Prime Fuzzy Submodules and Strongly Prime Fuzzy Submodules Over Near-ring

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الخلاصة

الغرض من هذا البحث هو توسيع المفاهيم (الاعتيادية) للموديولات الأولية إلى الموديولات الضبابية الأولية في الحلقة القريبة . قادتنا دراستنا لهذا المفهوم لتقديم ودراسة المفاهيم الضبابية القريبة منه والتشاكل الضبابي عليه

كذلك قمنا بدراسة مفهوم (الاعتيادية) الموديولات الأولية القوية وتحويله إلى الموديولات الضبابية الأولية القوية في الحلقة القريبة . قادتنا دراستنا لهذا المفهوم لتقديم ودراسة المفاهيم الضبابية القريبة منه والتشاكل الضبابي عليه .

بالإضافة لذلك ، درسنا علاقة الموديولات الضبابية الاولية في الحلقة القريبة مع الموديولات الضبابية الأولية القوية (الشديدة) في الحلقة القريبة .

ABSTRACT

The main aim of this paper is to extend and study the notions of (ordinary) prime submodules into prime fuzzy submodules over near-ring .This lead us to introduce and study other properties of fuzzy submodules of near-ring and fuzzy homomorphism about it .

Also , we study the notions of (ordinary) strongly prime submodules into strongly prime fuzzy submodules over near-ring .This lead us to introduce and study other properties of fuzzy submodules of near-ring and fuzzy homomorphism about it .

Moreover , we study the relationships of prime fuzzy submodules of near-ring and strongly prime fuzzy submodules of near-ring .

INTRODUCTION

The present paper introduces and studies prime fuzzy submodules of a near-ring and strongly prime fuzzy submodules of a near-ring .

Near-rings are one of the generalized structures of rings . The study and research near-ring is very systematic and continuous . In 1905 L.E. Dickson began the study of near-ring and later 1930 Wieland has investigated it . Further , material about a near-ring can be found . In 1965 Zadeh introduced the concept of fuzzy subset . Fuzzy ideal of ring were introduced by Liu 1982 . In 1990 , Mlik D.S. and Mordeson J. N. introduced the concept fuzzy ideals of ring and prime fuzzy ideals of a ring . In 1997 D.T.K. and Biswas introduced the concept fuzzy ideal of near-ring . In 2001 , Jari R.H. introduced the concept fuzzy submodule and prime fuzzy submodule . In 2009 , Naghiipour A.R. introduced the concept prime submodule and strongly prime submodule .

In section one , some basic definitions and results are recalled which will be needed later. In section two , We introduce the concept of the prime fuzzy submodules of a near-ring and we study some properties and theorems about it . In section three, we introduce the concept of the strongly prime fuzzy submodules of a near-ring. Throughout this paper N is commutative near-ring with unity , and every fuzzy subset A of a near-ring N is nonempty fuzzy subset .

S . 1 PRELIMINARY:

In this section , some basic definitions and results which we will be used in the next section .Let $(N,+,\cdot)$ be a nonempty set where N is a set and (+) and $(\cdot)$ are any binary operations on N such that (1) (N,+) is a group (2) multiplication is associative (3) left distributive law $n\cdot (b+c) = n\cdot b + n\cdot c$, for all $n , b, c \in N$. This **near-ring** will be termed as right near-ring if (b +c) ·n = b·n + c·n , for all n , b, c $\in$ N , **([AL-Abege A.M.-H , 2010],[Burton D.M. , 1967]).**If 1·n= n (n·1 = n) then N has a left identity (right identity) , if (N, +) is abelian , we call N **an abelian near-ring ,([Choudhary S.C. and Gajendra Purohit , 2009],[Vasantha W.B. , 2002,]) .** If (N, ·) is commutative we call N itself a commutative near-ring **,([Choudhary S.C. and Gajendra Purohit , 2009],[Vasantha W.B. , 2002,]) .**

Clearly if N is commutative near-ring then left or right distributive law is satisfied and 1·n = n·1 = n , N is called **unital commutative near-ring** , **([D.T.K .and Biswas , 1997],[Zadeh L.A., 1965]) .**

A **fuzzy subset of N** is a function from N into [0,1], ([10],[18]). Let A and B be fuzzy subset of N. We write **A $\subseteq$ B** if A(x) $\le$ B(x) , for all x $\in$ N. If A $\subseteq$ B and there exists x $\in$ N such that A(x) < B(x) , then we write **A $\subset$ B** and we say that A is a proper fuzzy subset of B , [**Dixit V.N., Kumar R. and Ajmal N., 1991**]. Note that **A = B** if and only if A(x) = B(x), for all x $\in$ N , **[Kumbhojkar H.V. and Bapat M.S., 1993].**

Let A be a fuzzy subset of N, A(0) $\ge$ A(x) , for all x $\in$ N, **([Bae J.Y. and Kim H. , 2002],[Liu W.J., 1982],[Vasantha W.B. , 2002]) .**Let f : N$\to$ N' ,A and B be two fuzzy subsets of N and N' respectively ,the fuzzy subset f(A) of N' defined by : f(A)(y) = sup A(y) if f(y) $\neq$ 0 , y$\in$ N' and f(A)(y) = 0 , otherwise .It is called **the image of A under f** and denoted by f(A) . The fuzzy subset f^{-1} (B) of N defined by : f^{-1} (B)(y) = B(f(x)) ,for all x $\in$N .Is called **the inverse image of B** and denoted by f^{-1} (B) **, ([Dixit V.N., Kumar R. and Ajmal N., 1991],[Kumbhojkar H.V. and Bapat M.S., 1991],[Martines L., 1995]) .**

The characteristic function of N denoted by λ_N which define by : $\lambda_N(x) = \begin{cases} 1 & if\ a \in N \\ 0 & if\ a \notin N \end{cases}$,it is called **singular fuzzy set in N ,[Zadehi M .M .and Mashinchi M. , 1992,] .**

Let N , N' be any sets and f : N$\to$ N' be any function .A fuzzy subset A of N is called **f-invariant** if f(x) = f(y) implies A(x) =A(y), where x ,y $\in$N **,([Kumar R.,1991] ,[Kumbhojkar H.V. ,1997],[Liu W.J., 1982]) .** For each t $\in$ [0,1], the set **A_t** = { x $\in$ N | A(x) $\ge$ t } is called **a level subset of N** and A=B if and only if A_t = B_t for all t $\in$ [0,1] , **([Dixit V.N., Kumar R. and Ajmal N., 1991],[Malik D.S. and Mordeson J.N. , 1990],[Martines L., 1995]).**

Let x $\in$ N and t $\in$ [0,1], let x_t denote the fuzzy subset of N defined by :

$x_t(y) = \begin{cases} t & if\ x = y \\ 0 & if x \neq y \end{cases}$, $\forall$y $\in$ N . x_t is called **a fuzzy singleton, [Malik D.S. and Mordeson J.N. , 1990].** If x_t and y_s are fuzzy singletons, then $x_t + y_s = (x + y)_\lambda$ and $x_t \circ y_s = (x . y)_\lambda$, where λ = min { t ,s } , **([Kumbhojkar H.V. and Bapat M.S., 1993],[Malik D.S. and Mordeson J.N. , 1990]).** Let {A_i | i $\in$ Λ} be a collection of fuzzy subset of N . Define the fuzzy subset of N (**intersection**) by ($\bigcap_{i \in \Lambda} A_i$) (x) = inf { A_i(x) | i $\in$ Λ }, for all x $\in$ N,**([Dixit V.N., Kumar R. and Ajmal N., 1991] ,[Malik D.S. and Mordeson J.N. , 1990]).** Let ϕ be denote ϕ(x) = 0 for all x $\in$ N, **the empty fuzzy subset** of N,**([Kumbhojkar H.V. and Bapat M.S., 1993],[Swamy U.M. and Swamy K.N., 1988]).**

Let A and B be a fuzzy subsets of N , **the product $A \circ B$** define by : $A \circ B(x) = \sup\{\min\{A(y), B(z)\} \mid x = y \cdot z\}\}$ y, z $\in$ N , for all x $\in$ N , **[Mukherjee T.K. and Sen M.K.,1989] .The addition A + B** define by $(A + B)(x) = \sup\{\min\{A(y), B(z) \mid x = y + z\}\}$ y, z $\in$ N , for all x $\in$ N , **[Mukherjee T.K. and Sen M.K.,1989]** .

(1) Let A be a fuzzy subset of N, A is called **a fuzzy subnear-ring** of N if for all x, y $\in$ N , $A(x + y) \geq \min\{A(x), A(y)\}$ and $A(x) = A(-x)$**,([AL-Abege A.M.-H , 2010],[Bae J.Y. and Kim H. , 2002],[Bae J.Y. and Kim H. , 2002],[Jun Y.B. and Kim H.S., 2002]) .**

Let A be a fuzzy subset of N , A is called **a fuzzy ring of N** if for all x, y $\in$ N , $A(x - y) \geq \min\{A(x), A(y)\}$ and $A(x \cdot y) \geq \min\{A(x), A(y)\}$, ([12],[23],[25],[26]).

(2) A non empty fuzzy subset A of N is called **a fuzzy ideal of N** if and only if $A(x - y) \geq \min\{A(x), A(y)\}$; $A(x \cdot y) \geq \min\{A(x), A(y)\}$; $A(y+x - y) \geq A(x)$; and $A(x\cdot y) \geq A(y)$, for all y, x $\in$ N , **([Jun Y.B. and Kim H.S., 2002],[Vasantha W.B. , 2003])** .

Let X be a fuzzy ring of N and A be a fuzzy ideal of N such that $A \subseteq X$, then A is a fuzzy ideal of X, **[Vasantha W.B. , 2002].** Let A be a fuzzy ideal of N . If for all $t \in [0, A(0)]$, then A_t is an ideal of N and **([Jun Y.B. and Kim H.S., 2002] , [Swamy U.M. and Swamy K.N., 1988]).**Let X be a fuzzy ring of N . A be a fuzzy subset of X is a fuzzy ideal of X if and only if A_t is an ideal of X_t ,for all $t \in [0,A(0)]$ **, ([Jun Y.B. and Kim H.S., 2002],[Mordeson J.N. and Malik D.S.,1998],[Swamy U.M. and Swamy K.N., 1988]).** Let A and B be fuzzy ideals of N, then AB is a fuzzy ideal of N,**([Jun Y.B. and Kim H.S., 2002],[Swamy U.M. and Swamy K.N., 1988])** .Let A and B be fuzzy ideals of N, then $A \cap B$, A + B are fuzzy ideals of N **,([Jun Y.B. and Kim H.S., 2002],[Swamy U.M. and Swamy K.N., 1988]).**

PROPOSITION 1. 1 [Abou-Draeb A.T. 2011]

Let A and B be two fuzzy subsets of N .Then :

1. $A \circ B \subseteq A \cap B$.
2. $(A \circ B)_t = A_t \cdot B_t$, $t \in [0,1]$ **,[Dixit V.N., Kumar R. and Ajmal N., 1991]** .
3. $(A \cap B)_t = A_t \cap B_t$, $t \in [0,1]$.

PROPOSITION 1 .2 ([Kumar R., 1992],[Kumar R.,1991]) :

Let $f : N_1 \rightarrow N_2$ be a homomorphism of rings . Let A and B be two fuzzy subsets of N_1 and C and D be two fuzzy subsets of N_2 .Then :

1. $A \subseteq f^{-1}(f(A))$ and $A = f^{-1}(f(A))$ whenever A is f-invariant .
2. $A = f^{-1}(f(A))$, whenever A is f-invariant .
3. $f(A \cap B) \subseteq f(A) \cap f(B)$.
4. $f(f^{-1}(C)) = C$, $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.
5. $A \subseteq B \Rightarrow f(A) \subseteq f(B)$.
6. $C \subseteq D \Rightarrow f^{1}(C) \subseteq f^{1}(D)$.

PROPOSITION 1. 3 ([Abou-Draeb A.T. 2011],[Kumbhojkar H.V. and Bapat M.S., 1991]) :

Let $A : N \rightarrow [0,1]$ a fuzzy ideal of N , $B : N' \rightarrow [0,1]$ be a fuzzy ideal of N', and $f : N \rightarrow N'$ be homomorphism between them, then :

1. $f(A)$ is a fuzzy ideal of N'.
2. $f^{-1}(B)$ is a fuzzy ideal of N.

PROPOSITION 1. 4 ([Abou-Draeb A.T. 2011],[Kumbhojkar H.V. and Bapat M.S., 1991]) :

If $X : N \rightarrow [0,1]$, $Y : N' \rightarrow [0,1]$ are fuzzy rings , let $f : N \rightarrow N'$ be homomorphism between them and $A : N \rightarrow [0,1]$ a fuzzy ideal of X , $B : N' \rightarrow [0,1]$ a fuzzy ideal of Y, then :

1. $f(A)$ is a fuzzy ideal of Y.

2. $f^{-1}(B)$ is a fuzzy ideal of X.

PROPOSITION 1.5 ([Mukherjee T.K. and Sen M.K., 1987],[Zadehi M .M .and Mashinchi M. , 1992,]) :

Let $\{A_i \mid i \in \Lambda\}$ be a family of fuzzy ideals of N. Then $\bigcap_{i\in\Lambda} A_i$ is a fuzzy ideal of N .

PROPOSITION 1. 6 ([Abou-Draeb A.T. 2011],[Kumbhojkar H.V. and Bapat M.S., 1991,]) :

Let A and B be two fuzzy subsets of N and $f : N \to N'$ is inverse image function of B .Then :

1- $f(A) \cap f(B) = f(A \cap B)$.

2- $f(A) \circ f(B) = f(A \circ B)$.

3- $f(A_t) = (f(A))_t$.

4- $f^{-1}(A_t) = (f^{-1}(A))_t$.

PROPOSITION 1 .7 ([Abou-Draeb A.T. 2011],[Liu W.J., 1982]) :

Let A and B be two fuzzy ideals of a fuzzy ring X of N_1 and C and D be two fuzzy ideals of a fuzzy ring Y of N_2 . Let $f : N_1 \to N_2$ be a homomorphism . Then :

1. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.
2. $f(A \cap B) = f(A) \cap f(B)$, where f is a monomorphism .

PROPOSITION 1 .8 ([Kumbhojkar H.V. and Bapat M.S.,1991],[Liu W.J., 1982,],[Martines L., 1995,]) :

A is a fuzzy ideal of N if and only if λ_N is a fuzzy ideal of N.

DEFINITION 1.9 ([AL-Abege A.M.-H , 2010],[Mordeson J.N. and Malik D.S.,1998]) :

Let $(M,+,\cdot)$ be a group . Suppose we have a map $N \times M \to M$ written as a scalar multiplication $(n, m) \to n\cdot m$ satisfying :

(1) $n\cdot m \in M$; (2) $(n_1 + n_2)\cdot m = n_1 \cdot m + n_2 \cdot m$; (3) $(n_1 \cdot n_2)\cdot m = n_1 (n_2 \cdot m)$; for all n, n_1 , $n_2 \in N$ and $m \in M$; then m is the structure of a near left N-module , and we can define the concept of a near right N-module . If M is a near left N-module and a near right N-module then M is a near-module (**M is a near-ring module**) .

REMARK 1.10 [Vasantha W.B. , 2003] :

Let A and B be fuzzy near-modules of an N-module M **.** Define **AB** by : $AB(x) = \sup\{\min\{\min\{A(y_i), B(z_i)\} \mid i=1,.., n\} \mid y_i, z_i \in M, i=1,.., n\} \mid x = \sum_{i=1}^{n} y_i \cdot z_i , n \in \mathbb{N}\}$, for all $x \in M$, ([26],[27]) . Let B be fuzzy submodule of an N-module M and x_t is fuzzy singleton of N , **the product $x_t \circ B$** define by : $x_t \circ B(a) = \sup\{x_t(y), B(z)\} \mid a = y\cdot z\}$ $y, z \in M$, $\forall a \in M$.

PROPOSITION 1.11 [Vasantha W.B. , 2003] :

Let W and K be near-modules over a fuzzy near-ring A in a near-ring N and f a homomorphism of W into K .

1- Let W be a fuzzy near-module in K , then the inverse image $f^{-1}(B)$ of B is a fuzzy near-module in W .

2- Let K be a fuzzy near-module in W that has the sup property , then the image f (B) of B is a fuzzy near-module in K .

DEFINITION 1.12 [Vasantha W.B. , 2003] :

An N-homomorphism f of an N-module M into an N-module M′ is a mapping from M to M′ such that $(s+n)f = sf + nf$ and $(mf)r = (mr)f$ for all s ,n , m $\in$ M and for all $r \in N$.

Now , we give some properties and theorems of fuzzy submodules .

DEFINITION 1.13 [Vasantha W.B. , 2003] :

A non empty fuzzy subset A of an additive group G is called **a fuzzy normal subgroup of G** if (1) $A(x+y) \geq \min \{ A(x) , A(y)\}$ for all x, y in G ; (2) $A(-x) = A(x)$ for all $x \in G$; (3) $A(y+x-y) = A(x)$ for all x , y in G .

DEFINITION 1.14 [Vasantha W.B. , 2003] :

Let A be a non empty fuzzy subset of an N-module M .Then A is called a **fuzzy submodule of M** if (1) A is a fuzzy normal subgroup of M and (2) $A[(x+y)r-xr] \geq A(y)$ for all $x , y \in M ,r \in N$.

PROPOSITION 1.15 [Vasantha W.B. , 2003] :

For a non empty fuzzy subset A of an N-module M . A is a fuzzy submodule of an N-module M if and only if A_t is a submodule of M , for all $t \in [0, 1]$.

DEFINITION 1.16 ([Kumar R.,1991], [Vasantha W.B. , 2003]) :

Let X be a non empty fuzzy module of an N-module M , and A be a non empty fuzzy subset of an N-module M .Then A is called a **fuzzy submodule of X** if A is a fuzzy submodule of a fuzzy module X of an N-module M .

PROPOSITION 1 .17 ([Kumbhojkar H.V. and Bapat M.S., 1991],[Liu W.J., 1982],[Martines L., 1995]) :

A is a fuzzy submodule of N-module M if and only if λ_N is a fuzzy submodule of N-module M **.**

PROPOSITION 1. 18 ([AL-Abege A.M.-H , 2010],[Mukherjee T.K. and Sen M.K.,1989]) :

Let $A : M \rightarrow [0,1]$ a fuzzy submodule of N-module M , $B : M' \rightarrow [0,1]$ be a fuzzy submodule of N-module M′, and $f : M \rightarrow M'$ be homomorphism between them, then:

1. f (A) is a fuzzy submodule of N-module M′.

2. $f^{-1}(B)$ is a fuzzy submodule of N-module M .

DEFINITION 1.19 [Jun Y.B. and Kim H.S., 2002]) :

Let A be a fuzzy ideal of a near-ring N .Then A is called **a prime fuzzy ideal of N** if A is not a constant function and for any fuzzy ideals B and C of N ,if $B \circ C \subseteq A$, then either $B \subseteq A$ or $C \subseteq A$.

THEOREM 1.20 [Jun Y.B. and Kim H.S., 2002] :

A is a fuzzy ideal of N .Then A is a prime fuzzy ideal if and only if A_t is a prime ideal of N for all $t \in [0,A(0)]$.

DEFINITION 1.21 [Zadehi M .M .and Mashinchi M. , 1992] :

Let A and B be two fuzzy submodule of N-module M . We define (A:B) by :
$(A: B) = \{ r_t \mid r_t$ fuzzy singleton of N such that $r_t B \subseteq A \}$ .Note that : $(A: B) (r) = \sup \{ t \in [0, 1] \mid r_t B \subseteq A$, for all $r \in N\}$ .And if $B = (b_k)$,then :$(A: (b_k)) = \{ r_t \mid r_t$ fuzzy singleton of N such that $r_t b_k \subseteq A \}$.

DEFINITION 1.22 **[Zadehi M .M .and Mashinchi M. , 1992] :**

Let A be a fuzzy submodule of X where X be a fuzzy module of N-module M and I be a fuzzy ideal of N , define (A:I) by $(A: I) = \{a_k \mid a_k \subseteq X , a_k I \subseteq A\}$ .Note that :$(A:_X I) (x) = \sup$

$\{t \mid t \in [0, 1] , Ix_t \subseteq A \} ,\forall x \in M$. And if $I = (r_t)$,then $(A: (r_t))=\{ a_k \mid a_k \subseteq X ; r_t a_k \subseteq A \}$.

PROPOSITION 1.23 **[Zadehi M .M .and Mashinchi M. , 1992] :**

Let A and B be two fuzzy submodule of X where X be a fuzzy module of N-module M and I be a fuzzy ideal of N such that I(0) = 1, then :

1- $(A:_X I)$ is a fuzzy submodule of N-module M **.**

2- (A: B) is a fuzzy ideal of N .

PROPOSITION 1.24 **([D.T.K .and Biswas , 1997],[Martinez L., 1999]) :**

Let A be a fuzzy submodule of N-module M . |A is torsion-free fuzzy submodule if and only if A_t is a torsion-free submodule of M for all $t \in (0, 1]$.

PROPOSITION 1.25 **([D.T.K .and Biswas],[Martinez L., 1999]) :**

Let A be a fuzzy submodule of N-module M . |A is divisible fuzzy submodule if and only if A_t is a divisible submodule of M for all $t \in (0, 1]$.

S.2 PRIME FUZZY SUBMODULES OF NEAR-RING :

We shall fuzzify this concept in to prime submodule of near-ring by [**Kumar R.,1991**] . Some basic definitions and results of prime fuzzy submodule s of near-ring are recalled which will be needed later as **([Abou-Draeb A.T. 2012],[Jun Y.B. and Kim H.S., 2002],[Kash F. , 1982]) .**

DEFINITION 2 .1 :

Let X be a fuzzy module of an N-module M and A be a proper fuzzy submodule of X .Then A is called **a prime fuzzy submodule of an N-module M** if $r_t\, a_t \subseteq A$ for fuzzy singleton r_t of N and $a_t \subseteq X$, then either $r_t \subseteq (A:_N X)$ or $a_t \subseteq A$ .Note that : $(A:_N X) = \{ r_t \mid r_t X \subseteq A , r_t \subseteq X \}$.

THEOREM 2 .2 :

Let X be a fuzzy module of an N-module M and A be a proper fuzzy submodule of X .Then A is called **a prime fuzzy submodule of an N-module M** if for all fuzzy submodule B of X and for any fuzzy ideal I of N , if $IB \subseteq A$, then $B \subseteq A$ or $I \subseteq (A:X)$ .Note that : $(A: X) = \{ r_t \mid r_t X \subseteq A , r_t$ fuzzy singleton of N $\}$.

REMARK 2 .3 :

The fuzzy ideal $P = (A: (a_k))$, for all $a_k \subseteq X$ such that $a_k \not\subset A$, where $(A: (a_k)) = \{ r_t \mid r_t\, a_k \subseteq A , r_t$ is a fuzzy singleton of N $\}$, then $P = (A:_N X)$.

PROPOSITION 2 .4 :

The definition (2.1) and theorem (2.2) are equivalent.

Proof:

If definition (2.1) is true , to prove definition (2.2) .If B is a fuzzy submodule of X and I is a fuzzy ideal of N such that $IB \subseteq A$. Suppose $B \not\subset A$, then there exists $b_k \subseteq B$ but $b_k \not\subset A$, hence for all $r_t \subseteq I$, $r_t\, b_k \subseteq IB \subseteq A$. So that $r_t \subseteq (A :X)$ Since A is a prime fuzzy submodule and $b_k \not\subset A$. Therefore , $I \subseteq (A:X)$ .If theorem (2.2) is true , to prove definition (2.1) .

Let $r_t\, a_t \subseteq A$ and $a_t \not\subset A$, then $r_t \subseteq (A: (a_k))$. But $(A: (a_k)) = P = (A:_N X)$ by remark (2.3) .Therefore , $r_t \subseteq (A:_N X)$. A is a prime fuzzy submodule of X .

PROPOSITION 2 .5 :

If A is a prime fuzzy submodule of X where X is a fuzzy module of an N-module M , then (A: X) is a prime fuzzy ideal of N .

Proof:

By proposition (1.23) , (A: X) is a fuzzy ideal of N. To prove (A: X) is a prime fuzzy ideal of N . Let r_t , y_k be two fuzzy singletons of N such that $r_t\, y_k \subseteq (A: X)$. Then $r_t\, y_k\, X \subseteq A$ and $r_t\, y_k\, m_s \subseteq A$ for all $m_s \subseteq X$ implies that $r_t\, (y_k\, m_s) \subseteq A$, so either $y_k\, m_s \subseteq A$ or $r_t \subseteq (A: X)$ Since A is a prime fuzzy submodule . If $y_k\, m_s \subseteq A$, then $y_k \subseteq (A: X)$, then either $r_t \subseteq (A: X)$ or $y_k \subseteq (A: X)$. Therefore , (A: X) is a prime fuzzy ideal of N .

REMARK 2 .6 :

The converse of proposition (2.5) is not true in general for example :

Let $M = Z\oplus Z$ as a Z-module , let $X: M \to [0, 1]$, $A:M\to [0,1]$ defined by :

$$X(a,b)=\begin{cases} 1 & if\ (a,b)\in 2Z \oplus Z \\ 0 & otherwise \end{cases} \qquad A(a,b)=\begin{cases} 1 & if\ (a,b)\in 4Z\oplus <0> \\ 0 & otherwise \end{cases}$$

It is clear that X is a fuzzy module of M and A is a fuzzy submodule of X .Since $2_{1/2}\,(2, 0)_{1/2} = (4, 0)_{1/2}$ and $(4, 0)_{1/2} \subseteq A$ Since $A(4, 0) = 1 > 1/2$, $(2, 0)_{1/2} \subseteq X$ Since $X(2, 0) = 1 > 1/2$ and $(2, 0)_{1/2} \not\subset A$ Since $A(2, 0) = 0 \neq 1/2$.

Since $2_{1/2} \not\subset (A:X)$ Since there exists $(2, 3)_{1/2} \subseteq X$ and $2_{1/2}\,(2, 3)_{1/2} = (4, 6)_{1/2} \not\subset A$ because $A(4, 6) = 0 \neq 1/2$ .Then A is not a prime fuzzy submodule of X .Since $(A_t:X_t) = (4Z\oplus 0$, 2Z⊕Z) = {0} , for all t > 0 .Hence $(A:X)_t$ = {0} for all t > 0 . and (A:X)(r) =

$$= \begin{cases} 1 & if\ r = 0 \\ 0 & otherwise \end{cases}$$ = 0_1 , which can be easily shown a prime fuzzy ideal of Z .

REMARK 2 .7 :

The converse of proposition (2.7) is not true in general for example :

Let M = N = Z , let X: M → [0, 1] , defined by :

$$X(a) = \begin{cases} 1 & if\ a \in 4Z \\ 1/2 & if a \in 2Z - 4Z \\ 1/3 & otherwise \end{cases}$$ X is a fuzzy module of M . A:M→ [0,1] defined by :

$$A(a) = \begin{cases} 1 & if\ a = 0 \\ 1/2 & if a \in 2Z - \{0\} \\ 0 & otherwise \end{cases}$$. A is a fuzzy submodule of N-module M .

A is not a prime fuzzy submodule of X , since $5_{1/3}\, 4_{2/3} = 20_{1/3} \subseteq A$ and $4_{2/3} \subseteq X$. But A(4) = 1/2 ≠ 2/3 and $4_{2/3} \not\subset A$, $5_{1/3} \not\subset (A:X)$,Since there exists $3_{1/4} \subseteq X$, but $5_{1/3} 3_{1/4} = 15_{1/4} \not\subset A$ .We have $A_{1/2} = 2Z$ and $X_{1/2} = 2Z$, $A_t = \{0\}$ and $X_t = 4Z$, for all t > 1/2 .And $A_t = 2Z$ and $X_t = Z$, for all t < 1/2 .$A_t \neq X_t$ and A_t is a prime submodule of X_t .

PROPOSITION 2 .8 :

A is a prime fuzzy submodule of N-module M if and only if λ_N is a prime fuzzy submodule of N-module M.

Proof:

By proposition (1.18) , A is a fuzzy submodule of N-module M if and only if λ_N is a fuzzy submodule of N-module M . Suppose that A is a prime fuzzy submodule of N-module M, we must prove that λ_N is a prime fuzzy submodule of N-module M.

Let A(ra) ≥ t ,for some r ∈ N , so $(ra)_t \subseteq A$, implies that $r_t a_t \subseteq A$. (since A is a prime fuzzy submodule of N-module M) . Since $\lambda_N(0) = 1$

Let r ∈ N and a ∈ M ,then $\lambda_N(ra) = 1 = \lambda_N(r)$ or $\lambda_N(ra) = 1 = \lambda_N(a)$. Let r ∈ N and a ∉ M ,then $\lambda_N(ra) = 1 = \lambda_N(r)$ or $\lambda_N(ra) = 0 = \lambda_N(a)$. Let r ∉ N and a ∈ M ,then $\lambda_N(ra) = 0 = \lambda_N(r)$ or $\lambda_N(ra) = 1 = \lambda_N(a)$ .Let r ∉ N and a ∉ M ,then $\lambda_N(ra) = 0 = \lambda_N(r)$ or $\lambda_N(ra) = 0 = \lambda_N(a)$.

Hence λ_N is a prime fuzzy submodule of N-module M.Suppose that λ_N is a prime fuzzy submodule of N-module M, we must prove that A is a prime fuzzy submodule of N-module M .Let ra ∈ A_t for some r ∈ N , a ∈ X_t . Then, $\lambda_N(ra) = 1$ then $\lambda_N(r) = 1$ or $\lambda_N(a) = 1$ (since λ_N is a prime fuzzy submodule of N-module M).Thus A(ra) =1 ≥ t implies that A(ra) ≥ t , so $(ra)_t \subseteq A$, implies that $r_t a_t \subseteq A$. Hence A is a prime fuzzy submodule of N-module M.

PROPOSITION 2 .9 :

Let M = N and A be a fuzzy ideal of N . Then A is a prime fuzzy submodule of N-module M if and only if A is a prime fuzzy ideal of N .

Proof:

Let A be a prime fuzzy submodule of N-module M , to prove A is a prime fuzzy ideal of N.

Let A be a prime fuzzy submodule of N-module M , then A is a fuzzy ideal of N by **[Zadehi M .M .and Mashinchi M. , 1992]**.Let $a_r b_t \subseteq A$. Suppose that $a_r \not\subset A$, then $b_t \subseteq (A:X)$ since A is a prime fuzzy submodule of N-module M .

Thus $b_t\ \lambda_N \subseteq A$ and $(b_t \lambda_N)(b) \leq A(b)$, b ∈N since b= 1.b = b.1 , and

$$(b_t\lambda_N)(w) = \begin{cases} t & if\ w = bc for some c \in N \\ 0 & otherwise \end{cases}$$ .Hence $(b_t\lambda_N)(w) = t$ implies that t ≤ A(b) , $b_t \subseteq A$. Hence A is a prime fuzzy ideal of N .

Conversely, Let A is a prime fuzzy ideal of N , to prove A is a prime fuzzy submodule of N-module M . Let A is a prime fuzzy ideal of N , by [30] A is a fuzzy submodule of N-module M .

Let $r_t a_s \subseteq A$. Suppose that $a_s \not\subseteq A$, then $r_t \subseteq A$. Hence $r_t \lambda_N \subseteq A$ since for all $w \in M = N$.

$(r_t\lambda_N)(w) = \begin{cases} t & if\ w = rc for some c \in N \\ 0 & otherwise \end{cases}$ .If w = rc , then $(r_t\lambda_N)(w) = t$, but $A(w) = A(rc) \geq A(r) \geq t$ .Hence $(r_t\lambda_N)(w) \leq A(w)$ .If $(r_t\lambda_N)(w) = 0$, then $0 = (r_t\lambda_N)(w) \leq A(w)$. Thus $r_t\lambda_N \subseteq A$ implies that $r_t \subseteq (A: \lambda_N)$ .Hence A is a prime fuzzy submodule of N-module M .

PROPOSITION 2 .10 :

Let A be a fuzzy submodule of a fuzzy module X of N-module M . Then A is a prime fuzzy submodule if and only if $(A:_X I)$ is a prime fuzzy submodule of X for every fuzzy ideal I of N .

Proof:

Let A be a prime fuzzy submodule , to prove $(A:_X I)$ is a prime fuzzy submodule of X for every fuzzy ideal I of N .

Let A be a prime fuzzy submodule of X , by proposition (1.23(1)) $(A:_X I)$ is a fuzzy submodule of X .To prove $(A:_X I)$ is a prime fuzzy submodule of X .

Let $r_t x_s \subseteq (A: I)$, then $(r_t x_s) I \subseteq A$, then $(r_t a_k x_s) \subseteq A$ for all $a_k \subseteq I$.

But A is a prime fuzzy submodule of X , so either $x_s \subseteq A$ or $r_t a_k \subseteq (A:X)$ .If $x_s \subseteq A$, then $x_s \subseteq (A:I)$ by [30] .If $r_t a_k \subseteq (A:X)$ for all $a_k \subseteq I$, then $r_t I \subseteq (A:X)$, So $r_t IX \subseteq A$ .Hence $r_t X \subseteq (A:I)$, thus $r_t \subseteq ((A:I):X)$, so $(A:_X I)$ is a prime fuzzy submodule of X .

Conversely , let $(A:_X I)$ is a prime fuzzy submodule of X for every fuzzy ideal I of N , to prove A is a prime fuzzy submodule .Suppose that $(A:_X I)$ is a prime fuzzy submodule of X for all fuzzy ideal I of N . Let $I = \lambda_N$, where $\lambda_N(a) = 1$, for all $a \in N$.

$A \subseteq (A: \lambda_N)$ by [30] , and let $x_t \subseteq (A: \lambda_N)$, implies that $x_t\lambda_N \subseteq A$, then

$(x_t\lambda_N)(w) = \begin{cases} t & if\ w = ax for some a \in N \\ 0 & otherwise \end{cases}$ $\leq A(w)$,for all $w \in M$.

Consequently , $(x_t\lambda_N)(x) = t \leq A(x)$, implies that $x_t \subseteq A$, hence $(A: \lambda_N) \subseteq A$. Thus $A = (A: \lambda_N)$ .Therefore, A is a prime fuzzy submodule of X .

LEMMA 2 .11 :

Let A be a fuzzy submodule of N-module M and let $x \in N$. Then $A(x) = \sup\{ t \mid x \in A_t\}$, $t \in (0,A(0)]$.

Proof:

Since $A_t = \{ x \in N \mid A(x) \geq t\}$. Let $x \in N$. Then for each $t \in (0,1]$, $A(x) \geq t \Leftrightarrow x \in A_t$. Therefore , A(x) is the least upper bound of the set $\{t \mid x \in A_t\}$. Thus $A(x) = \sup\{t \mid x \in A_t\}$, $t \in (0,A(0)]$.

PROPOSITION 2. 12 :

Let $f : M \to M'$ be an epimorphism of modules of near-ring N , then :

1. $A : M \to [0,1]$ be an f-invariant prime fuzzy submodule of N-module M . Then f (A) is a prime fuzzy submodule of N-module M' .
2. $B : M' \to [0,1]$ a prime fuzzy submodule of N-module M', then $f^{-1}(B)$ is a prime fuzzy submodule of N-module M .

Proof:

1- Since f (A) is a fuzzy submodule of N-module M' by [27] To prove f (A) is a prime fuzzy submodule of N-module M' .

Suppose $r_t y_k \subseteq f(A)$ for a fuzzy singleton r_t of N and $y_k \subseteq M'$, then $(ry)_\lambda \subseteq f(A)$ where $\lambda = \min\{ t, k\}$ implies that $f(A)(ry) \geq \lambda$ .But f is an epimorphism , there exists $a \in M$ such that $y = f(a)$. Hence $f(A)(rf(a)) \geq \lambda$. Thus $f(A)(rf(a)) \geq \lambda$ implies that $[f^{-1} f(A)](ra) \geq \lambda$.$A(ra) \geq \lambda$ .implies that $(ra)_\lambda \subseteq A$, thus $r_t a_k \subseteq A$.

On the other hand $y_k \subseteq M'$ implies that $M'(y) \geq k$. Hence $M'(f(a)) \geq k$, so $M(a) \geq k$ that mean $a_k \subseteq M$. Thus $r_t a_k \subseteq A$, $a_k \subseteq X$. But A is a prime fuzzy submodule of X , hence either

$a_k \subseteq A$ or $r_t \subseteq (A:X)$ .If $a_k \subseteq A$, then $A(a) \geq k$ implies that $y_k \subseteq f(A)$ since $(f(A))(y) = f(A)f(a) = (f^{-1} f(A))(a) = A(a)$ since A is f-invariant .Thus $f(A)(y) \geq k$ implies that $y_k \subseteq f(A)$.

If $r_t \subseteq (A:X)$, then $r_t X \subseteq A$, hence $f(r_t X) \subseteq f(A)$, but $f(r_t X) = r_t X'$, implies that $(f(r_t X))(y) = \sup \{ (r_t X)(a) \mid f(a) = y\}$

$$= \sup \begin{cases} \sup\{\inf(t, X(s)\} & if\ a = rs \\ 0 & otherwise \end{cases} = \begin{cases} \sup\{\inf(t, X'f(s)\} & if\ f(a) = rf(s) \\ 0 & otherwise \end{cases}$$

$= (r_t X')f(a) = r_t X'(y)$.Hence $f(r_t X) = r_t X'$, implies that $r_t X' \subseteq f(A)$, thus $r_t \subseteq (f(A):X')$, then f (A) is a prime fuzzy submodule of N-module M′ .

2- Since $f^{-1}(B)$ is a fuzzy submodule of N-module M′ by [27] To prove $f^{-1}(B)$ is a prime fuzzy submodule of N-module M .Suppose $r_t a_k \subseteq f^{-1}(B)$ for a fuzzy singleton r_t of N and $a_k \subseteq M$, then $(ra)_\lambda \subseteq f^{-1}(B)$ where $\lambda = \min \{ t, k\}$ implies that $f^{-1}(B)(ra) \geq \lambda$. Then $B(f(ra)) \geq \lambda$ implies that $B(rf(a)) \geq \lambda$. Hence $(rf(a))_\lambda \subseteq B$. But B is a prime fuzzy submodule of N-module M′ , so either $(f(a))_k \subseteq B$ or $r_t \subseteq (B:X')$ .If $(f(a))_k \subseteq B$, then $B(f(a)) \geq k$ implies that $f^{-1}(B)(a) \geq k$, hence $a_k \subseteq f^{-1}(B)$ .If $r_t \subseteq (B:X')$, then $r_t X' \subseteq B$, hence $r_t \subseteq (f^{-1}(B):X)$.

$$(r_t X)(w) = \begin{cases} \sup\{\inf(t, X(a)\} & if\ w = ra, w \in M \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} \sup\{\inf(t, X'f(a)\} & if\ f(w) = rf(a) \\ 0 & otherwise \end{cases} = (r_t X')f(w) \leq B(f(w)) = f^{-1}(B)(w) .$$

Hence $r_t X \subseteq f^{-1}(B)$, implies that $r_t \subseteq (f^{-1}(B):X)$ .Thus $f^{-1}(B)$ is a prime fuzzy submodule of N-module M .

PROPOSITION 2. 13 :

Let A and B be two prime fuzzy submodules of N-module M. Then $(A \cap B)$ is a prime fuzzy submodule of N-module M.

Proof:

Let A and B be two prime fuzzy submodules of N-module M. To prove $A \cap B$ is a prime fuzzy submodule of N-module M.

$A \cap B$ is a fuzzy submodule of N-module M by proposition (1.6) .

To prove $A \cap B$ is a prime fuzzy submodule of N-module M .

Let $r_t a_t \subseteq A$ for fuzzy singleton r_t of N and $a_t \subseteq X$, then either $r_t \subseteq (A:_N X)$ or $a_t \subseteq A$.

Let $r_t a_t \subseteq B$ for fuzzy singleton r_t of N and $a_t \subseteq X$, then either $r_t \subseteq (B:_N X)$ or $a_t \subseteq B$.

Thus if $r_t a_t \subseteq (A \cap B)$ for fuzzy singleton r_t of N and $a_t \subseteq X$, then either $r_t \subseteq (A:_N X)$ and $r_t \subseteq (B:_N X)$ implies that $r_t \subseteq ((A \cap B) :_N X)$.or $a_t \subseteq A$ and $a_t \subseteq B$ implies that $a_t \subseteq (A \cap B)$ either $r_t \subseteq ((A \cap B):_N X)$ or $a_t \subseteq (A \cap B)$ .Then $A \cap B$ is a prime fuzzy submodule of N-module M.

S.3 STRONGLY PRIME FUZZY SUBMODULES :

In this section , we introduce and study the concept of the strongly prime fuzzy submodule. We give some basic properties and theorems of this concept .The idea of fuzzifying the concept of strongly prime submodule was appeared in [Kash F. , 1982] as the following :-

DEFINITION 3.1 :

Let A be a fuzzy submodule of N-module M .Then A is called **a strongly prime fuzzy submodule of N-module M** if $A(rx) = A(x)$ where $r \in N$, $r \neq 0$ and $x \in M$.

THEOREM 3 .2 :

Let A be a fuzzy submodule of N-module M. Then A is a strongly prime fuzzy submodule if and only if A_t is a strongly prime submodule of N-module M , for all $t \in [0,A(0)]$.

Proof:

Suppose A is a strongly prime fuzzy submodule of N-module M. Let $t \in [0, A(0)]$, to prove that A_t is a strongly prime submodule of N-module M. Let $rx \in A_t$ where r

$\in$ N , r $\neq$ 0 and x $\in$ M , then A(rx) $\geq$ t . But A is a strongly prime fuzzy submodule of N-module M, then A(rx) =A(x) $\geq$ t .Thus A(x) $\geq$ t . Hence x $\in$ A_t .Therefore , A_t is a strongly prime submodule of N-module M for all t $\in$ [0,A(0)].

Conversely , suppose A_t is a strongly prime submodule of M for all t$\in$ [0, A(0)] , to prove that A is a strongly prime fuzzy submodule of N-module M.Let r $\in$ N , r $\neq$ 0 and x $\in$ M , suppose A(rx) = t , then rx $\in$ A_t and rx $\notin$ A_k for all k > t by lemma (2.12) .But A_t is a strongly prime submodule of M implies that x $\in$ A_t and x $\notin$ A_k for all k > t ,then A(x) = t. That mean A(rx) =A(x) = t by lemma (2.12) .Therefore , A is a strongly prime fuzzy submodule of N-module M .

REMARK 3 .3 :

Note that the converse of proposition (3.3) is not true in general , for example : Let N be the Z-module $Z\oplus Z_2$: $X(a,b)=\begin{cases} 1 & if\ (a,b)\in 2Z\oplus Z_2 \\ 0 & otherwise \end{cases}$ $A(a,b)=\begin{cases} 1 & if\ (a,b)\in 2Z\oplus <0> \\ 0 & otherwise \end{cases}$,X is not strongly prime fuzzy submodule of N-module M and proposition(3.2) but A is a strongly prime fuzzy submodule of N-module M by [9] and proposition(3.2) .

PROPOSITION 3 .4 :

The intersection of two strongly prime fuzzy submodules of N-module M is strongly prime fuzzy submodule of N-module M .

Proof:

Let A and B be two strongly prime fuzzy submodules of N-module M. To prove A∩B is a strongly prime fuzzy submodule of N-module M.A∩ B is a fuzzy submodule of N-module M by proposition (1.6) . A∩B is a prime fuzzy submodule of N-module M by proposition (2.14).

Let r $\in$ N , r $\neq$ 0 and x $\in$ M , (A∩B) (rx) = min {A(rx) , B(rx) } .If A(rx) =A(x) and B(rx) =B(x) , then(A∩B) (rx) = min {A(x) , B(x) }= (A∩B) (x) .
Then A∩B is a strongly prime fuzzy submodule of N-module M.

PROPOSITION 3 .5 :

Let I be a submodule of an N-module M . Then I is a strongly prime submodule of N-module M if and only if λ_I is a strongly prime fuzzy submodule of N-module M.

Proof:

Suppose that I is a strongly prime submodule of N-module M, we must prove that λ_I is a strongly prime fuzzy submodule of N-module M. Let r $\in$ N , x $\in$ M, if rx $\in$ I , then x $\in$ I (since I is a strongly prime submodule of N-module M). Hence λ_I(rx) =1 = λ_I(x) . If rx $\notin$ I , then x $\notin$ I (since I is a strongly prime) .Hence λ_I(rx) =0 = λ_I(x) . Therefore , λ_I(rx) = λ_I(x) for all r $\in$N and x $\in$ M.Hence λ_I is a strongly prime fuzzy submodule of N-module M.

Conversely , let λ_I is a strongly prime fuzzy submodule of N-module M, we must prove that I is a strongly prime submodule of N-module M. Let r $\in$ N , x $\in$ M such that rx $\in$ I , then λ_I(rx) = 1 . But λ_I(rx) = λ_I(x) (since λ_I is a strongly prime fuzzy submodule of N-module M). Then λ_I(rx) = 1 implies that x $\in$ I .. Hence A is a strongly prime fuzzy submodule of N-module M .

PROPOSITION 3 .6 :

Let A be a fuzzy submodule of N-module M and I is submodule of N-module M . Then I is a strongly prime submodule of N-module M if and only if A_I is a strongly prime fuzzy submodule of N-module M .

Proof:

Suppose that I is a strongly prime submodule of N-module M , to prove A_I is a strongly prime fuzzy submodule of N-module M. We have to show that A_t and A_s are strongly prime submodules of M, for all t, s $\in$[0,A(0)] . Let r $\in$ N , r $\neq$ 0 and x $\in$ M , rx $\in A_t$, then A(rx) $\geq$ t . But by lemma (2.12) A(rx) = sup{t| xy $\in A_t$ }= t, thus A(rx) = t implies that rx $\in$I . Thus x $\in$I (since I is a strongly prime submodule of M, by [27] .

Therefore , A(x) = t , implies that x$\in A_t$ .Hence A_t is a strongly prime submodule of M.

Similarly, A_s is a strongly prime submodule of M. Hence, A_I is a strongly prime fuzzy submodule of N-module M by theorem (3.2) .

Conversely, suppose that A_I is a strongly prime fuzzy submodule of N-module M , to prove I is a strongly prime submodule of N-module M. Let r $\in$ N , r $\neq$ 0 and x $\in$ M, rx $\in$ N, then A(rx) = t implies that rx $\in A_t$ .But A is a strongly prime fuzzy submodule of M, then A_t is a strongly prime submodule of M for all t $\in$ [0,A(0)] by theorem (3.2) .Thus x $\in A_t$ implies that A(x) $\geq$ t and according by Lemma (2.12) , A(x) = t .Thus x $\in$ I. Hence I is a strongly prime submodule of N-module M.

REMARK 3 .7 :

Note that the converse of proposition (3.8) is not true in general , for example :

Let N = Z , M = Q$\oplus$Q , I = Q$\oplus$ <0> ; I is a strongly prime of N-module M .

A: Q$\oplus$ Q $\rightarrow$[0,1] such that : $A(x)=\begin{cases} 1 & if\ x\in I \\ 1/2 & otherwise \end{cases}$ It is easily seen that A is a fuzzy submodule of N-module M.. A is a strongly prime fuzzy submodule of N-module M.by proposition (3.7) . But A is not singular fuzzy submodule of Q$\oplus$Q .

PROPOSITION 3.8 :

Let N be a torsion integral domain and M be a torsion N-module . If A is a strongly prime fuzzy submodule of N-module M.such that A(0)=1 , then A is the singular fuzzy submodule of N-module M. .

Proof:

Let x $\in$ M x $\neq$ 0, then rx =0 ,for some r$\in$ N ,r $\neq$ 0 . A(rx)= A(0) =1 . But A(rx) = A(x) for all x $\in$ M because A is a strongly prime fuzzy submodule of N-module M. Hence A(x)=1, for all x$\in$ M .Therefore , A is the singular fuzzy submodule of N-module M.

THEOREM 3.9 :

Let N be an integral domain and M be a torsion N-module. If A is a divisible torsion-free fuzzy submodule of N-module M..Then A_t is a strongly prime submodule of N-module M., for all t $\in$(0,A(0)] .

Proof:

Since A is a divisible torsion-free fuzzy submodule of N-module M.. Then A_t is a divisible torsion-free submodule of M., for all t $\in$(0,A(0)] by proposition(1.24) and proposition(1.25) .Let x$\in$ M, r $\in$ N and r $\neq$ 0 such that rx $\in A_t$ .Since A_t is divisible , then there exists z $\in A_t$ such that rx =rz and hence r(x-z) =0 . But A_t is torsion –free and r$\neq$0 . Then x-z =0 which implies that x = z and hence x $\in A_t$. Therefore , A_t is strongly prime submodule of M by definition (3.1) .

PROPOSITION 3.10 :

Let f : M$\rightarrow$ M′ be an epimorphism of a modules of N ,if A and B are strongly prime fuzzy submodules of M and M′ respectively **,** then :

1. f (A) is a strongly prime fuzzy submodule of M′ provided that A is f-invariant .

2. $f^{-1}(B)$ is a strongly prime fuzzy submodule of M .

Proof:

1- Since f(A) is a fuzzy submodule of M' by proposition (1.18(1)) and f(A) is a prime fuzzy submodule of M' by proposition (2.13(1)) .Then we must prove that f(A)(ry) = f(A)(y), for all $y \in M'$, $r \in N$, $r \neq 0$. f(A)(ry) = f(A)(rf(x)) , since $y \in M'$, then there exists $x \in M$= such that y = f(x) and f is an epimorphism . = f(A)(f(rx)) ,[since f is an homomorphism] .

= f^{-1} (f(A))((rx)) ,by definition of the inverse image .

= A(rx) , by proposition (1.2, (2)) .

= A(x) , (since A is a strongly prime fuzzy submodule)

= f^{-1} (f(A))((x)) , by proposition (1.2, (2))

= f(A)(f(x)) ,by definition of the inverse image .

= f(A)(y) , for all $y \in M'$, $r \in N$, $r \neq 0$.Thus f(A) is a strongly prime fuzzy submodule of N-module M' .

2- Since $f^{-1}(B)$ is a fuzzy submodule of M' by proposition (1.18(2)) and $f^{-1}(B)$ is a prime fuzzy submodule of M' by proposition (2.13(2))Then we must prove that $f^{-1}(B)(ra) = f^{-1}(B)(a)$,

= (B) (rf (a)) ,[since f is an homomorphism] .

= (B) (f (a)) , (since B is a strongly prime fuzzy submodule)

= $f^{-1}(B)$ (a), by definition of the inverse image , for all $a \in M$, $r \in N$, $r \neq 0$.

Thus $f^{-1}(B)$ is a strongly prime fuzzy submodule of N-module M .

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