P-F Fuzzy Rings and Normal Fuzzy Ring(II)

Areeg T.H. Abou –Dareb
Department of Mathematics, College of Education for Girls, University of Kufa

Abstract

In 1965, (Zadeh) introduced the concept of fuzzy subset. Since that time many papers were introduced in different mathematical scopes of theoretical and practical applications.

In 1982, (Liu W.J.) formulated the term of fuzzy ring and fuzzy ideal of a ring R.

In 2004 ,(Hadi and Abou-Draeb, 2004**]** introduce and study P-F fuzzy rings and normal fuzzy rings and now we are complete it .

In this paper, the concepts P-F fuzzy rings and normal fuzzy rings have been investigated .Several basic results related to these concepts have given and studied . The relationship between them has also been given . Moreover, some properties of t-pure fuzzy ideal of a fuzzy ring have been given which are need it later .

الخلاصة

في عام ١٩٦٥ قدم (Zadeh) مفهوم المجموعات الضبابية(fuzzy sets) ومنذ ذلك الحين اجريت العديد من البحوث في مختلف المجالات الرياضية النظرية والتطبيقية حول هذا الموضوع.

وفي عام ١٩٨٢ صاغ ليو (Liu) مصطلح الحلقة الضبابية (fuzzy ring) وعرَّفها وعرَّف كذلك مفهوم المثالي الضبابي (fuzzy ideal).

وفي عام ٢٠٠٤ عرف ابو درب (Hadi and Abou–Dareb) مصطلح المثالي قدمنا مفهومي الحلقات الضبابية من النمط P–F والحلقات الضبابية السوية .

في هذا البحث قدمنا مفهومي الحلقات الضبابية من النمط P–F والحلقات الضبابية السوية بشكل اوسع ، وقد تم دراسة واعطاء العديد من الخواص الاساسية المتعلقة بهذين المفهومين وكذلك دراسة العلاقة بينهما .

وايضا دراسة العديد من الخواص الخاصة بالمثاليات الضبابية النقية في الحلقة الضبابية والتي نحتاجها .

Introduction

The present paper study P-F fuzzy rings and normal fuzzy rings .

In section one , some basic definitions and results are recalled which will be needed later .

In section two , several results about t-pure fuzzy ideals of a fuzzy ring ,are given which are necessary in proving some results in the following sections.

Section three is devoted for studying P-F fuzzy rings ,where a fuzzy ring X is called P-F fuzzy ring if F-ann(a_t) is a t-pure fuzzy ideal of X , for all $a_t \subseteq X$.

Next ,in the fourth section ,normal fuzzy rings are introduced , where a fuzzy ring X is called normal if every fuzzy ideal of X is t-pure .Many properties about this concept ,are given .Its relationship with P-F fuzzy ring is also given .

Throughout this paper R is commutative ring with unity ,and every fuzzy ideal A of a fuzzy ring X is a finite valued ,that is Im(A) is finite set . Finally , A(0) =X(0) ,for any fuzzy ideal A of a fuzzy ring X .

S.1 Basic Concepts:

This section contains some definitions and properties of fuzzy subset, fuzzy ring, fuzzy ideal which we will be used in the next section .

Preliminary Concepts

Let (R,+,·) be a commutative ring with identity. A **fuzzy subset of R** is a function from R into [0,1], ([Zadeh ,1965 ; Liu, 1982]).

Let A and B be fuzzy subset of R. We write $\mathbf{A \subseteq B}$ if $A(x) \leq B(x)$, for all $x \in R$. If $A \subseteq B$ and there exists $x \in R$ such that $A(x) < B(x)$, then we write $\mathbf{A \subset B}$ and we say that A is a proper fuzzy subset of B , [Liu, 1982]. Note that **A = B** if and only if A(x) = B(x), for all $x \in R$, [Zadeh, 1965].

Let R ,R' be any sets and f: R→ R' be any function ,A and B be two fuzzy subsets of R and R' respectively ,the fuzzy subset f(A) of R' defined by : f(A)(y) = sup A(y) if f(y) ≠ 0 , y∈ R' and f(A)(y) = 0 , otherwise .It is called **the image of A under f** and denoted by f(A) . The fuzzy subset f^{-1} (B) of R defined by : f^{-1} (B)(y) = B(f(x)) , for all x ∈R .Is called **the inverse image of B** and denoted by f^{-1} (B) , [Liu, 1982].

Let R ,R' be any sets and f : R→ R' be any function .A fuzzy subset A of R is called **f-invariant** if f(x) = f(y) implies A(x) =A(y), where x,y ∈R, [Dixit .,Kumar R. and Ajmal, 1991] .

For each $t \in [0,1]$, the set $\mathbf{A_t = \{ x \in R \mid A(x) \geq t \}}$ is called **a level subset of R** and the set $\mathbf{A_* = \{x \in R \mid A(x) = A(0) \}}$, and A=B if and only if $\mathbf{A_t = B_t}$ for all $t \in [0,1]$ [Liu, 1982; AL- Khamees and Mordeson, 1998].

Let λ_R denote the characteristic function of R defined by λ_R (x) =1 if $x \in R$ and λ_R (x) = 0 if $x \notin R$, [Dixit *et al.*, 1991; AL- Khamees and Mordeson, 1998] .

Let $x \in R$ and $t \in [0,1]$, let x_t denote the fuzzy subset of R defined by $x_t(y)$ =0 if x ≠ y and $x_t(y)$ = t if x = y for all $y \in R$. x_t is called **a fuzzy singleton**, [AL- Khamees and Mordeson, 1998] .

If x_t and y_s are fuzzy singletons, then $x_t + y_s = (x + y)_\lambda$ and $x_t \circ y_s = (x \,.\, y)_\lambda$, where λ = min { t ,s }, ([Zadeh, 1965; AL- Khamees and Mordeson, 1998).

Let $I^R = \{A_i \mid i \in \Lambda\}$ be a collection of fuzzy subset of R . Define the fuzzy subset of R (**intersection**) by $(\bigcap_{i\in\Lambda} A_i)$ (x) = inf { $A_i(x) \mid i \in \Lambda$ }, for all $x \in R$,([Liu W.J., 1982] ,[4]). Define the fuzzy subset of R (**union**) by $(\bigcup_{i\in\wedge} A_i)$ (x) = sup { $A_i(x) \mid i \in \Lambda$ }, for all $x \in R$,([Liu, 1982; AL- Khamees and Mordeson , 1998].

Let ϕ denote ϕ(x) = 0 for all $x \in R$, **the empty fuzzy subset** of R,([Zadeh, 1965; Martines, 1995].

Note that throughout our work any fuzzy subset is a nonempty fuzzy subset.

Let A and B be a fuzzy subsets of R , **the product A∘B** define by : A∘B (x) = sup {min{A(y), B(z) }| x = y · z} y, z ∈ R , for all x ∈ R , [Mukherjee and Sen, 1987].

The addition A + B define by (A + B) (x) = sup{min {A(y), B(z) | x = y + z} y, z ∈ R , for all x ∈ R , [Mukherjee and Sen, 1987] .

Let A be a fuzzy subset of R, A is called **a fuzzy subgroup** of R if for all x, y ∈ R, A(x + y) ≥ min {A(x), A(y)} and A(x) = A(-x), [Mukherjee and Sen, 1987].

Let A be a fuzzy subset of R , A is called **a fuzzy ring of R** if for all x, y ∈ R , A(x – y) ≥ min {A(x), A(y)} and A(x · y) ≥ min {A(x), A(y)},[Martines,1995; Mukherjee and Sen, 1987].

A fuzzy subset A of R is called **a fuzzy ideal of R** if and only if for all x, y∈ R , A(x - y) ≥ min {A(x), A(y)} and A(x · y) ≥ max {A(x), A(y)}, [Martines, 1995; Mukherjee and Sen, 1987] .

Let X be a fuzzy ring of R and A be a fuzzy ideal of R such that A ⊆ X. Then A is **a fuzzy ideal of the fuzzy ring X** [AL- Khamees and Mordeson, 1998].

But let X be a fuzzy ring of R and A be a fuzzy subset . A is called **a fuzzy ideal of the fuzzy ring X** if A ⊆ X (that is A(a) ≤ X(a), for all a∈ R) , A(b-c) ≥min {A(b) , A(c)} and A(bc) ≥min {A(b),A(c)},for all b ,c∈ R, [Martines, 1995].

And A is **a fuzzy ideal of the fuzzy ring X** of R if A(b-c) ≥min {A(b),A(c)} and A(bc) ≥min {max{A(b),A(c)},X(bc)} ,[Dixit *et al.*, 1991;Martines, 1995].

Let X be a fuzzy ring of R . A be a fuzzy subset of X is a fuzzy ideal of X if and only if A_t is an ideal of X_t ,for all $t \in [0,A(0)]$, [Martines, 1995].

Let A be a fuzzy ideal of R. If for all $t \in [0, A(0)]$, then A_t is an ideal of R and A_* is an ideal of R , [Dixit *et al.*, 1991; AL- Khamees and Mordeson, 1998].

Let $\{A_i \mid i \in \Lambda\}$ be a family of fuzzy ideals of R. Then $\bigcap_{i\in\Lambda} A_i$ is a fuzzy ideal of R and $(\bigcup_{i\in\wedge} A_i)$is a fuzzy ideal of R,[Martines, 1995; AL- Khamees and Mordeson, 1998].

Let A and B are fuzzy ideals of R ,then $A\circ B$ is a fuzzy ideal of R, [Martines, 1995] . Let A and B are fuzzy ideals of R, then $A \cap B$, $A + B$ are fuzzy ideals of R , [Martinez , 1999; Abou-Draeb, 2000].

Now , we give definition and some result about the fuzzy direct sum of X and Y as a definition of the direct sum by [Kash, 1982] .

Definition 1 .1 [Abou-Draeb, 2000 **] :**

Let X be a fuzzy ring of a ring R_1 and Y be a fuzzy ring of a ring R_2 . Let f: $R_1 \oplus R_2 \to [0,1]$ definite by $T(a ,b) = \min \{X(a),Y(b)\}$ for all $(a ,b) \in R_1\oplus R_2$. T is called **a fuzzy external direct sum of denoted by $X \oplus Y=T$.**

PROPOSITION 1. 2 [Abou-Draeb, 2000**] :**

If X and Y are fuzzy rings of rings R_1 and R_2 respectively ,then $T = X \oplus Y$ is a fuzzy ring of $R_1 \oplus R_2$.

Proposition 1. 3 [Abou-Draeb, 2000**] :**

If X and Y are fuzzy rings of R_1 and R_2 respectively ,then $T = X \oplus Y$ is a fuzzy ring of $R_1 \oplus R_2$. A be a fuzzy ideal of R_1 such that $A \subseteq X$ and B be a fuzzy ideal of R_2 such that $B \subseteq Y$ .Then :

1- $A \oplus B$ is a fuzzy ideal of $R_1 \oplus R_2$.

2- $(A \oplus B)_t = A_t \oplus B_t$, for all $t\in (0, (A \oplus B) (0 ,0)]$.

3- $A \oplus B$ is a fuzzy ideal of T , such that $(A \oplus B) \subseteq (X \oplus Y)= T$, where $(A \oplus B) (a ,b) = \min \{A(a),B(b)\}$, for all $(a ,b) \in R_1\oplus R_2$.

Proposition 1. 4 [Abou-Draeb, 2000**] :**

If X and Y are fuzzy rings of R_1 and R_2 respectively such that $T = X \oplus Y$. If A is a fuzzy ideal of $R_1 \oplus R_2$ such that $A \subseteq T$, then there exist B_1 and B_2 fuzzy ideals of R_1 and R_2 respectively such that $B_1 \subseteq X$ and $B_2 \subseteq Y$ and $A = B_1\oplus B_2$.

Corollary 1. 5 [Abou-Draeb, 2000**] :**

If X and Y are fuzzy rings of R_1 and R_2 respectively. If A ,C are fuzzy ideals of X and B , D are fuzzy ideals of Y .Then $A \oplus B= C \oplus D$ if and only if A=C , B =D .

Proposition 1. 6 :

Let A and B be two fuzzy subsets of a ring R .Then :

1- $A\circ B \subseteq A\cap B$.
2- $(A\circ B)_t = A_t\cdot B_t$, $t\in [0,1]$.
3- $(A\cap B)_t = A_t \cap B_t$, $t\in [0,1]$.

Proof :

1-To prove $A\circ B \subseteq A\cap B$, let $x \in R$,

$(A\circ B) (x) = \sup\{\min \{A(y),B(z)\}| x=y . z \}$ $y , z\in R$

$= \sup\{\min \{A(x),B(1)\},\min \{B(x),A(1)\},\{ \min \{A(y),B(z)\}| x=y .z \}\}$.

$\leq \min \{\max \{A(x),B(x)\},\max \{A(1),B(1)\}, \min \{A(y),B(z)\}| x=y .z\}\}$ by [Abou-Draeb, 2000] .

$\leq \min \{\min\{A(x),B(x)\},\min \{A(1),B(1)\}, \min \{A(y),B(z)\}| x=y .z\}$

$\leq \min\{A(x),B(x)\}$
$=(A\cap B)(x)$. Hence $A\circ B \subseteq A\cap B$.

2- $(A\circ B)_t = A_t \cdot B_t$, $t\in [0,1]$,[Hadi and Abou-Draeb, 2004] .

3- To prove $(A\cap B)_t = A_t \cap B_t$, $t\in [0,1]$, since $A_t = \{x : x \in R \mid A(x) \geq t\}$, $B_t = \{x : x \in R \mid B(x) \geq t\}$.

$(A \cap B)_t = \{x : x \in R \mid (A \cap B)(x) \geq t\}$.
$= \{x : x \in R \mid \min\{A(x),B(x)\} \geq t\}$.
$= \{x : x \in R \mid A(x) \geq t, B(x) \geq t\}$.
$= \min\{\{x : x \in R \mid A(x) \geq t\},\{x : x \in R \mid B(x) \geq t\}\}$.
$= A_t \cap B_t$. Hence $(A\cap B)_t = A_t \cap B_t$, $t \in [0,1]$.

Proposition 1.7 [Martines, 1995**] :**

Let $X : R \to [0,1]$, $Y : R' \to [0,1]$ are fuzzy rings $f : R \to R'$ be homomorphism between them and $A : R \to [0,1]$ a fuzzy ideal of X , $B : R' \to [0,1]$ a fuzzy ideal of Y, then :

1. f (A) is a fuzzy ideal of Y. 2. $f^{-1}(B)$ is a fuzzy ideal of X.

Proposition 1.8:

Let A and B be two fuzzy subsets of a ring R and $f : R \to R'$ is inverse image function of B .Then :

1- $f(A)\cap f(B)= f(A\cap B)$. 2- $f(A) \circ f(B)= f(A\circ B)$.
3- $f(A_t) = (f(A))_t$. 4- $f^{-1}(A_t) = (f^{-1}(A))_t$.

Proof:

1-To prove $f(A)\cap f(B)= f(A\cap B)$, let $x \in R$, $f(A\cap B)(x) = f[(A\cap B)(x)]$
$= f[\min\{A(x), B(x)\}] = \min\{f(A(x)), f(B(x))\} = f(A)\cap f(B)$.

2- To prove $f(A) \circ f(B)= f(A\circ B)$, let $x \in R$, $f(A\circ B)(x) = f[(A\circ B)(x)]$
$= f[\sup\{\min\{A(y),B(z)\} \mid x=y \cdot z\}]$ $y , z\in R$
$= \sup\{\min\{f(A)(y), f(B)(z)\} \mid x=y \cdot z\} = [f(A) \circ f(B)](x)$ by [Martines, 1995].

Hence, $f(A) \circ f(B)= f(A\circ B)$.

Note that $A(f^{-1}(x)) = f(A(x))$ and $A(f(x)) = f^{-1}(A(x))$ for each $x\in R$ since f is inverse image function .

3- To prove $f(A_t) = (f(A))_t$, let $x \in f(A_t)$ if and only if $f^{-1}(x) \in A_t$ if and only if $A(f^{-1}(x)) \geq t$ if and only if $f(A(x)) \geq t$ if and only if $x \in (f(A))_t$.

4- To prove $f^{-1}(A_t) = (f^{-1}(A))_t$, let $x \in f^{-1}(A_t)$ if and only if $f(x) \in A_t$ if and only if $A(f(x)) \geq t$ if and only if $f^{-1}(A(x)) \geq t$ if and only if $x \in (f^{-1}(A))_t$.

We give definition and some properties of quatient fuzzy ring.

Definition 1.9 [Abou-Draeb, 2000**]:**

Let X be a fuzzy ring of R and A is a fuzzy ideal in X.

Define $X / A : R / A_* \to [0,1]$ such that :

$$X / A(a + A_*) = \begin{cases} 1 & \text{if } a \in A_* \\ \sup\{X(a+b)\} & \text{if } a \notin A_*, b \in A_* \end{cases}$$

For all $a + A_* \in R / A_*$, X / A is called **a quatient fuzzy ring of X by A**.

Proposition 1.10 [Abou-Draeb, 2000**]:**

Let X be a fuzzy ring of R and A is a fuzzy ideal in X, then :
X / A is a fuzzy ring of R / A_* .

Proposition 1.11 [Abou-Draeb, 2000**]:**

Let X be a fuzzy ring of R and A is a fuzzy ideal in X, then $X / A (0 +A_*)=1$, $[0 + A_* = A_*]$.

Proposition 1.12 [Abou-Draeb, 2000**]:**

Let X is a fuzzy ring of R . A and B are fuzzy ideals in X such that A$\subseteq$ B .Then B / A : R / A_* $\rightarrow$ [0,1] such that :

$$B/A(a+A_*)=\begin{cases} 1 & \text{if } a\in A_* \\ \sup\{B(a+b)\} & \text{if } a\notin A_*, b\in A_* \end{cases}$$

For all a + A_* $\in$ R / A_*, B / A is a fuzzy ideal in X / A .

Proposition 1.13:

Let X be a fuzzy ring of R and A be a fuzzy ideal of X .Then :

1- $x_t + A_* = (x + A_*)_t$, $x_t \subseteq X$.

2- $a_t + A_* \subseteq b_k + A_*$ if and only if $a_t \subseteq b_k$, a_t, $b_k \subseteq X$.

Proof:

1- Since $A_t = \{x : x \in R \mid A(x) \geq t\}$ and $A_* = \{x : x \in R \mid A(x) = A(0)\}$, implies that $(A_*)_t = \{x : x \in R \mid A(x) = A(0) \geq t\} = \{x : x \in R \mid A(x) = A(0)\} = A_*$.

Then $x_t + A_* = x_t + (A_*)_t$.

2- It is easy .

Proposition 1.14:

Let X be a fuzzy ring of R and A, B, C be fuzzy ideals of X . Then :

1- (A$\cap$ B) / C = (A / C) $\cap$ (B / C) .

2- (A $\circ$B) / C = (A / C) $\circ$ (B / C) .

Proof:

1- Let $a+C_*\in R/C_*$,

$$[(A\cap B)/C](a+C_*)=\begin{cases} 1 & \text{if } a\in C_* \\ \sup\{A\cap B(a+b)\} & \text{if } a\notin C_*, b\in C_* \end{cases}$$

$$=\begin{cases} 1 & \text{if } a\in C_* \\ \sup\{\min A(a+b), B(a+b)\} & \text{if } a\notin C_*, b\in C_* \end{cases} \quad ---(1) .$$

$$[(A/C)\cap(B/C)](a+C_*)=\min\{A/C(a+C_*), B/C(a+C_*)\}$$

$$=\begin{cases} 1 & \text{if } a\in C_* \\ \min\{\sup\{A(a+b), B(a+b)\} & \text{if } a\notin C_*, b\in C_* \end{cases}$$

$$=\begin{cases} 1 & \text{if } a\in C_* \\ \sup\{\min A(a+b), B(a+b)\} & \text{if } a\notin C_*, b\in C_* \end{cases} \quad --(2)$$

From (1) and (2) ,(A$\cap$B) / C =(A/ C)$\cap$ (B/C) .

2- Let a + C_* $\in$ R / C_*,

$$[(A\circ B)/C](a+C_*)=\begin{cases} 1 & \text{if } a\in C_* \\ \sup\{A\circ B(a+b)\} & \text{if } a\notin C_*, b\in C_* \end{cases} \quad ---(1)$$

[(A / C) $\circ$(B/ C)] (a + C_*) = sup {inf {min { A / C(d_i + C_*) , B/ C (e_i + C_*)} | d_i e_i $\in$ R}| a + C_*= $\sum_{i=1}^{n}$ (d_i + C_*)(e_i + C_*)} .

[(A / C) $\circ$(B/ C)] (a + C_*) = sup {inf {min { A / C(d_i + C_*) , B/ C (e_i + C_*)} | d_i e_i $\in$ R}| a + C_*= $\sum_{i=1}^{n}$ (d_i e_i + C_*)} .

But $(A / C)(d_i + C_*) = \begin{cases} 1 & if\ d_i \in C_* \\ \sup\{\min A(d_i + b)\} & if\ d_i \notin C_*, b \in C_* \end{cases}$

$(B / C)(e_i + C_*) = \begin{cases} 1 & if\ e_i \in C_* \\ \sup\{\min B(e_i + b)\} & if\ e_i \notin C_*, b \in C_* \end{cases}$

Thus $[(A / C) \circ (B / C)](a+C_*) =$

$$\begin{cases} 1 & if\ d_i \in C_*\ or\ e_i \in C_* \\ \sup\{\inf\{\min\{\sup\{A(d_i+b)\},\sup\{B(e_i+b)\}\}\}\} & if\ d_i\ and\ e_i \notin C_*, b \in C_* \end{cases}$$

$$= \begin{cases} 1 & if\ d_i \in C_*\ or\ e_i \in C_* \\ \sup\{\inf\{\min\{\sup\{A(d_i+b)\},\sup\{B(e_i+b)\}\}\}\} & if\ d_i\ and\ e_i \notin C_*, b \in C_*, a = \sum_{i=1}^{n} d_i e_i \end{cases}$$

$$= \begin{cases} 1 & if\ d_i \in C_*\ or\ e_i \in C_* \\ \sup\{A \circ B(\sum_{i=1}^{n}(d_i+b)(e_i+b))\}\}\} & if\ d_i\ and\ e_i \notin C_*, b \in C_*\ a = \sum_{i=1}^{n} d_i e_i \end{cases}$$

$$= \begin{cases} 1 & if a \in C_* \\ \sup\{A \circ B[\sum_{i=1}^{n} d_i e_i + b\sum_{i=1}^{n} d_i + b\sum_{i=1}^{n} e_i + b^2]\} & if\ a \notin C_*, b \in C_* \end{cases}$$

$$= \begin{cases} 1 & if\ a \in C_* \\ \sup\{A \circ B(a + b_1)\} & if\ a \notin C_*, b_1 \in C_* \end{cases} \quad , b_1 = b\,\Sigma_{i=1} d_i + b\,\Sigma_{i=1} e_i + b^2 \text{---(2)}$$

From (1) and (2) , $(A \circ B) / C = (A / B) \circ (B / C)$.

Proposition 1.15 [Abou-Draeb, 2010**] :**

Let X be a fuzzy ring of R and A, B, C be fuzzy ideals of X . Then B / A = C / A if and only if B = C .

S.2 T-Pure Fuzzy Ideals:

Recall that an ideal I of a ring R is called a t-pure ideal if for each ideal J of R ,I∩J = I·J ,[Kash F. , 1982] .

A fuzzy ideal A of R is called t-pure fuzzy ideal of R if for each fuzzy ideal B of R , A ∩ B= A o B ,[Hadi and Abou-Draeb, 2004] .

We shall give the definition and some properties for t-pure fuzzy ideals which will be needed later .

Lemma 2.1 [Hadi and Abou-Draeb, 2004**]:**

Let X be a fuzzy ring of R and A ,B are fuzzy ideals of X . Then : for all t ∈ [0,A(0)] ,

1- $(A \text{ o } B)_t = A_t \cdot B_t$. 2- $(A \cap B)_t = A_t \cap B_t$.

definition 2.2 [Hadi and Abou-Draeb, 2004]:

If X is a fuzzy ring of a ring R ,a fuzzy ideal A of X is called a **t-pure fuzzy ideal of X** if for each fuzzy ideal B of X , $A \cap B = A \text{ o } B$.

Proposition 2. 3 [Hadi and Abou-Draeb, 2004]:

Let X be a fuzzy ring of R .Then $0_{X(0)}$ is a t-pure fuzzy ideal of X .

Proposition 2.4:

Let A be a fuzzy ideal of R .Then A is a t-pure fuzzy ideal if and only if A_t is a t-pure ideal of R ,for all $t \in (0,A(0)]$.

Proof:

Suppose A is a t-pure fuzzy ideal of R .To prove A_t is a t-pure ideal of R ,for all $t \in (0,A(0)]$.

We must show that $A_t \cap I = A_t \cdot I$,for any ideal I of R .

Let $B : R \to [0,1]$ defined by B(x)=1 if $x \in I$,B(x)=0 otherwise .

It can be easily shown that B is a fuzzy ideal of R and $I = (B)_t$, for all $t \in (0,A(0)]$. However $A \cap B = A \text{ o } B$, since A is a t-pure fuzzy ideal of R .

Hence $(A \cap B)_t = (A \text{ o } B)_t$, for all $t \in (0,A(0)]$. But $(A \cap B)_t = A_t \cap B_t = A_t \cap I$ and $(A \text{ o } B)_t = A_t \cdot B_t = A_t \cdot I$, by lemma (2.1) . Then $A_t \cap I = A_t \cdot I$.

Thus $A_t \cap I = A_t \cdot I$ and so A_t is a t-pure ideal of R ,for all $t \in (0,A(0)]$.

Conversely ,if A_t is a t-pure ideal of R ,for all $t \in (0,A(0)]$ .To prove A is a t-pure fuzzy ideal of R .

Let B be any fuzzy ideal of R , then B_t is a ideal of R , for all $t \in (0,A(0)]$ and so $A_t \cap B_t = A_t \cdot B_t$. Hence $(A \cap B)_t = A_t \cap B_t$ and $A_t \cdot B_t = (A \text{ o } B)_t$, for all $t \in (0,A(0)]$,by lemma (2.1) .

Thus $(A \cap B)_t = (A \text{ o } B)_t$, for all $t \in (0,A(0)]$, which implies $A \cap B = A \text{ o } B$. Therefore A is a t-pure fuzzy ideal of R .

Corollary 2. 5 :

Let I be an ideal of R .Then I is a t-pure ideal if and only if B is a t-pure fuzzy ideal of R where B(x)=1 if $x \in I$ and B(x)=0 otherwise .

Proof :

It follows directly by proposition (2.4) and in fact $B_t = I$, for all $t \in [0,B(0)]$.

Proposition 2.6 :

Let X be a fuzzy ring of R ,A be a fuzzy ideal of X .Then A is a t-pure fuzzy ideal of X if and only if A_t is a t-pure ideal of X_t ,for all $t \in (0,X(0)]$.

Proof :

The proof is similarly .

Proposition 2. 7 [Hadi and Abou-Draeb, 2004]:

Let X be a fuzzy ring of R such that X(0) = 1,for all $a \in R$ and let I be ideal of R .I is t-pure ideal of R if and only if $A(x) = \begin{cases} 1 & \text{if } x \in I \\ c & \text{otherwise} \end{cases}$ is a t-pure fuzzy ideal of X ,when $c \in (0,1]$.

Proposition 2. 8 [Hadi and Abou-Draeb, 2004]:

Let X and Y be fuzzy rings of R_1 and R_2 respectively and A and C be fuzzy ideals of X and B and D be fuzzy ideals of Y respectively .Then:

1- $(A \oplus B) \cap (C \oplus D) = (A \cap C) \oplus (B \cap D)$.

2- $(A \oplus B) \text{ o } (C \oplus D) = (A \text{ o } C) \oplus (B \text{ o } D)$.

Proposition 2. 9:

Let X and Y be fuzzy rings of R_1 and R_2 respectively and A and B be fuzzy ideals of X and Y respectively. Then A and B are t-pure fuzzy ideals if and only if $A \oplus B$ is t-pure fuzzy ideal of $X \oplus Y$.

Proof:

($\rightarrow$) Let $C \oplus D$ be any fuzzy ideal of $X \oplus Y$. We must prove $(A \oplus B) \cap (C \oplus D) = (A \oplus B) \circ (C \oplus D)$.

It is enough to prove $(A \oplus B) \cap (C \oplus D) \subseteq (A \oplus B) \circ (C \oplus D)$.

But C is a fuzzy ideal of X and D is a fuzzy ideal of Y implies that $A \cap C = A \circ C$ and $B \cap D = B \circ D$ since A and B are t-pure fuzzy ideals .

$(A \oplus B) \cap (C \oplus D) = (A \cap C) \oplus (B \cap D)$ by proposition (2.8) .

$= (A \circ C) \oplus (B \circ D = (A \oplus B) \circ (C \oplus D)$ by proposition (2.8)

Hence $A \oplus B$ is t-pure fuzzy ideal of $X \oplus Y$.

Conversely ,to prove A and B are t-pure fuzzy ideals of X and Y respectively.

Let C be any fuzzy ideal of X and D be any fuzzy ideal of Y .

To prove $A \circ C = A \cap C$ and $B \circ D = B \cap D$. But$(A \oplus B) \cap (C \oplus D) = (A \oplus B) \circ (C \oplus D)$.

Since $(A \oplus B)$, $(C \oplus D)$ are fuzzy ideals of $X \oplus Y$ and $(A \oplus B)$ is t-pure fuzzy ideal of $X \oplus Y$.

Note $(A \oplus B) \cap (C \oplus D) = (A \cap C) \oplus (B \cap D)$ by proposition (2.8) and $(A \oplus B) \cap (C \oplus D) = (A \circ C) \oplus (B \circ D)$ by defined (2.5) .

Thus $(A \circ C) \oplus (B \circ D) = (A \cap C) \oplus (B \cap D)$ implies that $A \circ C = A \cap C$ and $B \circ D = B \cap D$ by corollary (1.5) .

Hence A and B are t-pure fuzzy ideals of X and Y respectively .

Proposition 2.10:

Let $X : R_1 \rightarrow [0,1]$, $Y : R_2 \rightarrow [0,1]$ are fuzzy rings $f : R_1 \rightarrow R_2$be epimorphism function and $A : R_1 \rightarrow [0,1]$ is a fuzzy ideal of R_1 and $B : R_2 \rightarrow [0,1]$ is a fuzzy ideal of Y, then :

1. If A is t-pure fuzzy ideal of R_1, then f(A) is t-pure fuzzy ideal of R_2.

2. If B is t-pure fuzzy ideal of R_2, then $f^{-1}(B)$ is t-pure fuzzy ideal of R_1.

Proof:

To prove the first ,see [Hadi and Abou-Draeb, 2004] .

To prove the second ,let D be fuzzy ideal of R_1 .To prove $f^{-1}(B)$ is t-pure fuzzy ideal of R_1 ,that mean $f^{-1}(B) \cap D = f^{-1}(B) \circ D$.

It is enough prove $f^{-1}(B) \cap D \subseteq f^{-1}(B) \circ D$

Consider $f(f^{-1}(B) \cap D) = f(f^{-1}(B)) \cap f(D)$

$= B \cap f(D)$

$= B \circ f(D)$,since B is t-pure fuzzy ideal .

Hence $f^{-1}[f(f^{-1}(B) \cap D)] = f^{-1}[B \circ f(D)]$

$f^{-1}(B) \cap D = f^{-1}(B) \circ f^{-1}(f(D))$

$f^{-1}(B) \cap C = f^{-1}(B) \circ D$

Then $f^{-1}(B)$ is t-pure fuzzy ideal of R_1 .

Corollary 2. 11 :

Let $X : R_1 \rightarrow [0,1]$, $Y : R_2 \rightarrow [0,1]$ are fuzzy rings $f : R_1 \rightarrow R_2$be homomorphism function between them and onto and $A : R_1 \rightarrow [0,1]$ is a fuzzy ideal of X and $B : R_2 \rightarrow [0,1]$ is a fuzzy ideal of Y, then :

1. If A is t-pure fuzzy ideal of X, then f(A) is t-pure fuzzy ideal of Y.

2. If B is t-pure fuzzy ideal of Y, then f^{-1} (B) is t-pure fuzzy ideal of X.

Proposition 2. 12 :

Let X be a fuzzy ring of R .A , B are fuzzy ideals of X such that $A \subseteq B$ .X / A is a fuzzy ring of R / A_* and B / A is a fuzzy ideal in X / A . B/ A is a t-pure fuzzy ideal in X / A if and only if B is t-pure fuzzy ideal of X .

Proof:

Suppose B / A is a t-pure fuzzy ideal in X / A ,to prove B is t-pure fuzzy ideal of X .

Let C be fuzzy ideal of X , C/ A is fuzzy ideal in X / A .

(B / A)∩ (C / A) = (B / A) o (C / A) if and only if (B∩ C) / A = (B o C) / A by proposition (1.14) if and only if (B∩ C) = (B o C) by proposition (1.15) .Then B is t-pure fuzzy ideal of X .

Conversely , if B is t-pure fuzzy ideal of X ,to prove B / A is a t-pure fuzzy ideal in X / A .

Let C/ A is fuzzy ideal in X / A , C be fuzzy ideal of X .

(B / A)∩ (C / A) = (B∩ C) / A by proposition (1.14)
= (B o C) / A by definition (2.2) .
= (B / A) o (C / A) by proposition (1.14) .

Then B / A is a t-pure fuzzy ideal in X / A .

S.3 P-F Fuzzy Rings:

In this section ,we will fuzzify the concept of P-F ring into P-F fuzzy ring .Then we will investigate some basic results about P-F fuzzy rings .

Recall that any ring R is called a P-F ring if ann_R (a) is a t-pure ideal of R , for all $a \in R$,[Kash, 1982] .

Before giving the definition of P-F fuzzy ring , we have :

Definition 3. 1 [Abou-Draeb, 2000] :

Let A be a nonempty fuzzy ideal of R. **The Annihilator of A** denoted by (F-Ann A) is defined by $\{ x_t : x \in R , x_t \text{ o } A \subseteq 0_1 \}, t \in [0,A(0)]$.

Note that (F-Ann A) (a) = $\sup \{ t: t \in [0,A(0)] , a_t \text{ o } A \subseteq 0_1\}, a \in R$.
$= (0_1 : A)$.

Proposition 3.2 [Abou-Draeb, 2000]:

Let X be fuzzy ring of R and $a \in R$, $[\text{F-Ann } (a_t)]_k$ = ann (a) for all t, k (0,X(0)] .

Proposition 3. 3 [Abou-Draeb, 2000] :

The Annihilator of a fuzzy ideal A of R (F-Ann A) is a fuzzy ideal of R.

Proposition 3.4:

Let X be a fuzzy ring of R and $a \in R$,$t \in [0 ,X(0)]$,then :

F-Ann (a_t) is a fuzzy ideal of a fuzzy ring X of a ring R , when F-Ann $(a_t) \subseteq X$.

Proof

Let b, c $\in$R, to prove the following :

1- F-Ann (a_t) (b-c) $\geq$ min { F-Ann (a_t) (b), F-Ann (a_t) (c)}.

2- F-Ann (a_t) (b.c) $\geq$min {max { F-Ann (a_t) (b), F-Ann (a_t) (c)},X(b.c)}.

Now, let F-Ann (a_t)(b) = h and F-Ann (a_t)(c)= k, where h ,k $\in$ (0,X(0)] .

Let F-Ann (a_t)(b-c)= r ,to prove r $\geq$ min {h,k} = λ .

Since $b_h \text{ o } a_t \subseteq 0_{X(0)}$ and $c_k \text{ o } a_t \subseteq 0_{X(0)}$ and $b_h \subseteq b_\lambda$, $c_k \subseteq c_\lambda$,then $b_\lambda \text{ o } a_t \subseteq b_h \text{ o } a_t$, $c_\lambda \text{ o } a_t \subseteq c_k \text{ o } a_t$,[Hameed, 2000] ,implies that $b_\lambda \text{ o } a_t \subseteq 0_{X(0)}$, $c_\lambda \text{ o } a_t \subseteq 0_{X(0)}$, [Abou-Draeb, 2000].

Hence $(b_\lambda \text{ o } a_t) - (c_\lambda \text{ o } a_t) \subseteq 0_{X(0)} - 0_{X(0)} = 0_{X(0)}$,[Abou-Draeb, 2000] ,implies that $(b_\lambda - c_\lambda) a_t \subseteq 0_{X(0)}$ and $(b - c)_\lambda a_t \subseteq 0_{X(0)}$ .Then $(b - c)_\lambda \subseteq$ F-Ann (a_t) .

But F-Ann (a_t)(b-c)= $\sup \{ \beta: \beta \in (0,X(0)] , (b -c)_\beta \text{ o } a_t \subseteq 0_{X(0)}\}$ =r $\geq \lambda$.

The first condition is true .

Now , to prove the second condition . Let F-Ann (a_t)(bc)= r and X(bc) =q.

To prove $r \geq \min\{\max\{h,k\},q\}$ .Suppose that max {h ,k} =h , $(bc)_r$ o $a_t \subseteq 0_{X(0)}$ .But $(bc)_h$ o $a_t \subseteq ((bc)\ a)_\lambda \subseteq (c\ (ba))_\lambda \subseteq (c_h$ o $(b_h$ o $a_t) \subseteq c_h$ o $0_{X(0)} \subseteq 0_{X(0)}$ where λ= min {h ,t} . Since F-Ann (a_t)(bc)= sup { β: $\beta \in (0,X(0)]$, $(bc)_\beta$ o $a_t \subseteq 0_{X(0)}$} =r $\geq$ h .

Similarly ,if max{h ,k} =k ,then r $\geq$ k .

Hence r $\geq$ max{h ,k} ,then F-Ann (a_t) is a fuzzy ideal of R .

Since F-Ann (a_t)(bc) $\subseteq$ X(bc) ,then F-Ann (a_t)(bc) $\geq$ min{max{F-Ann(a_t)(b), F-Ann (a_t) (c)},X(bc)}.

Thus F-Ann (a_t) is a fuzzy ideal of a fuzzy ring X of a ring R .

We shall the definition as follows :

Definition 3.5 [Hadi and Abou-Draeb, 2004]**:**

Let X be a fuzzy ring of R. X is called **P-F fuzzy ring** if F-Ann (a_t) is a t-pure fuzzy ideal of X , for each $a_t \subseteq$ X and t ∈ (0,X(0)] .

That mean : if A is a fuzzy ideal of X , then F-Ann (a_t) o A$\subseteq$ F-Ann $(a_t)$$\cap$ A.

Now ,we give some properties and Theorems about this concept .

Theorem 3.6:

Let X be a fuzzy ring of R and $x_t \subseteq$ X ,for all t$\in$ (0,X(0)] . X is a P-F fuzzy ring if and only if X_t is a P-F ring ,for all t $\in$ (0,X(0)] .

Proof :

If X is a P- F fuzzy ring ,to prove X_t is a P-F ring , for all t $\in$ (0,X(0)] .

Let a $\in$ X_t ,then $a_t \subseteq$ X .Since X is P- F fuzzy ring , then F-ann $(a_t)$is a t-pure fuzzy ideal of X by definition (3.5). That mean F-ann (a_t) o B= F-ann $(a_t)$$\cap$ B for each B fuzzy ideal of X .

To prove ann (a) is t-pure ideal in X_t , for all t $\in$ (0,X(0)]. Let I be any ideal of X_t ,for all t $\in$ (0,X(0)] ,we must prove ann(a)$\cap$ I = ann(a) o I.

It is enough to show that ann(a)$\cap$ I$\subseteq$ ann(a) o I.

Define B:R $\rightarrow$ [0,1] such that B(y)= X(y) if y$\in$ I and B(y)= 0 otherwise . B is a fuzzy ideal of X by [Swamy and Swamy, 1988].

Let x$\in$ ann(a) $\cap$ I ,then x $\in$ ann(a) and x $\in$ I ,let X(x) = r , r $\in$ (0,X(0)] implies that x a = 0 and B(x) = X(x) = r implies that $x_r\ a_t \subseteq 0_{X(0)}$ and $x_r \subseteq$ B , then $x_r \subseteq$ F-ann (a_t) and $x_r \subseteq$ B .Hence $x_r \subseteq$ F-ann (a_t) $\cap$ B = F-ann (a_t) o B.

Since (F-ann (a_t) $\cap$ B)(x) = min { F-ann (a_t) (x) ,B(x) } $\geq$r , then x$\in$ F-ann (a_t) o B.

Let F-ann (a_t)= c ,c B(x) $\geq$ r and sup {inf {min{c(b_i),B(d_i)}|x=$\sum b_i d_i$}}$\geq$ r.

Note that min{c(b_i),B(d_i)} $\geq$r for all i= 1,2,--,n.,then c(b_i) $\geq$ r and B(d_i) $\geq$ r. Hence $(b_i)_r \subseteq$ F-ann (a_t) = c , then $(b_i)_r a_t \subseteq 0_{X(0)}$ for all i= 1,2,--,n. b_i a =0 for all i= 1,2,--,n . ,then $b_i \subseteq$ ann (a) for all i= 1,2,--,n .

Also , B(d_i) r = X(x), $d_i \in$ I for all i= 1,2,--,n . Hence x=$\sum b_i d_i \in$ ann (a) o I ,thus ann(a)$\cap$ I $\subseteq$ ann(a) o I and X_t is a P- F ring , for all t $\in$ (0,X(0)] .

Conversely ,since X_t is a P- F ring , for all t $\in$ (0,X(0)] , then to prove X is a P-F fuzzy ring .

Let $a_t \subseteq$ X, then a $\in$ X_t , since X_t is a P- F ring , for all t $\in$ (0,X(0)] , then ann (a) is t-pure ideal of X_t , that mean ann(a) o I = ann(a)$\cap$ I , for all I ideal of X_t .

To prove F-ann $(a_t)$is a t-pure fuzzy ideal of X , let B is a fuzzy ideal of X. We must prove F-ann (a_t) $\cap$ B= F- ann (a) o B. It is enough to show that F-ann (a_t) $\cap$ B $\subseteq$ F-ann (a_t) o B.

Let $x \in R$, $t \in (0,X(0)]$,note that B_t is ideal of X_t and $x_t \subseteq$ (F-ann $(a_t) \cap B$) , $x_t \subseteq$ F-ann (a_t) and $x_t \subseteq B$, then x_t o $a_t \subseteq 0_{X(0)}$ and $B(x) \geq t$ x a =0 and $x \in B_t$,then $x \in$ ann (a) and $x \in B_t$.

Hence $x \in$ ann (a) $\cap B_t$ = ann (a) o B_t .There exists $b_i \in$ ann (a) and $d_i \in B_t$ such that $x=\sum b_i d_i$, $b_i \in$ann (a) = (F-ann (a_t) $)_t$ by proposition (3.2)$(b_i)_t \subseteq$ F- ann (a_t) and $d_i \in B_t$,then $(d_i)_t \subseteq$ B, for all i= 1,2,--,n . Such that $B(d_i) \geq t$ and(F-ann (a_t) $(b_i)) \geq t$. Then min{ F-ann (a_t) (b_i), $B(d_i)$} $\geq t$, for all i= 1,2,--,n .

And inf { min{ F-ann (a_t) (b_i), $B(d_i)$} | $x=\sum b_i d_i$}}$\geq$ t.

Also, sup{ inf { min{ F-ann (a_t) (b_i), $B(d_i)$} | $x=\sum b_i d_i$}}}$\geq$ t. Hence (F-ann (a_t) o B)(x) = sup{ inf { min{ F-ann (a_t) (b_i), $B(d_i)$} | $x=\sum b_i d_i$}}} $\geq$ t, then $x_t \subseteq$ F-ann (a_t) o B . F-ann $(a_t) \cap B \subseteq$ F-ann (a_t) o B.

Thus X is a P-F fuzzy ring .

Remark 3. 7 [Hadi and Abou-Draeb, 2004]:

Let X be a fuzzy ring of R . If X is a P- F fuzzy ring of R , then R is not necessary P- F ring .

Example:

Let X : $Z_{12} \rightarrow [0,1]$ defined by X(a) = 1 if a {0,4,8 } and X(a) = 0 otherwise .

Then for all t >0, x_t ={0,4,8} which is a P-F ring . Hence X is a P-F ring of Z_{12} by theorem (3.6) . However Z_{12} is not a P-F ring .

Proposition 3.8:

Let X be a fuzzy ring over field F, then X is a P-F fuzzy ring of F .

Proof :

Let a_t X and $a \neq 0$, let $r_k \in$ F-ann (a_t) r_k o $a_t \subseteq 0_{X(0)}$,then a. r= 0 and r = 0.Then F-ann (a_t) =$0_{X(0)}$ is t-pure fuzzy ideal of X by proposition (2.3).

Hence X is a P-F fuzzy ring of F.

Note that ,if X is a fuzzy ring over an integral domain R , then X is a P-F fuzzy ring of R .

Proposition 3.9:

Let X be a fuzzy ring of R and A be a fuzzy ideal of X .If X is a P-F fuzzy ring of R , then (F-Ann A) is a t-pure fuzzy ideal of X .

Proof :

Since If X is a P-F fuzzy ring of R and A be a fuzzy ideal of X, then (F-Ann A) is a fuzzy ideal of X by proposition (3.3) .

To prove (F-Ann A) is a t-pure .That mean (F-Ann A) $\cap$ B = (F-Ann A) o B , for any B fuzzy ideal of X .

If X(0) = 1 and $a_t \subseteq$ A ,then F-ann (a_t) = F-Ann A by [Swamy and Swamy, 1988; Abou-Draeb, 2010].

[F-ann (a_t) o B = F-ann (a_t) $\cap$B] = [F-Ann A o B = F-Ann A $\cap$ B] by [Abou-Draeb, 2000] . Then (F-Ann A) is a t-pure fuzzy ideal of X .

Lemma 3.10:

Let R be a P-F ring , then every subring of R is P-F ring .

Theorem 3.11 [Hadi and Abou-Draeb, 2004]

Let X and Y be fuzzy rings of R_1 and R_2 respectively .Then X and Y are P-F fuzzy rings if and only if $X \oplus Y$ is a P-F fuzzy ring of $R_1 \oplus R_2$.

Theorem 3.12:

Let X be a fuzzy ring of R and A be a fuzzy ideal in X .X is a P- F fuzzy ring if X/A is a P-F fuzzy ring of R/A_* .

Proof:

If X/A is a P-F fuzzy ring of R/ A_*,we must prove X is a P- F fuzzy ring ,that mean F-ann (a_t) is t-pure fuzzy ideal of X , for all $a_t \subseteq X$,t ∈ (0,X(0)].

We must prove F-ann (a_t) ∩ B= F- ann (a) o B for every fuzzy ideal of X .It is enough to show that F-ann (a_t) ∩ B ⊆ F-ann (a_t) o B.

Since X/A is P-F fuzzy ring of R/ A_* , then F-ann (a_t) / A and B/ A are fuzzy ideals of X/A .

And (F-ann (a_t)/A) ∩(B/ A) =(F-ann (a_t)/ A) o (B/ A) ↔(F-ann (a_t) ∩ B) / A=(F-ann (a_t) o B) / A by proposition (1.14). ↔ (F-ann (a_t) ∩ B = F-ann (a_t) o B by proposition (1.15) .

Hence F-ann (a_t) is t-pure fuzzy ideal of X ↔ X is a P-F fuzzy ring of R.

Remark 3.13:

The converse of theorem (3.12) is not true as the following example shows.

Example:

Let X: Z→[0,1] such that: $X(a) = \begin{cases} 1 & \text{if } a \in 2Z \\ 0 & \text{otherwise} \end{cases}$

X is a fuzzy ring of Z and a fuzzy ideals A, B in X such that :

$A(a) = \begin{cases} 1 & \text{if } a \in 4Z \\ 1/2 & \text{if} a \in Z/4Z \end{cases}$ Note that A_* =4Z. And

$B(a) = \begin{cases} 1 & \text{if } a \in 4Z \\ 1/2 & \text{if} a \in 2Z/4Z \end{cases}$ and B(a) =0 otherwise .

Then X/A :Z/4Z →[0,1] such that: $X(a + A_*) = \begin{cases} 1 & \text{if } a \in 4Z \\ \sup(a+b) & \text{if} b \in 4Z, a \notin 4Z \end{cases}$

$= \begin{cases} 1 & \text{if } a + A* = A* \\ 1/2 & \text{if} a + A* = 2 + A* \end{cases}$ $X(a + A_*)$ =0 otherwise .

X is a P-F fuzzy ring and F- ann (2+ A)=B, B is not t-pure fuzzy ideal of X/A , since B∩B = B and B o B = $0_{X/A}$.

S.4Normal Fuzzy Rings :

In this section , we introduce the concept of a Normal fuzzy ring and we state some properties and Theorems about it .Also we shall give its relationship with P-F fuzzy ring .

As a generalization of definition in [Kash, 1982] , a ring R is called a normal if every ideal of R is t-pure ideal .

Definition 4.1 [Abou-Draeb, 2010**]:**

Let X be a fuzzy ring of R .X is called a **Normal fuzzy ring** if every fuzzy ideal of X is t-pure fuzzy ideal .

Now ,we give the following properties :

Proposition 4.2 [Hadi and Abou-Draeb, 2004**]:**

Let X be a fuzzy ring of R .X is a Normal fuzzy ring of R if and only if X_t is a normal ring , for all t ∈ (0,X(0)] .

Remark 4.3 [Hadi and Abou-Draeb, 2004]:

Let X be a fuzzy ring of a ring R . If R is a normal ring ,then X is not necessarily that Normal fuzzy ring.

Example:

Let R =Q is a normal ring and $X(a) = \begin{cases} 1 & if\ a \in Z \\ 0 & otherwise \end{cases}$

X is a fuzzy ring of Q. But X_t=Z for all t = 0 and Z is not normal ring .

Hence X is not Normal fuzzy ring by proposition (4.2).

Remark 4.4 :

The converse of proposition (4.3) is not true as the following example shows.

Example:

Let R =Z_{12} is not normal ring and $X(a) = \begin{cases} 1 & if\ a \in \{0,4,8\} \\ 0 & otherwise \end{cases}$

But X is a Normal fuzzy ring .

Proposition 4.5:

Let X : $R_1 \rightarrow [0,1]$, Y : $R_2 \rightarrow [0,1]$ are fuzzy rings . f : $R_1 \rightarrow R_2$ be epimorphism function .Then :

1. If X is Normal fuzzy ring ,then Y is Normal fuzzy ring.

2. If Y is Normal fuzzy ring ,then X is Normal fuzzy ring , where every fuzzy ideal of X is f-invariant .

Proof:

The proof follows directly by proposition (2.11) and definition (4.1) **.**

Proposition 4. 6 :

Let X and Y be fuzzy rings of R_1 and R_2 respectively. X and Y are Normal if and only if $X \oplus Y$ is Normal fuzzy ring of $R_1 \oplus R_2$.

Proof:

Let X and Y are Normal fuzzy rings ,to prove $X \oplus Y$ is Normal fuzzy ring . Since X and Y be fuzzy rings of R_1 and R_2 respectively ,then $(X \oplus Y)$ is fuzzy ring , by [Abou-Draeb, 2000] .

Let $(C \oplus D)$ be a fuzzy ideal of $(X \oplus Y)$,then C is fuzzy ideal of X and D is a fuzzy ideal of Y .

Since X and Y are Normal ,then C and D are t-pure fuzzy ideals of X and Y respectively $\Rightarrow$ $(C \oplus D)$ is a t-pure fuzzy ideal of $X \oplus Y$ by proposition (2.9).

Hence $X \oplus Y$ is Normal fuzzy ring of $R_1 \oplus R_2$.

Conversely , $X \oplus Y$ is Normal fuzzy ring ,we must prove X and Y are Normal fuzzy rings of R_1 and R_2 respectively.

Let A and B be fuzzy ideals of X and Y respectively ,then $(A \oplus B)$ is a fuzzy ideal of $(X \oplus Y)$.

Since $X \oplus Y$ is Normal, then $(A \oplus B)$ is a t-pure fuzzy ideal $\Rightarrow$ A and B are t-pure fuzzy ideals of X and Y respectively by proposition (2.9)

Hence X and Y are Normal fuzzy rings of R_1 and R_2 respectively.

Proposition 4.7:

Let X be a fuzzy ring of a ring R and A be a fuzzy ideal in X. Then X is a Normal fuzzy ring if and only if X/A is a Normal fuzzy ring of R/ A_* .

Proof:

Let X be a Normal fuzzy ring ,to prove X/A is a Normal fuzzy ring of R/ A_* . By [Abou-Draeb, 2000], X/A is a fuzzy ring of R/ A_* .

We must prove X/A is a Normal fuzzy ring of R/ A_* .Let B is a fuzzy ideal of X ,then B is t-pure by definition (4.1) implies that B/A is t-pure fuzzy ideal by proposition (2.12) that mean (B/A) ∩ (C/A) = (B/A) o (C/A) for all C/A fuzzy ideal of X/A .

Hence X/A is Normal fuzzy ring

Conversely , we must prove X is Normal fuzzy ring .Let B be any fuzzy ideal of X, to prove B is t-pure fuzzy ideal that mean B∩ C= B o C for all C fuzzy ideal of X.

But B/A , C/A are fuzzy ideals of X/A and X/A is Normal fuzzy ring, then (B/A) ∩ (C/A) = (B/A) o (C/A) .

By proposition(1.14) , (B/A) ∩ (C/A) = (B∩ C) /A and (B/A) o (C/A) = (B o C) /A , then (B∩ C) /A = (B o C) /A .

Thus B ∩ C =B o C by proposition(1.15) ,then B is t-pure fuzzy ideal .

Hence X is Normal fuzzy ring of R .

We give the relation about Normal fuzzy ring and P-F fuzzy ring.

Proposition 4.8 :

Let X be a fuzzy ring of a ring R .If X is a Normal fuzzy ring ,then X is P-F fuzzy ring of R.

Proof:

Since X is a fuzzy ring of a ring R ,that mean any fuzzy ideal of X is t-pure ,then F-ann (a_t) is t-pure fuzzy ideal of X ,for each $a_t \subseteq X \Rightarrow X$ is a P-F fuzzy ring of R.

Remark 4.9 :

The converse of proposition(4.8) is not true as the following example shows.

Example:

Let R =Q is a normal ring and $X(a) = \begin{cases} 1 & \text{if } a \in Z \\ 0 & \text{otherwise} \end{cases}$

X is a fuzzy ring of Q. But X_t=Z for all t ∈ (0,1] and Z is not normal ring .

Hence X is not Normal fuzzy ring by proposition (4.2).

$[\text{F-ann}(a_t)]_k$ = ann(a) for all k ∈ (0,1] .

= {0} is pure ideal of Z = X_t for all k ∈ (0,1] .

Then F-ann (a_t) is t-pure fuzzy ideal of X .

Hence X is a P-F fuzzy ring of R.

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