Effect of Tapered Thickness on the Logitudinal Free Vibrations of Cantilever Beam

Nawal H. Al – Raheimy
Babylon University/College of Engineering

Abstract

This paper deals with free longitudinal vibrations of nonuniform homogeneous cantilever beams. Cantilever of rectangular cross-section with constant width and tapered thickness variation are considered. Thickness at the clamped end is estimated while it changed with different values at free end at the ratio equal to the relation (thickness at free end hf / thickness at clamped end hc) where this ratio change from 0.05 to 0.9. The exact solution of differential equation in the linear case of free axial vibrations of nonuniform beam by using the analytic method by separation of variable in terms of Bessel function. Effect of thickness ratio between free end to clamped end (hf / hc) for different value of thickness of cantilever at clamped end and effect of different value of beam length on the characteristics of vibration (natural frequency and mode shape) are studied. Some of results are compared with approximation method which called Rayliegh's quotient. It is concluded that increasing the thickness of clamped end causes decrease in the natural frequency at any value of length of beam also increasing the thickness ratio and increasing the length of beam at assisted value of thickness at clamped end (hc) causing decreased in the value of natural frequency. On the other hand it is found that the value of mode shape of cantilever beam decrease when increase the thickness ratio (hf / hc) at any value of thickness of clamped end and at the same value of length of beam also the mode shape decreased with increasing thickness of clamped end (hc). Finally at the same value of (hc) the value of mode shape decreased with increasing length of beam..

Key words: Axial vibrations, cantilever beam, tapered, thickness ratio.

الخلاصة

يتناول هذا البحث دراسة الاهتزازات الحرة الطولية للدعامة الناتئة المثبتة من طرف واحد والتي تكون غير منتظمة التجانس. تم اعتبار الدعامة ذات مقطع مستطيل عند عرض ثابت وسمك يتغير بشكل متدرج, حيث تم تقدير قيمة السمك عند الطرف المثبت بينما السمك عند الطرف الحر يتغير عند النسبة المساوية للعلاقة (السمك عند الطرف الحر hf \ السمك عند الطرف المثبت hc) حيث انه هذه النسبة تتغير من 0.05 إلى 0.9 . الحل المضبوط للمعادلة التفاضلية في الحالة الخطية للاهتزازات الحرة المحورية للدعامة الغير منتظمة تمت بواسطة استخدام الطريقة التحليلية والمتمثلة بفصل المتغيرات والوصول إلى المصطلحات الخاصة بمعادلة بسل. تم دراسة تأثير نسبة السمك مابين النهاية الحرة إلى النهاية المثبتة(hf /hc) لقيم مختلفة لسمك الدعامة عند النهاية المثبتة وتأثير قيم مختلفة لطول العتبة على خصائص الاهتزاز (التردد الطبيعي وشكل النسق) ولقد تم مقارنة بعض النتائج مع الطريقة التقريبية والتي تسمى حاصل رايلي. تم الاستنتاج من ذلك بأنه عند زيادة سمك العتبة عند الطرف المثبت يسبب تناقص بالتردد الطبيعي عند أي قيمة من طول العتبة وكذلك زيادة نسبة السمك وزيادة طول العتبة عند قيمة معينة للسمك عند الطرف المثبت(hc) يسبب تناقص في قيمة التردد الطبيعي. من جهة أخرى وجد بان قيمة شكل النسق للعتبة الناتئة يقل عندما تزداد نسبة السمك (hf / hc) عند أي قيمة لسمك الطرف المثبت وعند نفس قيمة طول العتبة وكذلك شكل النسق يقل مع زيادة السمك للطرف المثبت (hc). أخيرا عن نفس قيمة ل(hc) فان قيمة النسق تقل مع زيادة طول العتبة.

List of Sympols

A(x) Area of cross section of beam at section x (m^2).
C_1 Arbitrary constant.
C_2 Arbitrary constant.
C_r Arbitrary constant
E Modulus of elasticity (N/m^2).
hc Thickness of beam at clamped end (cm).
hf Thickness of beam at free end (cm).
hx Thickness of beam at section x (cm).
L Length of beam (m).

$J_{1/2}$ Bessel function of order $\frac{1}{2}$.

$J_{-1/2}$ Bessel function of order $-\frac{1}{2}$.

J_{υ} Bessel function of order υ.

m(x) Mass of part of length of beam x (Kg).

P Axial force at section x (N).

$P+\frac{\partial P}{\partial x}$,Axial force at section (x+dx).

t Time (sec).

u(x,t) Displacement at any section x at time t.

U(x) Longitudinal displacement mode.

F(t) Function of time.

$U_r(x)$ Mode shape of order r.

$U_r'(x)$ Derivative of displacement of mode shape.

w Width of beam (cm).

x Length of part of beam (m)

z Parameter is equal to (hc-βx).

dz First derivative of parameter z w.r.t. x.

dz^2 Second derivative of parameter z w.r.t. x.

β Parameter define by Eq. (3).

ε Strain

λ Parameter equal to (density /modulus of elasticity).

ρ Density of material of beam (kg/m^3).

ω_1 Natural frequency of beam at mode 1 (rad/sec).

ω_r Natural frequency of beam at mode r (rad/sec).

Γ Gama function.

1. Introduction

Free vibration or stability analysis of structures is one of the main required tasks for an engineer to accomplish in the engineering design. Cantilevers of tapered thickness variation are important for studies regarding geometry influence on different phenomena. Cantilevers in general are key structures in many engineering applications. The fact that nonuniform cantilevers can be, under specific circumstances, more sensitive than uniform cantilevers is an important result. In particular they are extensively used as resonator sensors. Results regarding nonuniform cantilevers of particular geometry used as resonator sensors have been already reported in the literature. **Sanger, 1968** studied the characteristics of free transverse of beam, the differential equation of motion is solved analytically in terms of Bessel function, the beam which has rectangular cross section for constant width and tapered thickness. **Goel, 1976** applied an analytical method to obtain exact solution for the determination of modes and frequencies of nonuniform rectangular cross section beam in terms of Bessel function for free transverse vibration in pyramids thickness. **Wright, 1982** obtained solutions of differential equation for natural frequency an analytical method in terms of power series by Frobenius method for beam of constant thickness was dedicated to beam of one end sharp. **Storti and Aboelanga, 1987** performed study for nonuniform beams in hypergeometric series of circular cross section were linearly tapered. **De Rosa, 1994** the general case of a stepped beam with a single step has been solved, and the free vibration frequencies of a slender Euler-Bernoulli stepped beam with two elastic ends are calculated. **Auciello, 1996** presented a detailed study an exact analysis of free vibration of of rectangular tapered beam with amass at the tip and flexible constraint.. The rotary inertia of the concentrated mass is considered along with its eccentricity. **Lavendelis and . Zakrhevsky, 2000** presented the exact solution of differential equation in the linear case of free bending vibrations of nonuniform beam with rectangular cross-section using the factorization method. This beam with constant width and parabolic thickness is a good approximation of the gear tooth profile. The case of the beam with a sharp end is considered. **Turner and Wiehn 2001** considered the dynamics of atomic force microscope (AFM) cantilevers in terms of flexural vibrations. They investigated the sensitivity of a nonuniform cantilever beam (triangular with constant width) against a uniform cantilever, and found that for values of a studied parameter (the normal contact stiffness relative to the stiffness of the cantilever) greater than 100, the overall sensitivity of the triangular cantilever is greater than or equal to that of the uniform beam. **Caruntu 2004,** studied free vibration of nonuniform rotating beam which has circular cross section and the differential equation in term of hypergeometic function **Caruntu, 2007** studied the transverse vibration of beam in two cases of cross section where the first case of circular cross section which had both ends sharp and the second case was rectangular cross section for beam of one end sharp. The differential equation was solved analytically in term of orthogonal polynomials. **Dumirtu I, Caruntu, 2009** This paper deals with free transverse vibrations of nonuniform homogeneous beams. Cantilevers of rectangular (or elliptical) cross-section with parabolic thickness variation, and cantilevers of circular cross-section with parabolic radius variation, are considered. Factoring their fourth order differential equations of transverse vibrations into a pair of second order differential equations leads to general solutions in terms of hypergeometric functions.Exact natural frequencies and exact mode shapes are reported for sharp parabolic cantilevers of various dimensionless lengths. **Omer and Baki, 2010** The current study presents a mathematical model and numerical method for free vibration of tapered piles embedded in two-parameter elastic foundations. The method of Discrete Singular Convolution (DSC) is used for numerical simulation. Bernoulli-Euler beam theory is considered. Various numerical applications demonstrate the validity and applicability of the proposed method for free vibration analysis. The results prove that the proposed method is quite easy to implement, accurate and highly efficient for free vibration analysis of tapered beam-columns embedded in Winkler- Pasternak elastic foundations.

In this paper, frequency equation, mode shape are obtained in analytic form of cantilever beam which have tapered thickness and constant width for different ratio of thickness ration between free end to clamped end and estimate the characteristics of vibrations at different value of thickness at clamped end and different value of length also compare some of results with approximate method which called Raylieghs quotient.

2. Theoretical analysis

Consider an abruptly varying thickness of cantilever beam of length L and the thickness at any position of part of length of beam can be derived as shown in(Fig. 1).

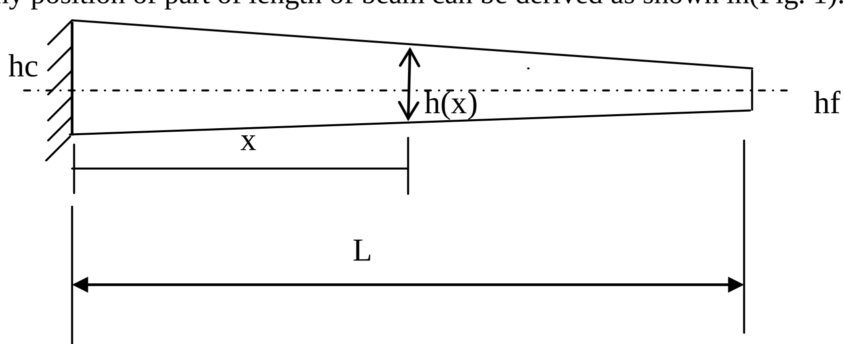

Fig.(1): Varying thickness of cantilever beam

$$\frac{(hc-hf)/2}{L} = \frac{(hx-hf)/2}{L-x} \qquad (1)$$

After simplified above relation yields:-

$$h(x) = hc - \frac{(hc - hf)}{L}x \qquad (2)$$

Let $\beta = \frac{hc - hf}{L}$ (3)

Therefore eq.(2) becomes as shown below:

$$h(x) = hc - \beta x \qquad (4)$$

The area of cross section at a distance x from length of beam can be writing as follow:

A(x) = w * h(x) (5)

Substitute eq.(4) in the eq. (5) yields:

A(x) = w *(hc-βx) (6)

Now we can be derive the natural frequency and mode shape for longitudinal motion of tapered thickness of cantilever beam. For extensional vibration it is assumed that cross section, which are initially plane and perpendicular to the axis of the beam, remain plane and perpendicular to that axis and that the normal stress in the axial direction is the only component of stress. The axis of the beam coincides with the X-axis; the displacement at any section x is denoted by u **[Warburton, 1976]**.

Consider the schematic of the beam on Fig.2. Displacements, strains, and stresses are assume uniform at a given cross section. From the figure, force P acts to the left and this force plus an undetermined increment dP acts to the right where dP=0 for static equilibrium. For dynamics problem, sum of the forces equals the product of mass and acceleration. Let the element have a mass per unit length of m(x) (or alternatively m(x) = ρ(x) .A(x), where ρ(x) is the density and A(x) is the area of te cross section at x).

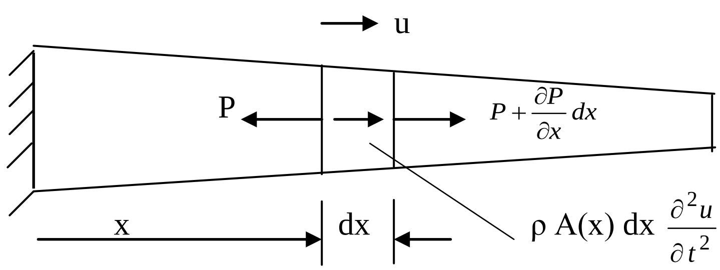

Fig.(2): Schematic for the longitudinal vibration of beam

Then, by Newton's second law of motion for an element of length dx,

$$\left[P + dP\right](x,t) - P(x,t) = m(x)\, dx \frac{\partial^2 u(x,t)}{\partial t^2}. \qquad (7)$$

From the strength of materials, P = A E ε = A E $\partial u(x,t)/\partial x$. Therefore, the force differential is

$$dP(x,t) = \frac{\partial P(x,t)}{\partial x} dx, \qquad (8)$$

$$\text{dp(x,t)} = \frac{\partial}{\partial x}\left(A(x)\, E \frac{\partial u(x,t)}{\partial x} \right) dx, \qquad (9)$$

and

$$\frac{\partial}{\partial x}\left(A(x)\, E \frac{\partial u(x,t)}{\partial x} \right) dx = m(x)\, dx \frac{\partial^2 u(x,t)}{\partial t^2} \qquad (10)$$

After substituting eq. (6) in the eq.(10) and then differential eq. yields the relation:

$$(hc - \beta x)\frac{\partial^2 u(x,t)}{\partial x^2} - \beta \frac{\partial u(x,t)}{\partial x} = \lambda (hc - \beta x)\frac{\partial^2 u(x,t)}{\partial t^2} \qquad (11)$$

where $\lambda = \rho/E$

Let $u(x,t) = U(x) . F(t)$ (12)

Differential eq. (12) for x & t and then substitute in eq. (11) can be obtained the following equation,

$$(hc-\beta x)\frac{d^2U(x)}{dx^2}-\beta\frac{dU(x)}{dx}-\omega^2\lambda(hc-\beta x)U(x)=0 \tag{13}$$

Let $z = hc - \beta x$ then find dz & d^2z , this substitute in the eq. (13) and after arranged yields

$$z\frac{d^2U(z)}{dz^2}-\beta\frac{dU(z)}{dz}+\frac{\omega^2\lambda}{\beta}U(z)=0 \tag{14}$$

To apply this to our problem it is necessary to determine appropriate boundary conditions to be applied at the clamped and free ends of beam., where $U(x)_{x=0} = U(z)_{z=hc} = 0$ and $(dU/dx)_{x=0}=(dU/dz)_{z=hc} =0$.

Identifying equation (14) with the general equation of Bessel function (**Wylli, 1987),** therefore the final solution can be writing in form:

$$U(z)=C_1\, z^{1/2}\, J_{1/2}\left(\sqrt{\frac{\lambda}{\beta}}\ \omega z\right)+C_2\, z^{1/2}\, J_{-1/2}\left(\sqrt{\frac{\lambda}{\beta}}\ \omega z\right) \tag{15}$$

For small z, $J_\upsilon(z)\approx\frac{1}{\Gamma(1+\upsilon)}\left(\frac{z}{2}\right)^{\upsilon}$ and $J_{-\upsilon}(z)\approx\frac{1}{\Gamma(1-\upsilon)}\left(\frac{z}{2}\right)^{-\upsilon}$

Now equation (15) becomes as follow:

$$U(z)\approx C_1\, z^{1/2}.\frac{1}{\Gamma\left(\frac{3}{2}\right)}\left(\frac{1}{2}\sqrt{\frac{\lambda}{\beta}}\ \omega z\right)^{1/2}+C_2\, z^{1/2}.\frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{1}{2}\sqrt{\frac{\lambda}{\beta}}\ \omega z\right)^{-1/2} \tag{16}$$

$$U(z)\approx C_1\, z\ .\frac{1}{\Gamma\left(\frac{3}{2}\right)}\left(\frac{1}{2}\sqrt{\frac{\lambda}{\beta}}\ \omega\right)^{1/2}+C_2\ .\frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{1}{2}\sqrt{\frac{\lambda}{\beta}}\ \omega\right)^{-1/2} \tag{17}$$

$$U^{\cdot}(z)\approx C_1\ \frac{1}{\Gamma\left(\frac{3}{2}\right)}\left(\frac{1}{2}\sqrt{\frac{\lambda}{\beta}}\ \omega\right)^{1/2} \tag{18}$$

But to satisfy the second boundary condition $(dU/dz)_{z=hc} = 0$, we must set $C_1 = 0$, causing solution eq.(15) to reduce to

$$U(z)=C_2\, z^{1/2}\, J_{-1/2}\left(\sqrt{\frac{\lambda}{\beta}}\ \omega z\right) \tag{19}$$

Applying the remaining boundary condition U(hc) = 0 to (19) gives

$$0=C_2\, hc^{1/2}\, J_{-1/2}\left(\sqrt{\frac{\lambda}{\beta}}\ \omega hc\right) \tag{20}$$

And this satisfied if either $C_2 = 0$ or $J_{-1/2}\left(\sqrt{\frac{\lambda}{\beta}}\ \omega hc\right)= 0$, the first result in the trivial solution u(z)=0 , which is no use. The second equation defines $J_{-1/2}$, that is , $C_1\neq 0$ and $J_{-1/2}\left(\sqrt{\frac{\lambda}{\beta}}\ \omega hc\right)= 0$

From the division:- $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos(x)$, therefore

$$J_{-1/2}\left(\sqrt{\frac{\lambda}{\beta}}\ \omega hc\right)=\sqrt{\frac{2}{\pi\sqrt{\frac{\lambda}{\beta}}\,\omega hc}}\ \cos\left(\sqrt{\frac{\lambda}{\beta}}\,\omega hc\right) \tag{21}$$

From eq.(20) and (21) we obtained $0 = C_2\, hc^{1/2} \sqrt{\dfrac{2}{\pi \sqrt{\frac{\lambda}{\beta}}\, \omega\, hc}}\, \cos(\sqrt{\frac{\lambda}{\beta}}\, \omega\, hc)$, that yields to

$\cos(\sqrt{\frac{\lambda}{\beta}}\, \omega\, hc) = 0$, therefore , $(\sqrt{\frac{\lambda}{\beta}}\, \omega_r\; hc) = (2r-1)\frac{\pi}{2}$ (22)

From eq.(22), we can solve for ω ,

$$\omega_r = \frac{(2r-1)\,\pi}{2}\sqrt{\frac{\beta}{\lambda\, hc^2}} \tag{23}$$

Equation (15) for the rth mode becomes

$$U(z) = C_r\, z^{1/2}\, J_{-1/2}\left(\sqrt{\frac{\lambda}{\beta}}\;\; \omega\, z\right) \qquad r= 1, 2, \ldots \tag{24}$$

From divisions of z eq.(24) becomes

$$U_r(x) = C_r\, (hc-\beta x)^{1/2}\, J_{-1/2}\left(\sqrt{\frac{\lambda}{\beta}}\;\; \omega_r\, (hc-\beta x)\right) \tag{25}$$

Since the modes can only be specified to a constant, it is customary to normalize them according to $\int_0^L m(x)\, U_r^2(x)\, dx = 1$. [**Benaroya, 1998**] (26)

Substituted eq.(25) in the eq.(26)

yields: $$\int_0^L m(x)\left[C_r\;\, (hc-\beta x)^{1/2}\, J_{-1/2}\left(\sqrt{\frac{\lambda}{\beta}}\;\; \omega_r\, (hc-\beta x)\right)\right]^2 dx = 1 \tag{27}$$

$$C_r^2 \int_0^L m(x)\left[(hc-\beta x)^{1/2}\, J_{-1/2}\left(\sqrt{\frac{\lambda}{\beta}}\;\; \omega_r\, (hc-\beta x)\right)\right]^2 dx = 1 \tag{28}$$

Put m(x) = ρ. A(x), mass per unit length, substituted m(x) in eq.(28) and integrated by part we can obtained the constant C_r :-

$$C_r = \sqrt{\frac{\pi(\lambda\,\omega/\beta)^{1/2}}{m\;\, L}} \tag{29}$$

Where, m= ρ w (hc + hf)/2 it is represented mass per unit length. Substituted eq.(29) in the eq.(25) and simplified we yields the modes shapes of vibrations in the following form

$$U_r(x) = \sqrt{\frac{\pi(\lambda\,\omega/\beta)^{1/2}}{m\;\, L}}\, (hc-\beta x)^{1/2}\, J_{-1/2}\left(\sqrt{\frac{\lambda}{\beta}}\;\; \omega_r\, (hc-\beta x)\right) \tag{30}$$

That is shown the natural frequency and mode shape of longitudinal vibration in analytical method but now we can obtain the natural frequency by using Rayliegh' s quotient, where the natural frequency is equal potential energy divided by kinetic energy. In the longitudinal motion this relation can we shown below

$$\omega_r^2 = \frac{\int_0^L E\, A(x)\left[U_r'(x)\right]^2 dx}{\int_0^L m(x)\left[U_r(x)\right]^2 dx} \tag{31}$$

Usually, one can only guess at the function $U_r(x)$. The better the guess, the closer (from above) will the approximate frequency be to the actual value. For the first mode, we can obtain reasonable approximate for ω_1. For a cantilever beam, guess for $U_1(x)$ a function equal zero at x=0, that is, $U_1(0) = 0$. At the free end, the deflection and slope must be not zero, and our guess $U_1(x)$ must be such that $U_1(L) \neq 0$ and $U'1(L) \neq 0$. We can arrive at several possible guesses for the eigenfunction, but a simple one is, [**Benaroya, 1998**]

$$U_1\,(x) = a\left[1 - \cos\frac{\pi\,x}{2\,L}\right] \qquad (32)$$

Where a is a constant, substitute the expression for $U_1(x)$ into the equation (32) and then integrate the formula to find the approximation to be

$$\omega_1 = \frac{7.55}{L}\sqrt{\frac{E\,(hc+hf)}{\rho\left(0.75\,(hc+hf) - \frac{4}{\pi}\,hf\right)}} \qquad (33)$$

3. Results and Discussion:

Table(1) shows the properties and dimensions of cantilever beam and table (2) shows the natural frequency of the first mode of the analytic method and Rayliegh's quotient for different value of thickness at clamped end and different value of length of beam. The approximate value is above the actual value since the approximate mode is always stiffer than the actual mode. Figures (3 to 8) show the natural frequency of the first two mode of vibration as a function of the thickness ratio of beam for different value of clamped thickness and different value of length of beam. It is shown that there is decreased in natural frequency with increasing the thickness ratio and increasing the clamped thickness which is cause increasing the mass according to the general relation in free vibrations $\omega_n=\sqrt{stifness/mass}$ declares effect the mass on the natural frequency, in the other wise the natural frequency also decreased with increasing the length of beam where the length effect directly on stiffness of beam and when increased causing decreased the stiffness of beam and finally caused decreased the natural frequency of beam at the same clamped thickness and thickness ratio, we can note that the frequency increased with increasing the number of mode (r=1,2), also can be seen the difference between three curves diminish when the thickness ratio (hf/hc) increases, this behavior can be explained by the fact the geometrical structure of beam approaches to uniform section when the ratio reaches to one, therefore the natural frequency equal to $(1.57\sqrt{E/\rho L^2})$, can be written $(1.57\sqrt{EA/mL^2})$ that is mean the constant properties and the length of beam effect on natural frequency . Figs (9&10) show the natural frequency as a function of length of beam for different value of the thickness ratio and different thickness of clamped end it is noted also the natural frequency is decreased with increasing the clamped thickness and thickness ratio at the same value of length. It may be observe from figures the natural frequency quickly decreased until approaches to (3.5m) where slowly variation occurs, this is attributed to the fact of the structures for length less (3.5m) have very low stiffness when compare with increasing mass, but for high length the structure approaches to stability where the stiffness decreasing uniformly with increasing the mass. The main features of the mode shapes associated with the first two of natural frequency as a function of the length of cantilever beam are shown in Figs.(11 to 22) for variable clamped thickness and variable length of beam. It can be note that for two mode shapes the amplitude is decreased with increasing the length of beam, thickness ratio and thickness of clamped beam associated to equation (30), we obtained that the behavior of all mode are wave which is to describe the motion of beam through vibration for all modes. The displacements of motion for all modes are equal to zero at minimum length of beam because of there is no motion at clamped end but the displacement is the maximum value at the other end because it is free motion. The behavior of wave of displacement isn't axisymetric because of the boundary condition for two end isn't similar in the other wise the mass and the stiffener is tapered on the length of beam.

4.Conclusions:

From the results obtained, the main conclusion can be summarized as; the natural frequencies of the tapered thickness of cantilever beam are decreased with increasing the clamped thickness, thickness ratio and length of beam

Table (1): Specifications of the tested models

Parameter	Symbol	Value	Units
Length	L	1- 5	m
Thickness of clamped end	hc	0.1, 0,15, 0.2	m
Thickness of free end	hf	0.1hc - 0.9hc	m
Width of beam	w	0.1, 0.15, 0.2	m
Modulus of elasticity	E	200	Gpa
Density	ρ	7800	kg / m^3

Table(2) :Natural frequencies of the first mode of varying thickness beam.

Length of beam (L) m	Clamped thickness hc(m)	Free thickness hf (m)	Rayliegh's quotient (R.M.)	Analytic method
1	0.1	0.75hc	12774	12576
	0.15	0.65hc	12210.86	12150
	0.2	0..6hc	11932.55	11248.71
	0.25	0.5hc	11381.69	11248.71
	0.3	0.45hc	11108.63	10769.82
2	0.1	0.9hc	6820.532	5624.353
	0.15	0.8hc	6530.123	6494.446
	0.2	0.755hc	6387.229	6288.221
	0.25	0.7hc	6245.707	6161.172
	0.3	0.655hc	6105.43	6074.99
3	0.1	0.95hc	3484.154	2812.174
	0.15	0.9hc	3410.265	3247.225
	0.2	0.85hc	3410.266	2812.177
	0.25	0.8hc	3337.257	3080.585
	0.3	0.75hc	3265.062	3247.223

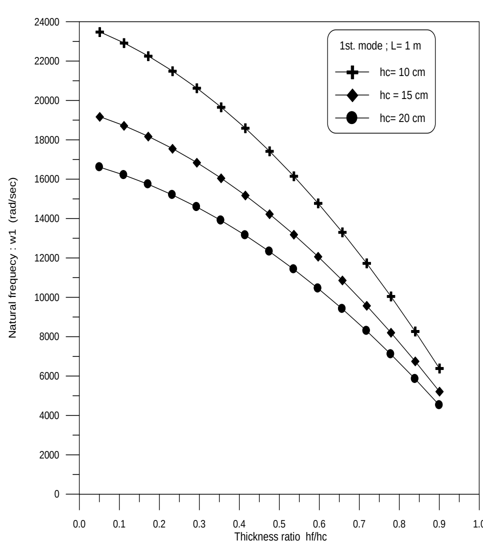

Fig. (3): Natural frequency as a function of thickness ratio of different value of clamped thickness for 1st mode at one meter length

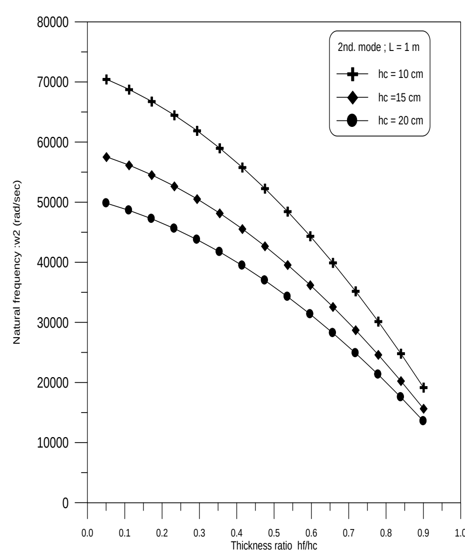

Fig. (4): Natural frequency as a function of thickness ratio of different value of clamped thickness for 2nd mode at one meter length

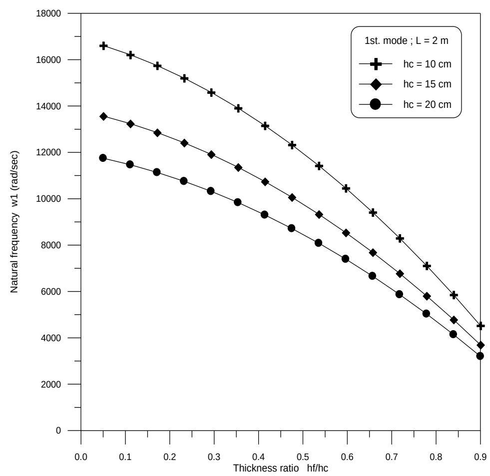

Fig. (5): Natural frequency as a function of thickness ratio of different value of clamped thickness for 1st mode at two meter length

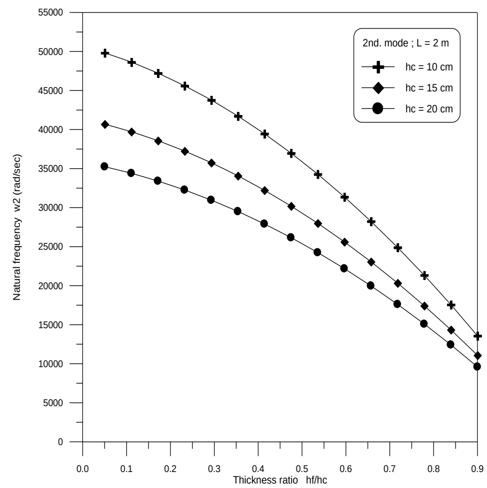

Fig. (6): Natural frequency as a function of thickness ratio of different value of clamped thickness for 2nd mode at two meter length

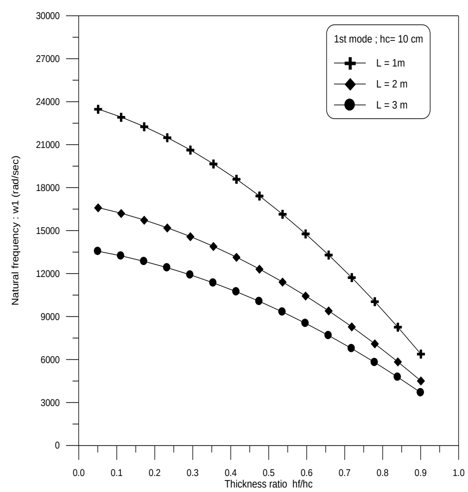

Fig. (7): Natural frequency as a function of thickness ratio of different value of length for 1st mode at clamped thickness = 10 cm

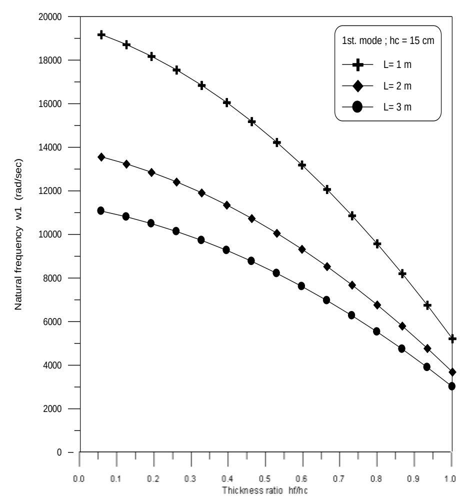

Fig. (8): Natural frequency as a function of thickness ratio of different value of length for 1st mode at clamped thickness = 15 cm

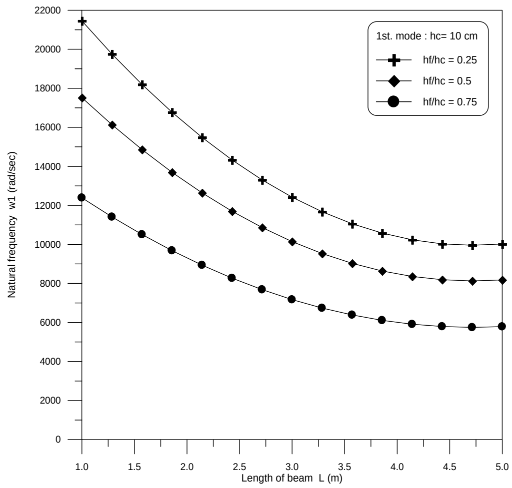

Fig. (9): Natural frequency as a function of length of different value of thickness ratio for 1st mode at clamped thickness = 10 cm

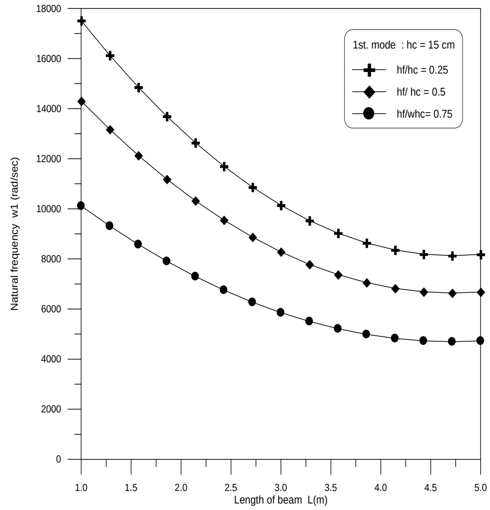

Fig.(10): Natural frequency as a function of length of different value of thickness ratio for 1st mode at clamped thickness = 15 cm

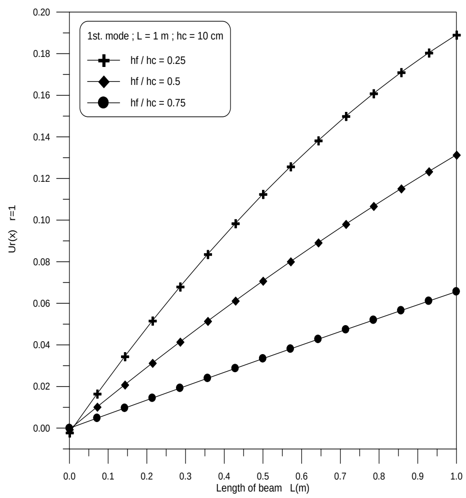

Fig.(11):Mode shapes associated with the first natural frequency of beam for different value of thickness ratio respect to one meter length at hc=10cm

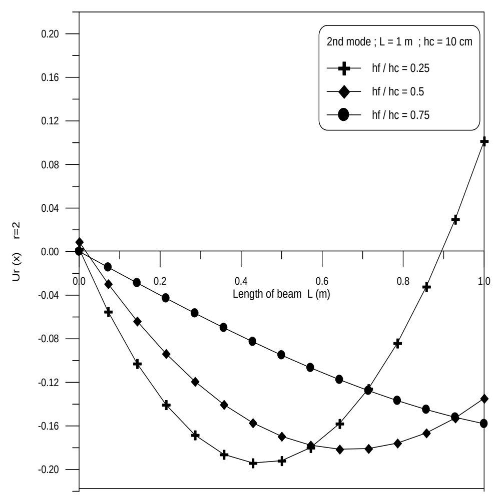

Fig.(12):Mode shapes associated with the second natural frequency of beam for different value of thickness ratio respect to one meter length at hc=10cm

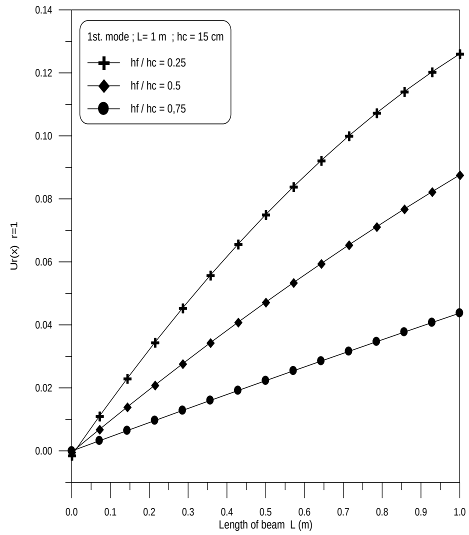

Fig.(13):Mode shapes associated with the first natural frequency of beam for different value of thickness ratio respect to one meter length at hc=15cm

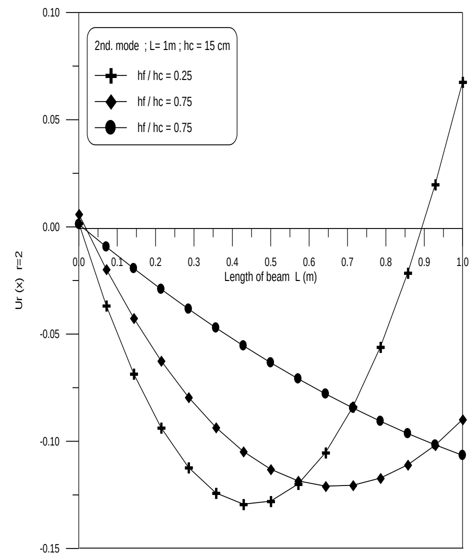

Fig.(14):Mode shapes associated with the second natural frequency of beam for different value of thickness ratio respect to one meter length at hc=15cm

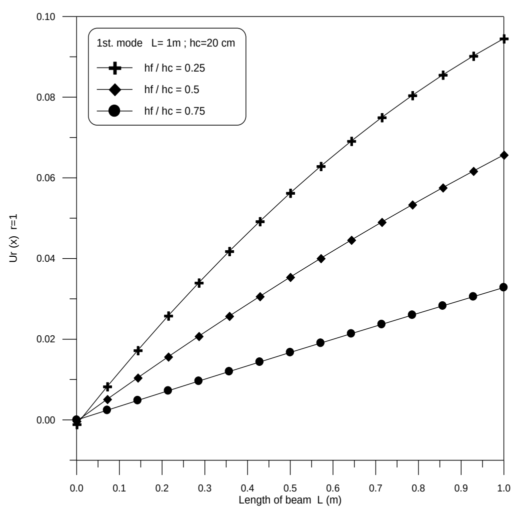

Fig.(15):Mode shapes associated with the first natural frequency of beam for different value of thickness ratio respect to one meter length at hc=20cm

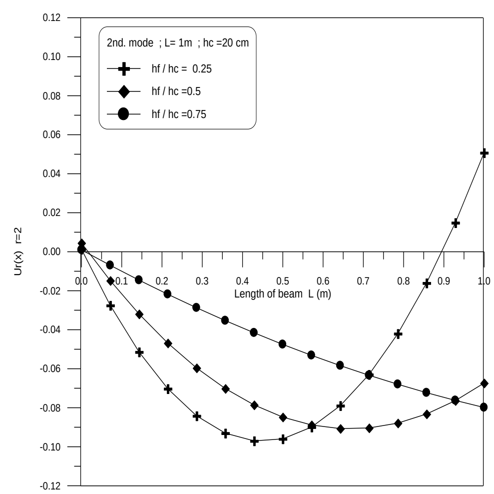

Fig.(16):Mode shapes associated with the second natural frequency of beam for different value of thickness ratio respect to one meter length at hc=20cm

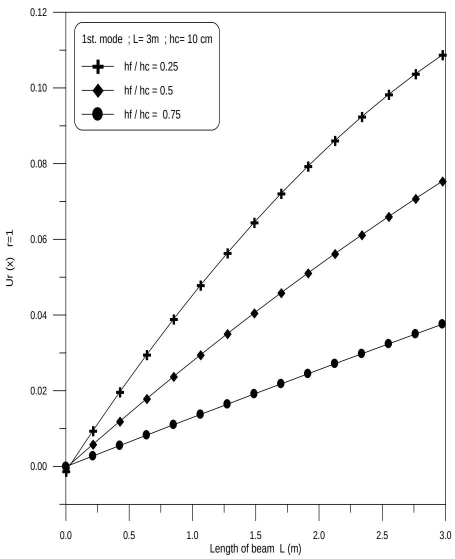

Fig.(17):Mode shapes associated with the first natural frequency of beam for different value of thickness ratio respect to three meter length at hc=10cm

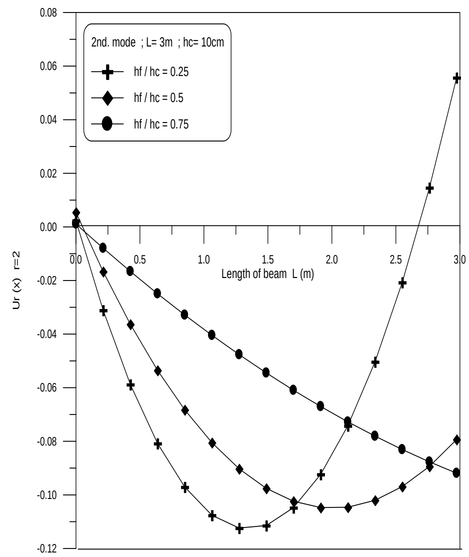

Fig.(18):Mode shapes associated with the second natural frequency of beam for different value of thickness ratio respect to three meter length at hc=10cm

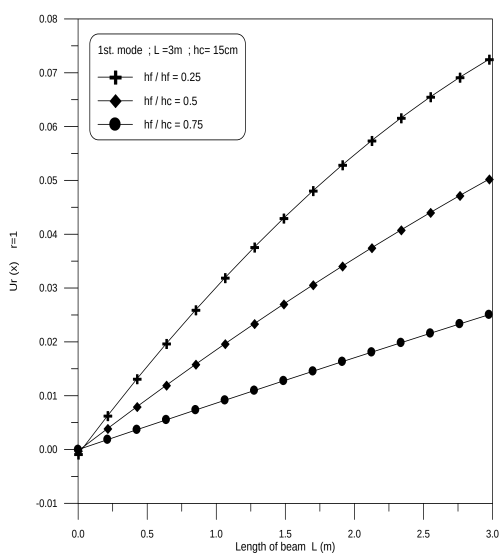

Fig.(19):Mode shapes associated with the first natural frequency of beam for different value of thickness ratio respect to three meter length at hc=15cm

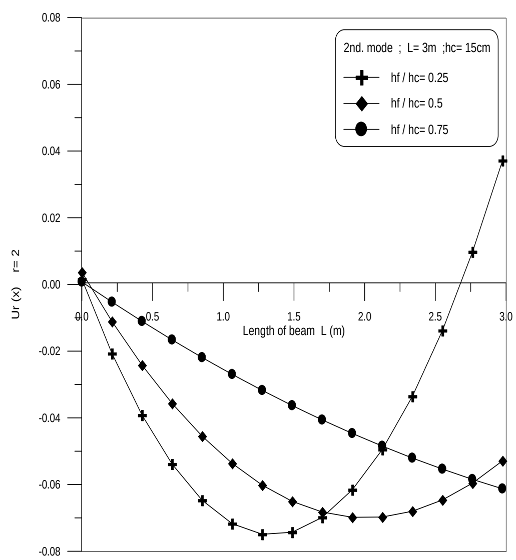

Fig.(20):Mode shapes associated with the second natural frequency of beam for different value of thickness ratio respect to three meter length at hc=15cm

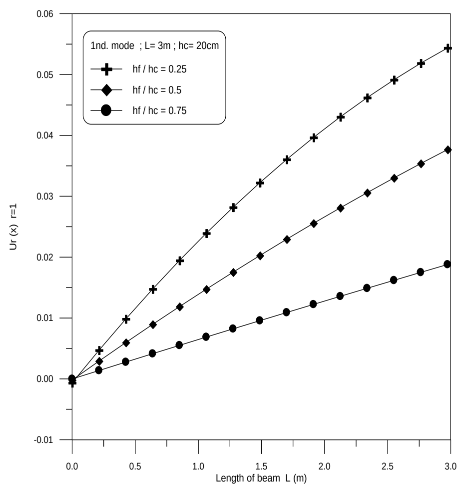

Fig.(21):Mode shapes associated with the first natural frequency of beam for different value of thickness ratio respect to three meter length at hc=20cm

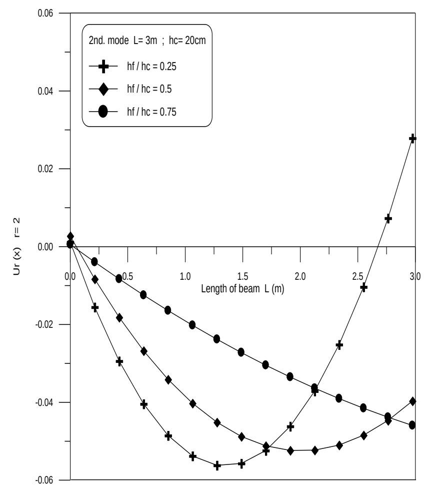

Fig.(22):Mode shapes associated with the second natural frequency of beam for different value of thickness ratio respect to three meter length at hc=20cm

5. References:

Auciello, "Transverse vibrations of a linearly tapered cantilever beam with tip mass at rotary inertia and eccentricity", Journal of Sound and Vibration, Vol. 194 (1), p.p. 25 – 34, 1996.

Benaroya, "Mechanical Vibration", Prentice – Hill, Inc., U.S.A., 1998.

Caruntu, "On non-axisymmetrical transverse vibrations of circular plates of convex parabolic thickness variation", Proceedings of ASME International Mechanical Engineering Congress and Exposition, Applied Mechanics Division, Vol. 225, pp. 335–340, 2004.

Caruntu, **D.I**.,. "Classical Jacobi polynomials, closed-form solutions for transverse vibrations", Journal of Sound and Vibration, Vol. 306 (3–5), p.p. 467–494, 2007.

D. Caruntu," On nonlinear vibration of nonuniform beam with rectangular cross section and parabolic thickness vibration" Journal of Solid Mechanics and its Applications, Vol. **73**, pp. 109-118, 2001

De Roza, "Free vibration of stepped beams with elastic ends", Journal of Sound and Vibration, Vol. 173 (4), p.p. 563–567, 1994.

Dumitru I. Caruntu, "Dynamic modal characteristics of transverse vibrations of cantilevers of parabolic thickness, Vol. 36, p.p.391–404, 2009.

Goel, R.P., "Transverse vibration of tapered beams", Journal of Sound and Vibration Vol. 47 (1), p.p.1–7, 1976.

Lavendelis and . Zakrhevsky, "On nonlinear vibration of nonuniform beam with rectangular cross section and parabolic thickness variation Solid Mechanics and its Applications, Vol.73, pp. 109-118,

Omer Civalek and Baki Ozturk, "Free vibration analysis of tapered beam-column withpinned ends embedded in Winkler-Pasternak elastic foundation", Vol. 2 (1) p.p.45-56, 2010.

Sanger, D.J, "Transverse vibration of a class of nonuniform beams", Journal of Mechanical Engineering Science, Vol. 16, p.p.111–120, 1968.

Storti, D., Aboelnaga, Y., "Bending vibrations of a class of rotating beams with hypergeometric solutions", Journal of Applied Mechanics Vol. 54, p.p.311–314, 1987.

Turner, J.A., Wiehn, J.S., "Sensitivity of flexural and torsional vibration modes of atomic force microscope cantilevers to surface stiffness variations nanotechnology", Vol. 12, p.p.322–330, 2001.

Warburton G. B. "The Dynamical Behavior of Shells", pergoman Press Ltd., England 1976.

Wright, A.D., Smith, C.E., Thresher, R.W., Wang, J.L.C., "Vibration modes of centrifugally stiffened beam", Journal of Applied Mechanics, Vol. 49, p.p.197–202, 1982.

Wylie, "Advanced engineering mathematics **"** 4th printing 1987**.**