does not exceed 1.5×10^{-4} radian. For the velocity, figure (3) plot the time response and again the maximum error, which does not exceed 6.5×10^{-3} radian per second, reveal the closeness between the velocity response and the desired velocity. The error phase plot is found in figure (4) where the state reaches the ω limit set of the origin point. The ω limit set forms here a non-smooth time varying limit cycle and accordingly, the error state will oscillate for all future time within certain amplitude. The oscillation amplitude has an upper bound decided early by the choice of the permissible error.

The positively invariant set formed by the sliding mode controller, as given by (32), enables the same controller to regulate the state when it is started within this set. This situation is verified in figure (5) for two starting points where the state reaches the ωlimit set corresponding to each point.

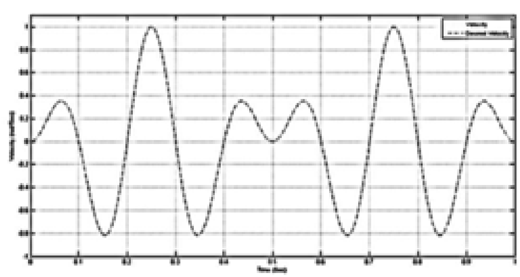


(a)

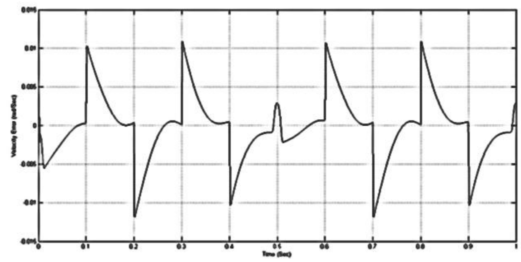


(b)

Figure (2)a) Position and the desired position vs. time (equation (29)). b) The position error for 5 second



(a)



(b)

Figure (3) a) Velocity and the desired velocity vs. time (equation (29)). b) The velocity error for 5 second.

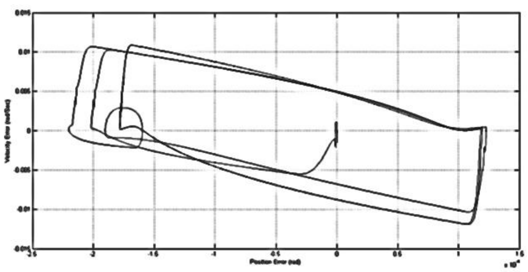
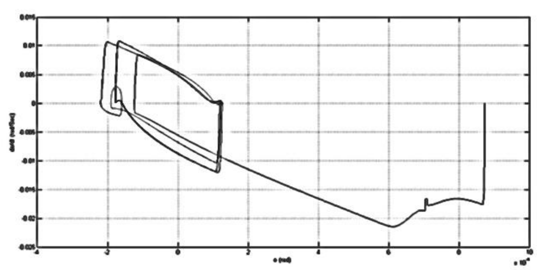
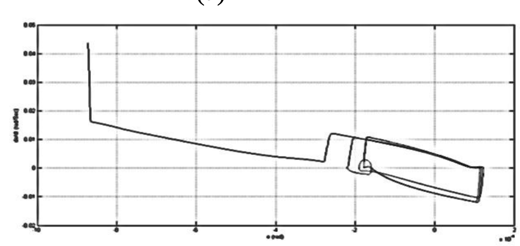


Figure (4) The phase plane plot when the error started at the origin.



(a)



(b)

Figure (5) The phase plane plot a) when the error started at $\left(e,\frac{de}{dt}\right)=\left(\frac{\pi}{3600},0\right)$ b) when the error started at$\left(e,\frac{de}{dt}\right)=\left(-\frac{\pi}{3600},2\frac{\pi}{144}\right)$.

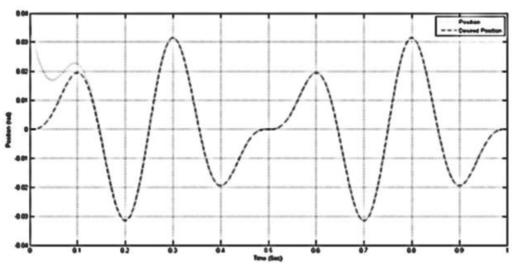
For the second case the sliding mode controller, as calculated in appendix (B), is

$$\left.\begin{aligned} u &= -45 * \mathrm{sgn}(s) \\ s &= (\dot{x} - \dot{x}_d) + 25 * (x - x_d) \end{aligned}\right\} \quad (33)$$

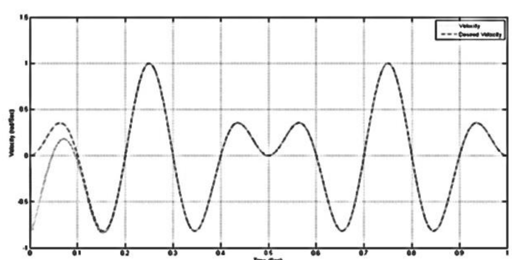
The controller will be able to regulate the error to the origin if it initiated in the following positively invariant set:

$$\Omega = \{x \in \mathcal{R}^2 : |s(t)| < 0.875, |x - x_d| \leq 0.035\} \quad (34)$$

The simulation results for the position and the velocity when the state starting at $(x, \dot{x}) = (0.035, 0)$ are shown in figure (6). In this figure the position and the velocity track the desired response after a period of time not exceeding 0.12 second.



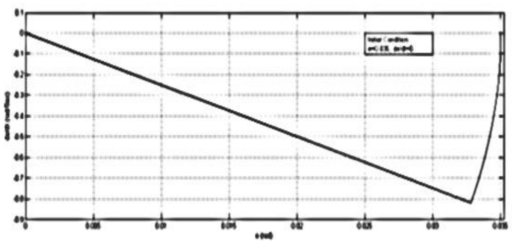
(a)



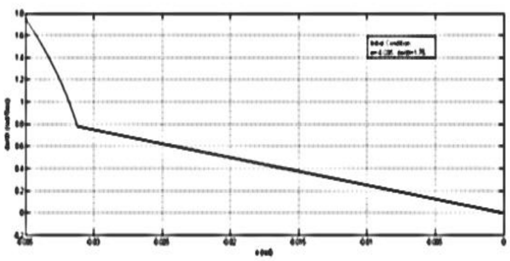
(b)

Figure (6) Servo actuator response for the initial condition $\left(e, \frac{de}{dt}\right) = (0.035, 0)$ a) The position vs. time b) velocity vs. time.

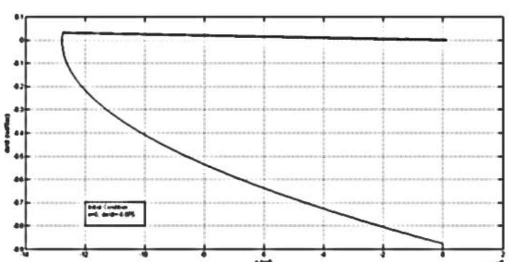
As for the sliding mode controller in (31), the sliding mode controller in (33) will create a positively invariant region (34) such that if the state initiated inside this set, it will be regulated to the origin. This situation is confirmed in figure (7) for three different starting points including the case of figure (6).



(a)



(b)



(c)

Figure (7) Error phase plot for different initial conditions a) $\left(e, \frac{de}{dt}\right) = (0.035, 0)$ b) $\left(e, \frac{de}{dt}\right) = (-0.035, 1.75)$ c) $\left(e, \frac{de}{dt}\right) = (0, -0.875)$.

If it is required to remove the chattering that exists in the system response for the second case, we again replace the segnum function by the arc tan function. In this case we replace the gain $k = 45$ by the following quantity:

$$k = 45 * 1.25 = 57, \quad \beta = 1.25$$

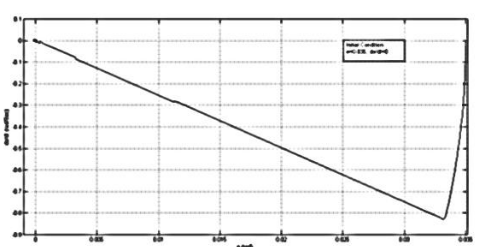
Then, we obtain

$$u = -\left(\frac{114}{\pi}\right) tan^{-1}(141 * s) \qquad (35)$$

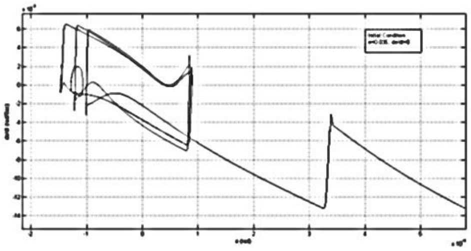
The sliding mode controller in (35) creates a positively invariant set equal to the set given in (34), but in this case the controller will not regulate the error to the origin. Indeed, the controller will regulate the error to enter the 2nd positively invariant set that was given in (32). Mathematically, the 1st and the 2nd sets in (34) and (32) are two positively invariant sets created by the sliding mode controller in equation (35) but with a different set level (see reference [1] for the definition of set level), namely

$$\Delta_\delta \subset \Omega$$

As in figure (7), the phase plane plot for the initial condition $\left(e, \frac{de}{dt}\right) = (0.035, 0)$is plotted in figure (8) but without chattering around the switching manifold due to replacing the segnum function in equation (33) by the approximate form in (35). Accordingly, the state will enter a smaller positively invariant set and then reach the ω limit set as in case one.



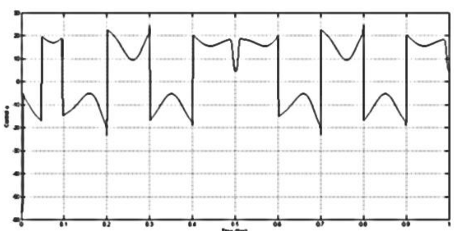
(a)



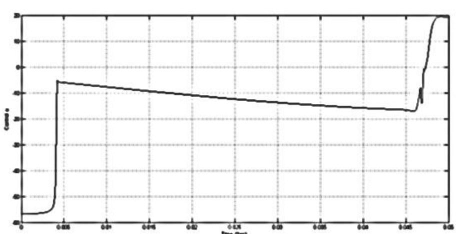
(b)

Figure (8) the phase plane plot when using the controller in (35) a) full phase plot b) small plot around the origin showing the oscillation behavior.

Finally, the chattering behavior is removed due to a continuous control action, where the continuity is revealed in figure (9) with a magnitude that lies between $\pm 25\, N.m$ after a period of time not exceeding 0.05 second.



(a)



(b)

Figure (9) The control action vs. time a) plot for 1 second b) plot for 0.05 second.

7-Conclusions

The positively invariant set for a second order affine system that uses a sliding mode controller has been derived. The size of the invariant sets are found functions to the initial condition and consequently to the controller gain and design parameters. The derived sets have been used to calculate the sliding mode controller gain for the servo actuator. The simulation results prove the invariant property of the derived set and the effectiveness of using them in the calculation of the sliding mode controller. The ability of the approximate sliding mode controller, a continuously and differentiable controller, has been verified when used to attenuate the effect of a nonsmooth disturbances (the friction) that acts on the servo actuator system. The controller maintains the maximum error (the difference between the actual and the desired state) very close to zero and according to the permissible error value specified previously.

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Appendix (A)

To design the approximate sliding mode controller we need, first, to calculate h as it is given in (6):

$$h = \max\left|\frac{f(e) + \lambda e_2}{g(e)}\right| = \left\{\frac{\max\left|\left(\frac{-F-T_L}{J}\right) - \ddot{x}_d + \lambda e_2\right|}{\min\left(\frac{1}{J}\right)}\right\}$$

$$= \max|F| + \max|T_L| + (\max J) * \max|\ddot{x}_d| + \lambda (\max J) * \max|e_2| \quad \text{(A-1)}$$

From the set Δ_δ, the following bounds are estimated:

$\max|e_2| = 2\delta$ and,
$\max|\dot{x}| = \max|e_2| + \max|\dot{x}_d| = 2\delta + 1$

The term $\max|\dot{x}|$ enables the calculation of $\max|F|$ as follows:

$$\max|F| = 1.2\left\{F_{so}e^{-\left(\frac{2\delta+1}{\dot{x}_s}\right)^2} + F_{co}\left(1 - e^{-\left(\frac{2\delta+1}{\dot{x}_s}\right)^2}\right) + \sigma_o(2\delta + 1)\right\} \le 1.2\big(F_{co} + \sigma_o(2\delta + 1)\big)$$

where F_{so}, F_{co}, and σ_o are the nominal friction parameter values also, we multiply their values by 1.2 to take into account the uncertainty in system parameters as assumed previously. In addition, we have

$$\max|J| = 1.2 * J_o \text{ and } \min|J| = 0.8 * J_o$$

again J_o is the nominal moment of inertia value and finally the load torque is bounded by

$$|T_L| \le T_{L_{max}} = 1.2T_{L_0}$$

Therefore, h becomes a function to the slope of the switching manifold λ and the boundary layer δ.

In sliding mode controller design, we are mainly concerned in calculating suitable value for the gain k after a proper selection to the switching function $s(x)$ (by proper we mean that the origin is an asymptotically stable after the state reaches the switching manifold $s(x) = 0$). Now, if we set the permissible error and λ as in the following

$$e_{per.} = 0.05 \text{ deg.} = \frac{\pi}{3600} rad, \quad \lambda = 25$$

then from (24), we have

$$\delta = \lambda * e_{per.} = \frac{\pi}{144} \Rightarrow |e_1| \le e_{per.}$$

Accordingly, to find the gain k, we first compute $h$as follows:

$$\max|F| \le 1.2\big(F_{co} + \sigma_o(2\delta + 1)\big) = 20.84$$
$$\Rightarrow h = 20.84 + 2.4 + 0.24 * 12\pi + 0.24 * 25 * 2\frac{\pi}{144} = 32.55$$

and then for $\beta = 1.25$, we get

$$k = \alpha * 1.25 * 32.55 = 42, \quad \alpha > 1$$

Also, from equation (20), γ equal to

$$\gamma = \frac{144}{\pi} tan\frac{\pi}{2.5} = 141$$

Finally, the sliding mode controller to the servo actuator is

$$u_{approx.} = -\frac{84}{\pi} tan^{-1}(141 * s)$$
$$s = (\dot{x} - \dot{x}_d) + 25 * (x - x_d) \quad \text{(A-2)}$$

The sliding mode controller will be able to prevent the state leaving the positively invariant set Δ_δ. That means the error $(x - x_d)$ is less than the permissible limit specified earlier.

Appendix (B)

In this case we consider the same desired position and velocity as in equation (29) with the following initial condition

$x = 0.035\, rad$, $\dot{x} = 0\ rad/sec.$
$\Rightarrow e(0) = (e_1, e_2) = (0.035,0)$

Also, consider the same switching function as in case one ($s = e_2 + 25e_1$). Then, accordingly, the invariant set is given by

$$\Theta = \{x \in \mathcal{R}^2 : 0 \le s(t) < 0.875 \,, |e_1(t)| \le 0.035\} \qquad \text{(B-1)}$$

In addition we have

$$|e_2(t)| \le 1.75 \Rightarrow \max|\dot{x}| = \max|e_2| + \max|\dot{x}_d| = 2.75\, rad/sec.$$

And then we can calculate $\max|F|$ as in the following:

$$\max|F| \le 1.2(F_{co} + 2.75 * \sigma_o) = 22.2$$

Thus, as in the first case, h is equal to

$$h = 22.2 + 2.4 + 0.24 * 12\pi + 0.24 * 25 * 1.75 = 44.15$$

The sliding mode controller gain from equation (6) is taken equal to

$$k = 45 > h$$

Finally, the sliding mode controller for the second case is given by

$$u = -45 * \mathrm{sgn}(s)$$
$$s = (\dot{x} - \dot{x}_d) + 25 * (x - x_d) \qquad \text{(B-2)}$$

If the state initiated inside the positively invariant set as given in (B-1), the sliding mode controller will regulate the error state to the origin irrespective to the uncertainty and the non-smooth components in the servo actuator model.