Are The Orlicz Spaces Generated By Dilatory Function And Their Duals Are Banach Spaces

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Abstract

The Orlicz spaces generated by dilatory functions are only quasi-Banach spaces contrast to those generated by Orlicz functions which are Banach spaces, and their duals are Banach space also.

الخلاصة

أثبتنا في هذا البحث ان فضاءات اورلسز المتولدة بالدالة الموسعة (dilatory functions) تكون فضاءات بناخ كاذبة على العكس من فضاءات اورلسز المتولدة بدالة اورلسز بأنها فضاءت بناخ بينما ثنائيات تلك الفضاءات تمثل فضاءات بناخ .

1-Introduction

We shall introduce a background of the Orlicz space the word Orlicz came from the name of the mathematician Wiadyslaw Roman Orlicz.

Orlicz spaces are generalization of L_p space their definition are very well known :if $(\Omega,\mathcal{F},\mu)$ is a measure space ,and $1 \leq P \leq \infty$ then for any measurable function $f:\Omega\to\mathbb{C}$ the L_p-norm is defined to be

$$\| f \|_p = \left(\int_\Omega |f(w)|^p \, d\mu(w) \right)^{\frac{1}{p}} \qquad \text{for } p < \infty$$

And $\|f\|_\infty = \text{ess sup}_{w\in\Omega} |f(w)|$ for $p = \infty$

Then we define the Banach space $L_p(\Omega,\mathcal{F},\mu)$ to be the vector space of all measurable function $f:\Omega \longrightarrow \mathbb{C}$ for which $\|f\|_p$ is finite.

Now. If $F:[0,\infty)\to[0,\infty)$ is an Orlicz function where F is non-decreasing convex with $F(0)=0$ then we define the Luxemburg norm by $\|f\|_F = \inf\left\{c: \int_\Omega F(\frac{|f(w)|}{c})d\mu \leq 1\right\}$ for all measurable function $f$and define Orlicz space $L_F(\Omega,\mathcal{F},\mu)$ to be those measurable function f for which $\|f\|_F$ is finite the Orlicz space L_F is a true generalization of L_p at least for $p < \infty$. if $F(t) = t^p$ then $L_F = L_p$ with quality norms.

we shall not work with this definition of the Orlicz space , however , but with different equivalent definition . this definition we give in the following section .

2- Definitions : We first define Φ- function . these replace the notion of Orlisz – function in our discussions .

Definition (2-1) [S.J. Montgomery ,4Dec 1999] : A Φ – function is a function $F:[0,\infty) \to [0,\infty)$ such that

i) $F(0) = 0$

ii) $\lim_{n\to\infty} F_{(t)} = \infty$

iii) F is strictly increasing

iv) F is continuous

However we will often desire that the function F has some control on its growth both from above and below for this reason we will often require that F be dilatory .

We will say that a Φ – function F is dilatory if for some $K_1, K_2 > 1$ we have $F(K_1 t) \geq K_2 F(t)$ for all $0 \leq t < \infty$

We will say that F satisfies the Δ_2-conditon if F^{-1} is dilatory

The definition of Φ-function is slightly more restrictive than that of an Orlicz function in that we insist that F be strictly increasing the notion of dilatory replaces the notion of convexity

Definition(2-2) [S.J. Montgomery] : if $(\Omega, \mathcal{F}, \mu)$ is a measure space and F is Φ – function then we define Luxemburg functional of a measurable function f by

$$\|f\|_F = \inf\left\{c: \int_\Omega F\left(\frac{|f(w)|}{c}\right) d\mu(w) \leq 1\right\}$$

for every measurable function f ,we define the Orlicz space L_F to be the vector space of measurable function f ,for which $\|f\|_F < \infty$ modulo functions that are zero almost everywhere .

Definition(2-3) [W.Cong ,And ,L.Yongjin , 2008] : quasi – norm on a(real or complex) vector space X is anon-negative real –valued function on X satisfying :

(i) $\|x\| = 0$ if and only if $x = 0$

(ii) $\|\lambda x\| = |\lambda|\ \|x\|$ for all $x \in X$ and $\lambda \in R$

(iii) $\|x + y\| \leq K[\|x\| + \|y\|]$ for some fixed $K \geq 1$ and all $x, y \in X$

3- Results

Theorem (3-1) : If $F(t)$ is Φ– function satisfy dilatory condition then L_F is a quasi – Banach space

Proof : -

(i) Let $\|x\|_F = 0$, since $c > 0$ and $\|x\|_F = 0$

then c is very small and greater than zero ,so x must equal to zero

If x =0 and $c > 0$ so $\|x\|_F$ must be zero

(ii) since F is dilatory ,then $F(K_1 w) \geq K_2 F(w)$ and since F is increasing so $K_1 = K_2$

hence $\int_\Omega F\left(\frac{|K_1 f(w)|}{c}\right) d\mu(w) \geq K_1 \int_\Omega F\left(\frac{|f(w)|}{c}\right) d\mu(w)$

$$\inf\left\{c: \int_\Omega F\left(\frac{K_1|f(w)|}{c}\right) d\mu(w)\right\} \leq K_1 \inf\left\{c: \int_\Omega F\left(\frac{|f(w)|}{c}\right) d\mu(w)\right\}$$

Since $\int_\Omega F\left(\frac{K_1|f(w)|}{c}\right) d\mu \leq 1$

So $K_1 \int_\Omega F\left(\frac{|f(w)|}{c}\right) d\mu \leq 1$

Hence $\|K_1 f\|_F = \inf\{c : \int_\Omega F\left(\frac{|K_1 f(w)|}{c}\right) d\mu(w) \leq 1\}$

$= |K_1| \inf\{c : \int_\Omega F\left(\frac{|f(w)|}{c}\right) d\mu(w) \leq 1\}$

$= |K_1|\ \|f\|_F$

iii) Since F is dilatory we have $K_2 \int_\Omega F\left(\frac{|x+y|}{\|x\| + \|y\|}\right) d\mu \leq \int_\Omega F\left(\frac{K_1|x+y|}{\|x\| + \|y\|}\right) d\mu$

Since $x + y > x$ then $K_1(x + y) > K_1(x)$

and $F(K_1(x + y)) > F(K_1(x))$

Since $\|x\| + \|y\| > \|x\|$

So $\frac{1}{\|x\| + \|y\|} < \frac{1}{\|x\|}$

Then $\int_\Omega F(\frac{|x+y|}{\|x\|+\|y\|})d\mu \leq \frac{1}{K_2}[(\int_\Omega F(\frac{K_1|x+y|}{\|x\|+\|y\|})d\mu]$

$\leq \frac{K_1}{K_2}[\int_\Omega F(\frac{|x|}{\|x\|})d\mu + \int_\Omega F(\frac{|y|}{\|y\|})d\mu]$

Since $\int_\Omega F(\frac{|x|}{\|x\|})d\mu \leq 1$ and $\int_\Omega F(\frac{|y|}{\|y\|})d\mu \leq 1$

So $\inf\{(\|x\|+\|y\|) : \int_\Omega F(\frac{|x+y|}{\|x\|+\|y\|})d\mu \leq 1\}$

Hence $\|x+y\| \leq \frac{K_1}{K_2}[\|x\|+\|y\|]$

Let $K = \frac{K_1}{K_2}$ So $\|x+y\| \leq K[\|x\|+\|y\|]$

Theorem (3-2): The set $B[L_F,R] = L_F{}^*$ of all bounded linear operators on a normed space L_F into R is Banach space.

Proof:

1. We introduce in $B[L_F,R]$ the operations of addition (+) and scalar multiplication $(\cdot)$ in the following manner: for $T,S \in B[L_F,R]$

(a) $(T+S)(f) = Tf + Sf$.

(b) $(\propto T)f = \propto Tf$,where α is a real number .

It can be easily verified that the following relations are satisfied by the above two operations:

(i) If $T,S \in B[L_F,R]$,then $T+S \in B[L_F,R]$

(ii) $(T+S)+U = T+(S+U)$, for all $T,S,U \in B[L_F,R]$.

(iii) There is an element $0 \in B(L_F,R)$ such that $0+T = T$ for all $T \in B[L_F,R]$

(iv) For every $T \in B[L_F,R]$,there is a $T_1 \in B[L_F,R]$ such that $T+T_1 = 0$

(v) $T+S = S+T$

(vi) $\propto T \in B[L_F,R]$ for all real $\propto$ and $T \in B[L_F,R]$.

(vii) $\propto(T+S) = \propto T + \propto S$.

(viii) $(\propto+\beta)T = \propto T + \beta T$

(ix) $(\propto\beta)T = \propto(\beta T)$.

(x) $1T = T$.

These relations mean that $B[L_F,R]$ is a vector space.

2. Now we prove that $\|T\| = \sup\left\{\frac{|Tf|}{\|f\|} / f \neq 0\right\}$ is a norm on $B[L_F,R]$and so $B[L_F,R]$ is a normed space with respect to this norm .It is clear that $\|T\|$ exist

Since $\left\{\frac{|Tf|}{\|f\|} / f \neq 0\right\}$ is a bounded subset of real numbers, its least upper bound or sup must exist .

(a) $\|T\| \geq 0$ as $\|T\|$ is the sup of nonnegative numbers. Let $\|T\|$ =0

Then $|Tf|$=0 or Tf=0 for all $f \in L_F$ or T=0 ,which is the zero element of $B[L_F,R]$

Conversely ,assume that T=0. Then $|Tf|=|0f|=|0|=0$.

This means that $\|T\| = \sup\left\{\frac{|Tf|}{\|f\|} / f \neq 0\right\}$=0.

Thus , $\|T\|$ =0 if and only if T=0

(b) $\|\propto T\| = \sup\left\{\frac{|\propto(T)f|}{\|f\|} / f \neq 0\right\}$

$= \sup\left\{\frac{|\propto(Tf)|}{\|f\|} / f \neq 0\right\}$

$$= \sup\left\{\frac{|\propto||Tf|}{\|f\|} / f \neq 0\right\}$$

By applying property (2) of the norm on R

$$\|\propto T\| = |\propto| \sup\left\{\frac{|Tf|}{\|f\|} / f \neq 0\right\} = |\propto|\|T\| .$$

$$\|T+S\| = \sup\left\{\frac{|(T+S)(f)|}{\|f\|} / f \neq 0\right\} = \sup\left\{\frac{|(Tf+Sf)|}{\|f\|} / f \neq 0\right\}$$

By the property (3) of the norm on R, we have

$|(Tf+Sf)| \leq |Tf| + |Sf|$

This implies that

$$\|T+S\| \leq \sup\left\{\frac{|Tf| + |Sf|}{\|f\|} / f \neq 0\right\}$$

$$= \sup\left\{\frac{|Tf|}{\|f\|} + \frac{|Sf|}{\|f\|} / f \neq 0\right\}$$

(By the Note: if A and B are two subsets of real numbers

Then $\sup(A+B) \leq \sup A + \sup B$) [S.Abul Hasan, 2004]

$$\|T+S\| \leq \sup\left\{\frac{|Tf|}{\|f\|} / f \neq 0\right\} + \sup\left\{\frac{|Sf|}{\|f\|} / f \neq 0\right\}$$

$=\|T\|+\|S\|$ or $\|T+S\| \leq \|T\|+\|S\|$

This proves that $B[L_F, R]$ is a normed space.

3. Now , we prove that , $B[L_F, R]$ is a Banach space provided R is a Banach space.

Let $\{T_n\}$ be a Cauchy sequence in $B[L_F, R]$. This means that for $\varepsilon > 0$, there exists a natural number N such that $\|T_n - T_m\| < \varepsilon$ for all n , m > N. This implies that , for any fixed function $f \in L_F$,we have

$|(T_n f - T_m f)| \leq \|T_n - T_m\|\|f\| < \varepsilon\|f\|$, for n , m > N;

That is , $\{T_n f\}$ is a Cauchy sequence in R. Since R is a Banach space , $\lim_{n\to\infty} T_n f = Tf$.

Now ,we verify that (a) T is a linear operator on L_F into R,

(b) T is bounded on L_F into R , and (c) $\|T_n - T\| \leq \varepsilon$ for $m > N$.

(a) Since T is defined for arbitrary $f \in L_F$ .it is an operator on L_F into R.

$$T(f_1 + f_2) = \lim_{n\to\infty} T_n(f_1 + f_2)$$

$=\lim_{n\to\infty}[T_n f_1 + T_n f_2]$, as all T_n 's are linear

$$= \lim_{n\to\infty} T_n f_1 + \lim_{n\to\infty} T_n f_2 = Tf_1 + Tf_2$$

$$T(\propto f) = \lim_{n\to\infty} T_n(\propto f) = \lim_{n\to\infty} \propto T_n f$$

since T_n's are linear , so that

$$T(\propto f) = \propto \lim_{n\to\infty} T_n f = \propto Tf .$$

(b) Since T_n's are bounded , there exists $M > 0$ such that $|T_n f| \leq M$ for all n .This implies that for all n and any $f \in L_F$,

$|T_n f| \leq \|T_n\|\|f\| \leq M\|f\|$.Taking the limit , we have

$\lim_{n\to\infty} |T_n f| \leq M\|f\|$,

Or

$|Tf| \leq M\|f\|$. This proves that T is bounded .

(c) Since $\{T_n\}$ is a Cauchy sequence , for each $\varepsilon > 0$ there exists a positive integer N such that $\|T_n - T_m\| < \varepsilon$ for all n , m > N .Thus , we have

$|T_n f - T_m f| \leq \|T_n - T_m\|\|f\| < \epsilon \|f\|$ for n , m > N .

Or

$\lim_{n\to\infty}\|T-T_m\| = \sup\left\{\frac{|(T-T_m)f|}{\|f\|} \,/\, f \neq 0\right\} \leq \in$ for all n , m > N

That is , $T_m \to T$ as $m \to \infty$.

Conclusions

1 . L_F is quasi-Banach Space .

2 . $B[L_F, R]$ is Banach Space .

References

Abul Hasan , S. 2004 , Applied Function Analysis , Marcel Dekker ,INC, NEW YORK. BASEL .

Montgomery , S.J. Orlicz – Lorentz Spaces , www.math.missouri.edu / ~ stephen /preprints/ orl-Lor- for conf.pdf.

Cong , W. and L. Yongjin , 2008 , On The Tringle Inequality in Quasi-Banach Spaces ,Journal of Inequalities in pur and applied mathematics ,9 ,2 , (1-4) .

Montgomery , S.J. 4Dec 1999 , comparison of Orlicz – Lorentz Spaces, arXiv: math /9201221v2 [math. FA] .