Development Of Finite Element –Based Space-Time Discretiztion For Solving The Stochastic Lagrangian Averaged Navier-Stokes

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Abstract:

The research aims to develop space-time estimation based on finite elements with the aim of solving the stochastic (LANS-α) Navier-Stokes equations with the multiplicative random effect and incompressible fluid turbulence, under conditions within a limited, non-periodic boundary. The polyhedral (or polygonal) domain of $\mathbb{R}d$, $d \in \{2,3\}$. For a fully discrete numerical scheme, the convergence analysis is studied and divided according to the spatial scale α into two cases, i.e. we first assume through the space partitioning step size that α is controlled such that It disappears when passing to the maximum, and then when α is constant, we present an alternative search.

Keywords: Stochastic Navier-Stokes, Stochastic Lagrangian averaged Navier-Stokes, Euler method, Finite element.

Introduction:

Navier-Stokes equations are used to analyze the velocity field in fluid flow, in the basics of fluid mechanics, in topics of mass conservation, momentum and continuity equations, as well as in fluid mechanics applications such as aerodynamics and turbos. These equations were firstly introduced and presented by Navier in 1822, then they were more completed by Stokes in special cases. Beside the mass conservation equation, the Navier-Stokes equations make the number of equations and unknowns equal and the problem could be solved theoretically. These equations are one of the most important fluid mechanics equations that are used in modelling of different phenomena related to fluid dynamics.In fact, Navier-Stokes equations present a mathematical model of nonlinear device of the motions, flows and fluids dynamics (both liquids and gases). These second order equations are nonlinear which are the most complex equations existing in fluid flow, that there is no exact answer in most models. To illustrate the significance of these equations, it is sufficient to state that Euler's and Bernoulli's equations, the most practical equations in the fluids motion, are derivatives of these equations which are obtained with a set of theories and simplifications. These equations can be applied to simulate and solve the flow fields of various practical problems, such as the flow in turbomachines, the flow around the drift and the wave impact on them, as well as the fuel and burning flow in internal and external combustion engines. Thus far, different types of these equations have been introduced. The thin shear layer Navier-Stokes is an example of these types of equations, a mathematical expression that actually expresses the physical conduct of a compressible fluid flow in the developing region of internal flows. To determine the heat transfer coefficient or the friction coefficient of compressible fluid flow inside a nozzle, the velocity and temperature variables in the vicinity of the wall can be obtained by solving these equations.The linear momentum equation form as

$$F = \left.\frac{dp}{dt}\right|_{sys}$$

whose derivative is the material derivative operator and F be the force acting on the fluid mass and P be the linear momentum. The integral form of this system is as

$$P = \int_{sys} V dm$$

By writing the integral form of this equation for a control volume and their summation, we will have:

$$\sum F_{\psi} = \frac{\partial}{\partial t}\int_{cv} V_{\rho} dV + \int_{cs} V_{\rho} V.\hat{n} dA$$

Here ψ is the components of the control volume. This equation can be used in many problems of fluid mechanics in the case of limited control volume.
for a system with mass δm, the following differential form of the linear momentum equation is obtained.

$$\delta F = \frac{d(V\delta m)}{dt}$$

where δF is the resultant of force acting on the mass δm. If we keep δm constant, we can write the equation as follows:

$$\delta F = \delta m \frac{dV}{dt}$$

Now, the velocity material derivative or dV/dt shows the calculated element acceleration, which is always indicated by a So we have:

$$\delta F = \delta m\, a$$

which is a simple representation and repetition of Newton's second law. Now, according to the relation between acceleration and velocity field

$$dF = dm\frac{dV}{dt} = dm\left[u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\frac{\partial V}{\partial z} + \frac{\partial V}{\partial t}\right]$$

In addition, the forces acting on a surface element can be divided into two categories, surface forces and mass forces. Mass forces are caused by gravity. Surface forces are divided into two parts, perpendicular force to the surface and tangential force to the surface. If we represent volume forces with the symbol dFb, according to Newton's second law we will have:

$$dF_b = dm.g$$

where g is the acceleration of gravity, however for the surface forces represented by the symbol of dFs, the perpendicular and tangential components can be introduced according to the Figure 1

As can be seen in Figure 1, dFs is divided into three components, dFn is perpendicular to the dA surface, $dF1$ and $dF2$ are parallel to the surface. The normal stress can be calculated as follows:

$$\sigma_n = \lim_{dA \to 0} \frac{dF_n}{dA}$$

and also the shear or tangential stresses which are produced by $dF1$ and $dF2$ on the surface are represented as follows:

$$\tau_1 = \lim_{dA \to 0} \frac{dF_1}{dA}$$

$$\tau_2 = \lim_{dA \to 0} \frac{dF_2}{dA}.$$

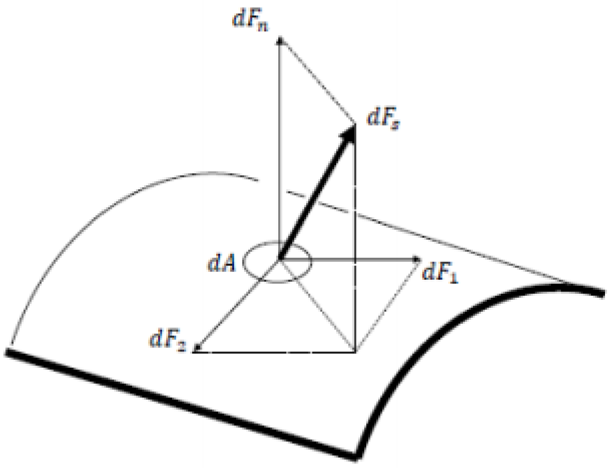


Figure 1: Surface force and its decomposition in an element

The same analysis can be done in three-dimensional space and obtain τxy, τxz, and τyz. So, to calculate the surface force acting on the element along the x-axis, we will have:

$$dF_{sx} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dxdydz$$

$$dF_{sy} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) dxdydz$$

$$dF_{sz} = \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) dxdydz$$

And finally, in all three directions of the 3D coordinate system

$$dF_s = dF_{sx}\vec{\imath} + dF_{sy}\vec{\jmath} + dF_{sz}\vec{k}$$

$$dF = dF_s + dF_b$$

$$dF_x = dm\, a_x$$

$$dF_y = dm\, a_y$$

$$dF_z = dm\, a_z$$

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

And the mass of the fluid element with the help of density is as follows:

$$dm = \rho\, dxdydz$$

The general form of the differential equation of motion in fluids is obtained by putting these formulas in the relationship between mass and forces in Newton's second law as follows:

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right).$$

Discretization of the Problem

we will first describe the finite elements method algorithm for solving the Navier-Stokes equation of the stochastic Lagrange averaged which was introduced , and then we will describe the results obtained regarding the convergence method.

To solve the problem of Navier-Stokes stochastic Lagrange averaged, we will discretize in both time and space dimensions.

Time Discretization

Suppose $M \in \mathbb{N}*$ and set $Ik= \{tl\}$: l=0 to M

is an equidistant partition of the interval $[0, T]$, where $t0 := 0$, $tM := T$ and $k := T/M$ are the time-step length, equal length step is not mandatory, but it helps in simplifying calculations. To build the time-step method, we define kmas a closed interval $[tm{-}1, tm]$ for $m \in \{1, \dots, M\}$.

Spatial Discretization

For simplicity, suppose that $\mathfrak{T}h$ is a quasi-uniform triangulation of the domain $D \subset \mathbb{R}d$, d = 1,2, whose maximum diameter is $h > 0$ and

$$\bar{D} = \bigcup_{K \in \mathfrak{T}_h} \bar{K}$$

We represent the vector field space of maximal polynomials of degree $n \in \mathbb{N}$ on an arbitrary set O with the symbol $\mathbb{P}n(O) = (pn(O))d$. Let's assume for $n1$, $n2 \in \mathbb{N}\backslash\{0\}$

$$\mathbb{H}_h := \{z_h \in \mathbb{H}_0^1 \cap [C^0(\bar{D})]^d | \quad z_h|_K \in \mathbb{P}_{n_1}(K), \forall K \in \mathcal{T}_h\},$$
$$L_h := \{q_h \in L_0^2(D) | \quad q_h|_K \in P_{n_2}(K), \quad \forall k \in \mathcal{T}_h\},$$
$$\mathbb{V}_h := \{z_h \in \mathbb{H}_h | \quad (div\, z_h, q_h) = 0, \quad \forall q_h \in L_h\}.$$

Be the spaces related to the finite elements functions. For fixed $n1$, $n2 \in \mathbb{N}\backslash\{0\}$, we assume that ($\mathbb{H}h$, Lh) applies to the discrete inf-sup condition, which means, there is a constant $\beta > 0$ independent of the mesh size h that

$$\sup_{z_h \in \mathbb{H}_h \backslash \{0\}} \frac{(div\, z_h, q_h)}{\|\nabla z_h\|_{\mathbb{L}^2}} \geq \beta \|q_h\|_{L^2}, \qquad \forall q_h \in L_h.$$

Let $z \in \mathbb{L}2$ be. We also represent the orthogonal projection operator in $\mathbb{L}2$ by $\Pi h : \mathbb{L}2 \rightarrow \mathbb{V}h$, which is determined by the unique solution of the following equation.

$$(z - \Pi_h z, \varphi_h) = 0, \qquad \forall \varphi_h \in \mathbb{V}_h$$

For $z \in \mathbb{H}01$, $\Delta h : \mathbb{H}01 \rightarrow \mathbb{V}h$ discrete Laplace operator is shown and with unique solution of

$$(\Delta^h z, \varphi_h) = -(\nabla z, \nabla \varphi_h), \forall \varphi_h \in \mathbb{V}_h$$

For $z \in \mathbb{H}01 \cap \mathbb{W}s,2$ there exists a positive constant C independent of h that

$$\sum_{j=0}^{1} h^j \|D^j(z - \Pi_{S_h} z)\|_{\mathbb{L}^2} \leq C h^s \|z\|_{\mathbb{W}^{s,2}}, \qquad 2 \leq s \leq n+1$$

where n is the degree of polynomials in $\mathcal{S}h$.
Also suppose that $\mathcal{S}h$ holds the following inequality.
(discrete differential filter) Suppose v is a vector field mentioned the discrete differential filter is represented by $\bar{u} \in \mathbb{V}h$ and it is given by:

$$\alpha^2(\nabla \bar{u}_h, \nabla \varphi_h) + (\bar{u}_h, \varphi_h) = (v, \varphi_h), \forall \varphi_h \in \mathbb{V}_h$$

Additional information is provided in the fourth chapter of the reference (Merdan and Manica, 2007) and we will only review some properties here.
Suppose $v = vh \in \mathbb{V}h$ and $\bar{u}h \in \mathbb{V}h$ is its differential filter, then

i. $v_h = \bar{u}_h - \alpha^2 \Delta^h \bar{u}_h$ *and* $\nabla v_h = \nabla \bar{u}_h - \alpha^2 \nabla \Delta^h \bar{u}_h$ almost everywhere on D

ii. $(\nabla v_h, \nabla \bar{u}_h) = \|\nabla \bar{u}_h\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h \bar{u}_h\|_{\mathbb{L}^2}^2$

We start with an initial point $U^0 \in \mathbb{H}_h$, for each $m \in \{1, \dots, M\}$ a quadruple stochastic process $(U^m, V^m, \Pi^m, \tilde{\Pi}^m) \in \mathbb{H}_h \times \mathbb{H}_h \times L_h \times L_h$ holds for all $(\varphi, \psi, \Lambda_1, \Lambda_2) \in \mathbb{H}_h \times \mathbb{H}_h \times L_h \times L_h$ in a P-almost surely in the following relations

$$\begin{cases} \circ(V^m - V^{m-1}, \varphi) + k\nu(\nabla V^m, \nabla\varphi) + k\tilde{b}(U^m, V^{m-1}, \varphi) - k(\Pi^m, div\ \varphi) \\ \qquad = k\langle f(t_{m-1}, U^{m-1}), \varphi\rangle + (g(t_{m-1}, U^{m-1})\Delta_m W, \varphi), \\ \circ(V^m, \psi) = (U^m, \psi) + \alpha^2(\nabla U^m, \nabla\psi) - (\tilde{\Pi}^m, div\ \psi), \\ \circ(div\ U^m, \Lambda_1) = (div\ V^m, \Lambda_2) = 0, \end{cases}$$

Where $\Delta_m W = W(t_m) - W(t_{m-1})$ *for all* $m \in \{1, \dots, M\}$.

Mode $\boldsymbol{\alpha} \leq \boldsymbol{Ch}$
Consider $\alpha = 10$^−3
h , $h \approx 0.03$ and $k = 10$^−3
we choose two different time values in the interval $[0, T]$ and plot the obtained solution in these two times, side by side.
In this way, the differences in the behavior of the solutions are shown. As expected, it is observed that the LANS-α and NSE solutions are similar with a minor variation. The reason for this minor variation is that here we are dealing with approximate computations, and generally the discretizatios step of the space h cannot be too close to zero, so this tiny variation cannot be omitted. In this case, code execution will be expensive.

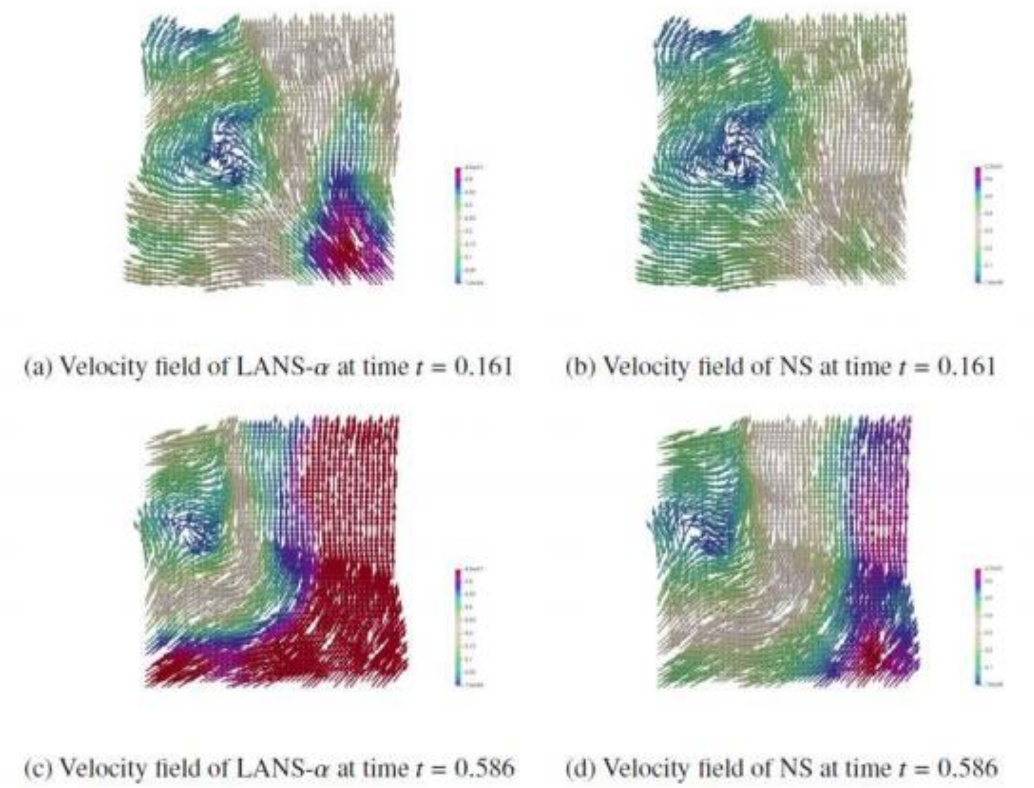
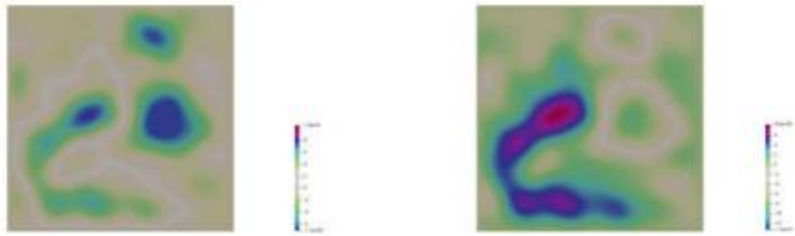
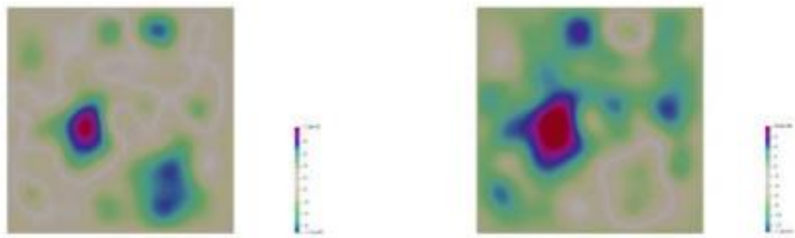


Figure 1: Numerical results obtained for the velocity field at two different times for LANS-α and NSE problems .



(a) LANS-α pressure at $t = 0.161$ (b) NS pressure at $t = 0.161$



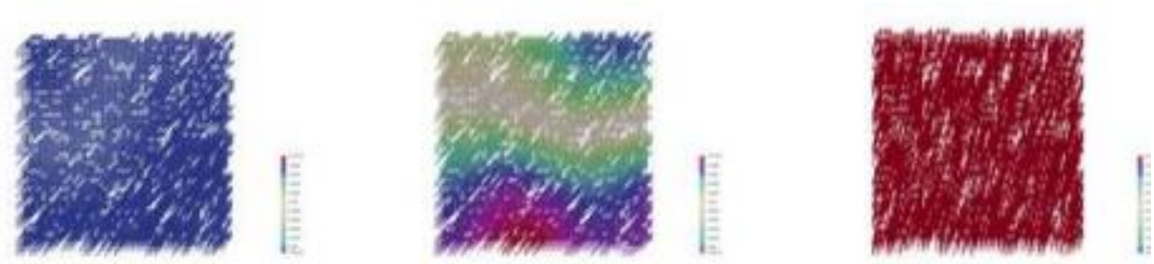
(c) LANS-α pressure at $t = 0.586$ (d) NS pressure at $t = 0.586$

Figure 2: Numerical results obtained for pressure at two different times for LANS-α and NSE problems

Mode $\boldsymbol{\alpha} \geq \boldsymbol{L} > \sqrt{\boldsymbol{k}}/\boldsymbol{h}$

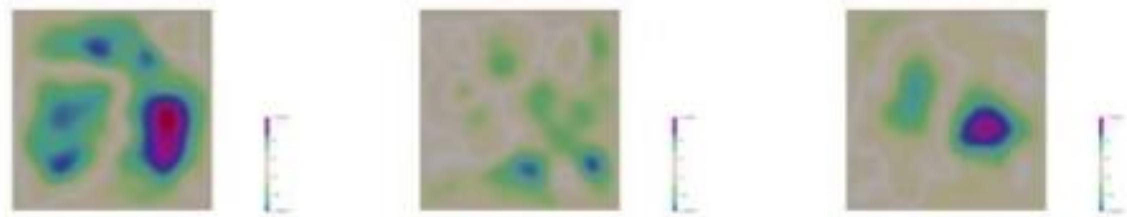
To check this case, we set $\alpha = 1$, $h \approx 0.03$. We take k as a term according to h, such as $k = 0.9h$ ^2. It also maintains the condition $\alpha > \sqrt{k}/h$.

With one realization, the present case result is obviously well-behaved, that means the terms of velocity field have higher smoothness. The stage of speed variations in the interval $[0.45 - \epsilon, 0.65 + \epsilon]$ is mentionable, $\epsilon \ll 1$ is where the velocity value changes from its lowest to its highest possible limit. In time less than $t = 0.45$ and more than $t = 0.65$, the velocity field maintains almost a constant value



(a) Time $t = 0.161$ (b) Time $t = 0.586$ (c) Time $t = 1$

Figure 3: Velocity field for the LANS-α problem in the second mode and at different times



(a) Time $t = 0.161$ (b) Time $t = 0.586$ (c) Time $t = 1$

Figure 4: Pressure obtained for the LANS-α problem in the second state and at different times

Here, we must point out that in both cases, the pressure is greatly affected by the noise. If we consider the obtained shape in terms of time, a random behavior will be noticed in each time node. This can be considered as the stochastic pressure presented in (Breit and Dodgson, 2021) in the two-dimensional case of the Navier-Stokes equations, a case where p can be divided into several terms, one of which can be in terms of W to form of Weiner process.

Conclusion

Here we briefly summarize the main results of this research:

Convergence of LANS-α to NS in 2D

Suppose $d = 2$ and $V0 \longrightarrow v0$ in L^2 (Ω; $\mathbb{L}$^2) when $h \rightarrow 0$. we get that v in (2.11) applies $\mathbb{P}$-almost surely, for all $t \in [0, T]$. In addition, by using a standard method, obviously see that

$v \in L$^2(Ω; C([0, T]; $\mathbb{H}$)). For more details, you can check the reference (Paradox, 1975).
Also, from the first part of Proposition 2, we have $v \in M\mathcal{F}t2(0, T; \mathbb{V})$.
Convergence for LANS-α
If $d \in \{2,3\}$ and $U0 \rightarrow \tilde{u}\,0$in L^2
(Ω; $\mathbb{H}1$) when $h \rightarrow 0$,
we may need to explain the convergence of
$\mathbb{E}\ [\int\ (\nabla\mathcal{V}k,h +,\ \nabla\varphi h\ )dtT0]$ in contrast to its continuous counterpart. For this purpose, we define the elliptic simulation Eh: $\mathbb{H}01 \rightarrow \mathbb{V}h$. This corresponding simulation is defined by the unique solution of the following problem: $(\nabla Eh\ z, \nabla\varphi h\ ) = (\nabla z, \nabla\varphi h\ ), \forall\varphi h \in \mathbb{V}h$.
The operator Eh applies to all $z \in \mathbb{H}2 \cap \mathbb{V}$ in $\Delta h\ Eh\ \ z = \Pi\ h\ \ \Delta z$
(H. Bessaih and A. Millet, 2021).
Therefore, it is true for all the equations (3.2), (3.3), preposition 3.14 and above relation (which is as follows).

$$(\nabla\mathcal{V}_{k,h}^{+}, \nabla E_h\varphi) = -(\mathcal{V}_{k,h}^{+}, \Delta^h E_h\varphi) = -(\mathcal{V}_{k,h}^{+}, \Pi_h\Delta\varphi) = -(\mathcal{V}_{k,h}^{+}, \Delta\varphi)$$

As a result, $\mathbb{E}\left[\int_0^T(\nabla\mathcal{V}_{k,h}^{+}, \nabla E_h\varphi)dt\right]$ converges to:

$$-\mathbb{E}\left[\int_0^T (v_\alpha(t), \Delta\varphi)dt\right] = -\mathbb{E}\left[\int_0^T (v_\alpha(t), A\varphi)dt\right] = \mathbb{E}\left[\int_0^T (u_\alpha(t) + \alpha^2 A u_\alpha(t), A\varphi)dt\right]$$

References

Banas L., et al. (2014). A convergent finite-element-based discretization of the stochastic
Landau-Lifshitz-Gilbert equation. IMAJNA, 34, 502–549.
Bessaih H., Millet A. (2021). Space-time Euler discretization schemes for the stochastic 2D
Navier-Stokes equations. Stochastics and Partial Differential Equations: Analysis and
Computations, 1-10.
Bjorland C., Schonbek M. E. (2008). On questions of decay and existence for the viscous
Camassa–Holm equations. Ann. Inst. H. Poincaré Anal. Non Linéaire, 25.5(0294-1449),

907 –936.
Breit D., Dodgson A. (2021). Convergence rates for the numerical approximation of the 2D stochastic Navier-Stokes equations. Numerische Mathematik, 147.3, 553–578.
Brenner S., Scott R. (2007). The mathematical theory of finite element methods. New York: Springer Sci. & Bus. Media. doi:10.1007/978-0-387-75934-0
Brzezniak Z., et al. (2013). Finite-element-based discretizations of the incompressible Navier–Stokes equations with multiplicative random forcing. IMAJNA, 33.3, 771–824.
Brzezniak Z., et. al. (2013). Finite-element-based discretizations of the incompressible Navier–Stokes equations with multiplicative random forcing. In: IMAJNA. 33.3 (Jan. 2013), pp. 771–824. issn: 0272-4979, 33.3(0272-4979), 771–824.
Çaˇglar A. (2009). Convergence analysis of the Navier–Stokes alpha model. Numer. Methods Partial Differential Equ., 26.5, 1154–1167.
Cao Y., Titi E. S. (2009). On the Rate of Convergence of theTwo-Dimensional α-Models of Turbulence to the Navier–Stokes Equations. Numer. Funct. Anal., 30.11-12, 1231–1271. doi:doi.org/10.1080/01630560903439189
Caraballo T. et al. (2006). On the existence and uniqueness of solutions to stochastic threedimensional Lagrangian averaged Navier-Stokes equations. Proc. R. Soc. A: Math., Phys. and Eng. Sci., 462.2066, 459–479.
Caraballo T., et al. (2006). On the existence and uniqueness of solutions to stochastic threedimensional Lagrangian averaged Navier-Stokes equations. Proc. R. Soc. A: Math., Phys. and Eng. Sci., 462.2066, 459–479. doi:10.1098/rspa.2005.1574
Cattabriga L. (1961). Su un problema al contorno relativo al sistema di equazioni di Stokes.

Rendiconti del Seminario Matematico della Università di Padova, 31, 308–340.
Chen S. et al. (1999). A connection between the Camassa–Holm equations and turbulent flows
in channels and pipes. Physics of Fluids, 11.8, 2343–2353.
Chueshov I., Millet A. (2009). Stochastic 2D Hydrodynamical Type Systems: Well Posedness
and Large Deviations. Appl. Math. Optim., 61.3(1432-0606), 379–420.
60
Connors J. (2010). Convergence analysis and computational testing of the finite element
discretization of the Navier–Stokes alpha model. Numerical Methods for Partial
Differential Equations, 26.6, 1328–1350.
Deugoue G., Sango M. (2009). On the Stochastic 3D Navier-Stokes-α Model of Fluids
Turbulence. Abstr. Appl. Anal. , 1-27.
Doghman J., Goudenège L. (2021). Numerical and convergence analysis of the stochastic
Lagrangian averaged Navier-Stokes equations. ArXiv: 2110.15611v1, math.NA, 1-31.
Foias C. et al. (2002). The Three Dimensional Viscous Camassa–Holm Equations, and Their
Relation to the Navier–Stokes Equations and Turbulence Theory. Journal of Dynamics
and Differential Equations, 14.1, 1–35.
Germano M. (1986). Differential filters for the large eddy numerical simulation of turbulent
flows. The Physics of Fluids, 29.6, 1755–1757.
Girault V., Raviart P. A. (2012). Finite element methods forNavier-Stokes equations: theory
and algorithms. Sci. & Bus. Media., 5, 1-13.
Goudenège L., Manca L. (2020). α-Navier-Stokes equation perturbed by space-time noise of
trace class.

Guermond J.L., Pasciak J. (2008). Stability of Discrete Stokes Operators in Fractional Sobolev
Spaces. J. Math. Fluid Mech. 10 (Nov. 2008), pp. 588–610., 10, 588–610.
Holm D. et al. (1998). The Euler-Poincaré Equations and Semidirect Products with
Applications to Continuum Theories. Adv. Math., 137, 1–81.
Ichikawa A. (1982). Stability of semilinear stochastic evolution equations. j. Math. Anal. Appl.,
90.1, 12-44.
Logg A., et al. (2012). Automated Solution of Differential Equations by the Finite Element
Method. Springer. doi:10.1007/978-3-642-23099-8
Marsden J.E., Shkoller S. (2001). Global well-posedness for the Lagrangian Navier-Stokes
(LANS-α) equations on bounded domains. Philosophical Transactions Of The Royal
Society B: Biological Sciences, 359.
Merdan S. K., Manica C. (2012). Convergence analysis of the finite element method for a
fundamental model in turbulence. Mathematical Models and Methods in Applied
Sciences, 12(11), 22-24.
Pardoux E. (1975). Equations aux dérivées partielles stochastiques non linéaires monotones.
Université Paris: PhD thesis.
Pierre Grisvard. (2011). Elliptic problems in nonsmooth domains. SIAM.
Vidar Thomée. (2007). Galerkin finite element methods for parabolic problems. Springer
Science & Business Media.
61
Xiuqing Chen, Jian-Guo Liu. (2012). Two nonlinear compactness theorems in Lp(0,T;B). Appl.
Math. Lett., 2252–2257