Theorems on Lorentz Space

Eman Samir Bhaya **Nada Mohammed Abbas**
Department Of Mathematics ,College Of Education , Ibin-Hayan Babylon University.

Abstract

In this paper we introduce a generalization of the Lorentz space .And prove some theorems about it.

الخلاصة

قدمنا في هذا البحث توسيع لفضاء لورنز مع بعض المبرهنات حول هذا التوسيع .

1-Introduction

Let f be a complex-valued measurable function defined on a σ- finite measure space $(X, \mathcal{A}, \mu)$. For $s \geq 0$,define μf the distribution function of f as
$\mu f(s) = \mu\{x \in X: |f(x)| > s\}$. [**Arora *et al.*, 2007**]
By f^*we mean the non –increasing rearrangement of f given as
$f^*(t) = inf\{s > 0: \mu f(s) \leq t\}, \qquad t \geq 0$.
For $t > 0$, let

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds .$$

For a measurable function f on X, define

$$\|f\|_{pq} = \left\{\frac{q}{p}\int_0^\infty \left(t^{1/p} f^{**}(t)\right)^q \frac{dt}{t}\right\} \quad 0 < q, p < 1$$

The Lorentz space $L(p, q)$ consists of those complex – valued measurable functions f on X such that $\|f\|_{pq} < \infty$. For more on Lorentz space one can refer to [**Bennet and Sharpley1988; Hunt 1966; Lorentz 1950; Stein and Weiss,1971**].

Let $\mathrm{T}: X \to X$ be a measurable ($\mathrm{T}^{-1}(E) \in \mathcal{A}$,for $E \in \mathcal{A}$) non-singular transformation $\left(\mu\left(\mathrm{T}^{-1}(E)\right) = 0 \text{ whenever } \mu(E) = 0\right)$ and u a complex –valued measurable function defined on X.

We define a linear transformation $\mathcal{W} = \mathcal{W}_{u,\mathrm{T}}$ on the Lorentz space $L(p, q)$ into the linear space of all complex – valued measurable functions by
$\mathcal{W}_{u,\mathrm{T}}(f)(x) = u\big(\mathrm{T}(x)\big)f\big(\mathrm{T}(x)\big), x \in X, f \in L(p,q)$.

If $\mathcal{W}$ is bounded with range in $L(p, q)$, then it is called a *weighted composition operator on* $L(p, q)$. if u $\equiv 1$,then $\mathcal{W} \equiv C_{\mathrm{T}}: f \to f \circ \mathrm{T}$ is called *composition operator* induced by T . If T is identity mapping , then $\mathcal{W} \equiv M_u : f \to u \cdot f$, a *multiplication operator* induced by u .The study of these operators on L_p -spaces has been made in **[Chan1992; Jabbarzadeh and Pourreza 2003; Jabbarzadeh 2005 ; Singh and Manhas 1993 ; Takagi 1993**] and references there in .Composition and multiplication operators on the Lorentz spaces were studied in [**Kumar ,2005 , Arora *et al.*, 2006**] respectively .In this paper a characterization of the non – singular measurable transformations T from X into itself and complex –valued measurable function u on X inducing weighted composition operators is obtained on the Lorentz space $L(p, q)$, $0 < q, p < 1$.

2.Characterizations

In this section we introduce our main results .

Theorem 2.1. $\|\cdot\|_{pq}$ is a quasi-norm on $L(p,q)$ for $0 < q,p < 1$.

Proof :

(i) Assume $\|f\|_{pq}$=0 ,we must prove f=0 ,to show that it is sufficient to prove,

$\int_0^\infty \left(t^{1/p} f^{**}(t)\right)^q \frac{dt}{t} = 0$,where t > 0 ,

So we must show $f^{**}(t) = 0$, i.e $\frac{1}{t}\int_0^t f^*(s)\, ds = 0$

Since $\frac{1}{t} > 0$,hence it is remain to show $f^*(s) = 0$ or $inf\{s > 0: \mu f(s) \le t\} = 0$, which is clear since s > 0 .

Now , if f=0 ,so $\mu f(s) = \mu\{x \in X: 0 > s\}$,which contracts since s > 0 ,so there is no $x \in X$ such that $|f(x)| = 0 > s$, this leads to $\mu\{x \in X: 0 > s\} = 0$ and $inf\{s > 0: \mu f(s) = 0 \le t\} = 0$ so $f^{**}(t) = 0$ hence $\|f\|_{pq}$=0 .

(ii) since $|\alpha f(x)| = |\alpha||f(x)|$, so $\|\alpha f\|_{pq} = |\alpha|\|f\|_{pq}$

(iii) To prove the triangular inequality , we must prove

$(f+g)^{**} \le c(f^{**} + g^{**})$; $(f+g)^* \le c\,(f^* + g^*)$;

$inf\{s > 0: \mu(f+g)(s) \le t\} \le c\; inf\{s > 0: \mu f(s) \le t\} + inf\{s > 0: \mu g(s) \le t\}$

And $\mu(f+g)(s) \le c\big(\mu f(s) + \mu g(s)\big)$ .... (1)

Thus from the definitions of $\mu f(s)$, $f^*(t)$, $f^{**}(t)$ it is sufficient to prove (1).

We have

$\mu(f+g)(s) = \mu\{x \in X: |f(x) + g(x)| > s\}$

$\qquad = \mu\{x \in X: |f(x)| + |g(x)| > |f(x) + g(x)| > s = s_1 + s_2\}$

Where s_1 and $s_2 > 0$, and chosen so that for a given $\epsilon > 0$,

$\mu\{x \in X: |f(x)| + |g(x)| > s_1 + s_2\}$

$\qquad \le c_1\mu\{x \in X: |f(x)| > s_1\} + c_2\mu\{x \in X: |g(x)| > s_2\}$ ... (2)

And $\mu\{x \in X: |f(x)| > s_1\} \le c_3\; \mu\{x \in X: |f(x)| > s\} + \frac{\epsilon}{2}$ ... (3)

And $\mu\{x \in X: |g(x)| > s_2\} \le c_4 \mu\{x \in X: |g(x)| > s\} + \frac{\epsilon}{2}$ ... (4)

Which are true for any $\epsilon > 0$, so combining between (2) and((3) and (4)), to get (1) where c=max $\{c_1c_3, c_2c_4\}$

Theorem 2.2. let $(X, \mathcal{A}, \mu)$ be a σ- finite measure space and $u: X \to \mathbb{C}$ be a measurable function . let $\mathrm{T}: X \to X$ be a non-singular measurable transformation such that the Radon-Nikodym derivative $f_{\mathrm{T}} = d(\mu \mathrm{T}^{-1})/d\mu$ is in $L_\infty(\mu)$.

Then $\mathcal{W}_{u,\mathrm{T}}: f \to u \circ \mathrm{T} \cdot f \circ \mathrm{T}$ is bounded on $L(p,q)$, $0 < q,p < 1$ if u$\in L_\infty(\mu)$.

Proof: Suppose b=$\|f_{\mathrm{T}}\|_\infty$, then for f in $L(p,q)$, the distribution function of $\mathcal{W}f$ satisfies , where $\mathcal{W}f = \mathcal{W}_{u,\mathrm{T}} = u \circ \mathrm{T} \cdot f \circ \mathrm{T}$, we have

$(\mathcal{W}f)^{**}(t) \le \|u\|_\infty f^{**}\left({}^{t}/_{b}\right)$ ….(1) [**Arora *et al.*, 2007**]

Then for $0 < q,p < 1$ we have

$$\|\mathcal{W}f\|_{pq}^q = \frac{q}{p}\int_0^\infty \left(t^{\frac{1}{p}}\, ((\mathcal{W}f)^{**}(t))\right)^q \frac{dt}{t}$$

Then by using (1) we have

$$\|\mathcal{W}f\|_{pq}^{q} \leq \|u\|_{\infty}^{q} \, \frac{q}{p} \int_{0}^{\infty} \left(t^{\frac{1}{p}} f^{**}\left(\frac{t}{b}\right) \right)^{q} \frac{dt}{t}$$

$$= \|u\|_{\infty}^{q} \, \frac{q}{p} \int_{0}^{b\infty} \left((bt)^{\frac{1}{p}} f^{**}(t) \right)^{q} \frac{b\,dt}{bt}$$

$$\leq \|u\|_{\infty}^{q} \; b^{\frac{q}{p}} \; \|f\|_{pq}^{q}$$

Thus

$$\|\mathcal{W}\|_{pq} \leq b^{\frac{1}{p}} \, \|u\|_{\infty}$$

References

Bennet.C and Sharpley R,1988,Interpolation of Operators ,Pure and Applied Mathematics ,129.Academic Press, Inc., Boston, MA.

Stein.E.M and Weiss .G,1971,Introduction to Fourier analysis on Euclidean spaces, Princeton Mathematical Series, no.32. Princeton University Press , Princeton ,N.J.

Lorentz. G.G, 1950, Some new functional spaces , Ann. of Math , (2), 37-55.

Takagi . H,1993,Fredholm Weighted Composition Operators ,Integral Equations Operator Theory , no.2,267-276.

Chan .J.T,1992,Anote on Compact Weighted Composition Operators on L^p ,Acta Sci.Math.no.1-2,165-168.

Jabbarzadeh . M.R,2005,Weighted Composition Operators between L^p - spaces Bull. Korean Math .Soc.no.2,369-378.

.Jabbarzadeh. M.R and Pourreza. E,2003,Anote on Weighted Composition Operators on L^p- spaces, Bull. Iranian Math.Soc.no.1,47-54.

Hunt. R.A,1966 ,on $L(p,q)$ spaces , Ensignement Math. (2), 249-276 .

Singh. R.K andManhas .J.S,1993, Composition Operators on function spaces ,North-Holland Mathematics Studies,179.North –Holland Publishing Co.,Amsterdam.

Kumar.R,2005, Composition Operators on Banach function spaces,Proc.Amer.Math.Soc.no.7,2109-2118.

Arora. S.C, Dat G. t and Verma S , 2006 , Multiplication operators on Lorentz spaces , Indian J. Math . 3 , 317-329 .

Arora. S.C, Datt .G and Verma. S, 2007 , Weighted composition operators on Lorentz spaces , Bull. Korean. Math. Soc. no. 4 , 701-708 .