Some Properties On Orlicz Sequence Λ_M and $\Lambda_M(\Delta)$ Spaces

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ABSTRACT :

This paper is devoted to the study of the general properties of Λ_M and $\Lambda_M(\Delta)$ Orlicz sequence space.

الخلاصة

-في هذا البحث قمنا بدراسة الخصائص العامة لفضاء متتابعات اورلسز Λ_M و $\Lambda_M(\Delta)$.

1-Introduction :

A complex sequence whose k^{th} term is x_k will denoted by (x_k) or simply x .A sequence $x = (x_k)$ is said to be analytic if $sup_k|x_k|^{\frac{1}{k}} < \infty$. The vector space of all analytic sequence will be denoted by Λ .A sequence $x = (x_k)$ is said to be entire if $\lim_{k\to\infty}|x_k|^{\frac{1}{k}} = 0$ The vector space of all entire sequence will be denoted by Γ.A sequence $x = (x_k)$ is said to be gai sequence if $\lim_{k\to\infty}(K!|x_k|^{\frac{1}{k}}) = 0$. The vector space of all gai sequence will be denoted by χ. Kizmaz[H.Kizmaz,(1981)]defined the following difference sequence spaces $Z(\Delta) = \{x = (x_k): \Delta x \in Z\}forZ = \ell_\infty, c, c_0$where $\Delta x = (\Delta x)_{k=1}^{\infty} = (x_k - x_{k+1})_{k=1}^{\infty}$ and showed that these are Banach spaces with norm $\|x\| = |x_1| + \|\Delta x\|_\infty$.

Orlicz [W.Orlicz,(1936)]used the idea of Orlicz function to construct the space (L^M) . J. Lindenstrass and L. Tzafriri [J.Lindenstrauss and L.Tzafriri, (1971)] investigated Orlicz sequence spaces in more detail , and they proved that every Orlicz sequence spaces ℓ_M contains a subspace isomorphic to ℓ_p $(1 \le p < \infty)$. Subsequently different classes of sequence spaces were defined by Parashar and Choudhary [S.D.Parashar and B.Choudhary,(1994)] , Mursaleen et al. [M.Mursaleen ,M.A.Khan and Qamaruddin,(1999)] , Bektas and

Altin [C.Bektas and Y.Altin,(2003)] , Tripathy et al. [B.C.Tripathy ,M.Etand Y.Altin ,(2003)], Rao and Subramanian [K.Chandrasekhara Rao and N.Subramanian , (2004)] , and many others . the Orlicz sequence spaces are the special cases of Orlicz spaces studied in [M.A.Krasnoselskii and Y.B.Rutickii, (1961)].

An Orlicz function is a function $M:[0,\infty) \to [0,\infty)$ which is continuous , nondecreasing , and convex with $M(0)=0, M(x)>0 \; for \; x>0$, $and \; M(x) \to \infty \; as \; x \to \infty$. if the convexity of Orlicz function M is replaced by $M(x+y) \le M(x)+M(y)$,then this function is called a modulus function , defined and discussed by Ruckle [W.H.Ruckle,(1973)] and Maddox [I.J.Maddox,(1986)] .
An Orlicz function M is said to satisfy the Δ_2- condition for all values of u if there exists a constant $K>0$ such that $M(2u) \le KM(u) \; (u \ge 0)$. the Δ_2- condition is equivalent is equivalent to $M(\ell u) \le K.\ell M(u)$, for all values of u and for $\ell > 1$ .Lindenstrass and L. Tzafriri [J.Lindenstrauss and L.Tzafriri, (1971)] used the idea of Orlicz function M to construct the sequence spaces ℓ_M of all sequence such that

$$\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, for \; some \; \rho > 0 \right\} \dots (1.1)$$

where ω = {all complex sequences}
The space ℓ_M becomes a Banach space with the norm.

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\} \dots (1.2)$$

which is called an Orlicz sequence space.

2-Basic Definition :-

In this section we introduce the definition of Orlicz sequence space Λ_M , normal function and the difference Orlicz space of analytic sequence that we need in all our work .

Definition(2.1):- [J.Lindenstrauss and L.Tzafriri, (1971)]
Let M be modulus function and x be any sequence in ω . Then the space which consisting of all sequences x in ω such that

$$sup_k \left(M\left(\frac{|x_k|^{\frac{1}{k}}}{\rho} \right) \right) < \infty \dots (2.1.1)$$

for some arbitrary fixed $\rho > 0$ .is denoted by Λ_M.
Definition(2.2):- [M.Mursaleen ,M.A.Khan and Qamaruddin,(1999)]
Let ω denote the set of all complex sequence $x = (x_k)_{K=1}^{\infty}$, $\Delta : \omega \to \omega$ be the difference operator defined by
$\Delta x = (x_k - x_{k+1})_{k=1}^{\infty}$, and $M:[0,\infty) \to [0,\infty)$ be an Orlicz function then $\Lambda_M(\Delta) = \{x \in \omega : \Delta x \in \Lambda_M\}$
Definition(2.3) :- [K.Chandrasekhara Rao and N.Subramanian , (2004)]

Let F be a sequence space and x, y be the arbitrary of F. Then F is called solid or normal if $(\propto_k \ x_k) \in F$ whenever $(x_k) \in F$ and for all sequences of scalars $(\propto_k)$ with $|\propto_k| \leq 1$

3-Main Results

In this section we introduce our main results .

Proposition(3.1):- if M is modulus function, then Λ_M is a linear set over the set of complex numbers $\mathbb{C}$.

Proof :- Let $x, y \in \Lambda_M$ and $\propto, \beta \in \mathbb{C}$.

Then there exist positive real numbers ρ_1, ρ_2 such that

$$sup_k \left(M\left(\frac{|x_k|^{\frac{1}{k}}}{\rho_1}\right)\right) < \infty \ldots (3.1.1)$$

$$sup_k M\left(\frac{|x_k|^{\frac{1}{k}}}{\rho_2}\right) < \infty \ldots (3.1.2)$$

Since M is a non decreasing and modulus function then we have

$$sup_k \left(M\left(\frac{|\alpha x_k + \beta y_k|^{\frac{1}{k}}}{\rho_3}\right)\right) \leq sup_k \left(M\left(\frac{|\alpha x_k|^{\frac{1}{k}} + |\beta y_k|^{\frac{1}{k}}}{\rho_3}\right)\right)$$

$$\leq sup_k \left(M\left(\frac{|\alpha|^{\frac{1}{k}} \, |x_k|^{\frac{1}{k}}}{\rho_3} + \frac{|\beta|^{\frac{1}{k}} \, |y_k|^{\frac{1}{k}}}{\rho_3}\right)\right)$$

$$\leq sup_k \left(M\left(\frac{(\alpha)|x_k|^{\frac{1}{k}}}{\rho_3} + \frac{(\beta)|y_k|^{\frac{1}{k}}}{\rho_3}\right)\right)$$

Let ρ_3 be any positive real numbers such that $\frac{1}{\rho_3} = \min\left\{\frac{1}{|\alpha|}\frac{1}{\rho_1}, \frac{1}{|\beta|}\frac{1}{\rho_2}\right\}$

$$sup_k \left(M\left(\frac{|\alpha x_k + \beta y_k|^{\frac{1}{k}}}{\rho_3}\right)\right) \leq sup_k \left(M\left(\frac{|x_k|^{\frac{1}{k}}}{\rho_1} + \frac{|y_k|^{\frac{1}{k}}}{\rho_2}\right)\right)$$

$$\leq sup_k \left[M\left(\frac{|x_k|^{\frac{1}{k}}}{\rho_1}\right) + M\left(\frac{|y_k|^{\frac{1}{k}}}{\rho_2}\right) \right]$$

$$\leq sup_k \left(M\left(\frac{|x_k|^{\frac{1}{k}}}{\rho_1}\right)\right) + sup_k \left(M\left(\frac{|y_k|^{\frac{1}{k}}}{\rho_2}\right)\right)$$

$$< \infty$$

By (3.1.1)and (3.1.2) we have $sup_k \left(M\left(\frac{|\alpha x_k+\beta y_k|^{\frac{1}{k}}}{\rho_3}\right)\right) < \infty$

So $(\alpha x_k + \beta y_k) \in \Lambda_M$. Therefore Λ_M is a linear space .

Proposition (3.2):-let M and M_1 be two Orlicz functions if M satisfies the Δ_2- condition then $\Lambda_{M_1} \subseteq \Lambda_{M \circ M_1}$

Proof :- let $x \in \Lambda_{M_1}$, then we have $sup_k \left(M_1\left(\frac{|x_k|^{\frac{1}{k}}}{\rho}\right)\right) < \infty$ for some $\rho > 0$

Since M is non decreasing and satisfies the $\Delta_2 -$ condition , so we have

$$sup_k \left(M\left(M_1\left(\frac{|x_k|^{\frac{1}{k}}}{\rho}\right)\right)\right) \leq C \; sup_k \left(M_1\left(\frac{|x_k|^{\frac{1}{k}}}{\rho}\right)\right) < \infty$$

Hence $x \in \Lambda_{M \circ M_1}$. therefore $\Lambda_{M_1} \subseteq \Lambda_{M \circ M_1}$

Proposition (3.3):- let M_1 and M_2 be two Orlicz functions , then $\Lambda_{M_1} \cap \Lambda_{M_2} \subseteq \Lambda_{M_1+M_2}$

Proof :- let $x \in \Lambda_{M_1} \cap \Lambda_{M_2}$ .Then there exist ρ_1 and ρ_2 such that

$$sup_k \left(M_1\left(\frac{|x_k|^{\frac{1}{k}}}{\rho_1}\right)\right) < \infty \ldots (3,3.1)$$

And $sup_k \left(M_2\left(\frac{|x_k|^{\frac{1}{k}}}{\rho_2}\right)\right) < \infty \ldots (3.3.2)$,Put $\rho = \min[\frac{1}{\rho_1}, \frac{1}{\rho_2}]$

Since M is non decreasing and modulus function then we have

$$sup_k\left((M_1+M_2)\left(\frac{|x_k|^{\frac{1}{k}}}{\rho}\right)\right)\leq sup_k\left(M_1\left(\frac{|x_k|^{\frac{1}{k}}}{\rho_1}\right)\right)+sup_k\left(M_2\left(\frac{|y_k|^{\frac{1}{k}}}{\rho_2}\right)\right)$$

By (3.3.1) and (3.3.2) we have $sup_k\left((M_1+M_2)\left(\frac{|x_k|^{\frac{1}{k}}}{\rho}\right)\right)<\infty$. So $x\in\Lambda_{M_1+M_2}$

Proposition (3.4):- $\Lambda_M(\Delta)$ is a complete metric space under the metric

$$d(x,y)=\inf\left\{\rho>0: sup_k\left(M\left(\frac{|\Delta x_k-\Delta y_k|^{\frac{1}{k}}}{\rho}\right)\right)\leq 1\right\}$$

Where $x=(x_k)\in\Lambda_M(\Delta)$ and $y=(y_k)\in\Lambda_M(\Delta)$

Proof :- let $(x^{(n)})$ be a Cauchy sequence in $\Lambda_M(\Delta)$

Then given any $\varepsilon>0$ there exists a positive integer N depending on ε such that

$d(x^{(n)},x^{(m)})<\varepsilon$ for all $n,m\geq N$. So that ,

$$d(x^{(n)},x^{(m)})=\inf\left\{\rho>0: sup_k\left(M\left(\frac{|\Delta x^{(n)}{}_k-\Delta x^{(m)}{}_k|^{\frac{1}{k}}}{\rho}\right)\right)\leq 1\right\}<\varepsilon,\forall n\geq N, m\geq N$$

Consequently, $\left\{M\left(\frac{|\Delta x^{(n)}{}_k|}{\rho}\right)\right\}$ is a Cauchy sequence in the metric $\mathbb{C}$ of complex numbers.

But $\mathbb{C}$ is complete , so there exist$(x_k)\in\mathbb{C}$ such that $\left\{M\left(\frac{|\Delta x^{(n)}{}_k|}{\rho}\right)\right\}\rightarrow\left\{M\left(\frac{|\Delta x_k|}{\rho}\right)\right\}$

as $n\rightarrow\infty$. Hence for any $\varepsilon>0$ there exists a positive integer $n_\circ$ such that

$$d(x^{(n)},x)=\inf\left\{\rho>0: sup_k\left(M\left(\frac{|\Delta x^{(n)}{}_k-\Delta x_k|^{\frac{1}{k}}}{\rho}\right)\right)\leq 1\right\}<\varepsilon, \text{ for all } n\geq n_\circ$$

In particular , we have $\left(M\left(\frac{|\Delta x^{(n)}{}_k-\Delta x_k|^{\frac{1}{k}}}{\rho}\right)\right)<\varepsilon$ ... (3.4.1)

Now , $sup_k\left(M\left(\frac{|\Delta x_k|^{\frac{1}{k}}}{\rho}\right)\right) \leq sup_k\left(M\left(\frac{|\Delta x^{(n)}{}_k - \Delta x_k|^{\frac{1}{k}}}{\rho}\right)\right) + sup_k\left(M\left(\frac{|\Delta x^{(n)}{}_k|^{\frac{1}{k}}}{\rho}\right)\right)$

Since $(x^{(n)}{}_k)$ is bounded and by (3.4.1) implies , $sup_k\left(M\left(\frac{|\Delta x_k|^{\frac{1}{k}}}{\rho}\right)\right) \in \Lambda_M(\Delta)$

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