Existence of Minimal Blocking Sets of Size 31 in the Projective Plane PG(2,17)

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Abstract

In this paper, we show that by an example existence minimal blocking set of Rédei-type of size 27 representing protectively triangle in PG(2,17) example (2-5).We prove that the projective plane PG(2,17)having minimal blocking set of size 31 contain 14-secant and not contain i-secant ;14<i<q theorem (3-1) ,also we finding important propositions about minimal blocking set of size 31 in the projective plane PG(2,17)in theorems which are(3-3) into(3-19).

الخلاصة

في هذا البحث قمنا بإثبات بمثال وجود مجموعة قالبيه أصغريه من النوع _ ريدي ذات حجم ٢٧ وتمثل مثلثا اسقاطيا في PG(2,17) مثال(٢-٥).وأثبتنا أن المستوي ألإسقاطي PG(2,17) يمتلك مجموعة قالبيه أصغريه ذات الحجم ٣١ والتي تضم قاطع – ١٤ ولا تضم قاطعا–i ; q >i< ١٤ مبرهنة (٣-١).وأيضا وجدنا الخواص المهمة حول المجاميع القالبية الاصغرية ذات حجم ٣١ في المستوي الاسقاطي PG(2,17) في المبرهنات التي هي من (٣-٣) الى (٣-١٩).

1. Introduction

Let GF(q)be denote the Galois field of q elements and V(3, q) be the vector space of row vectors of length three with entries in GF(q).

Let PG(2, q) be the corresponding projective plane. The points of PG(2, q) are the non-zero vectors of V(3, q) with the rule that X = $(x_1; x_2; x_3)$ and Y = $(\lambda x_1; \lambda x_2; \lambda x_3)$ are the same point, where $\lambda \in$ GF(q)\{0}. Since any non-zero vector has precisely q-1non-zero scalar multiples, the number of points of PG(2, q) is $\frac{q^3-1}{q-1} = q^2+q+1$.

If the point P(X) is the equivalence class of the vector X, then we will say that X is a vector representing P(X). A subspace of dimension one is a set of points all of whose representing vectors form a subspace of dimension two of V(3, q). Such subspaces are called lines. The number of lines in PG(2, q) is $q^2 + q + 1$. There are q + 1 points on every line and q + 1 lines through every point. Also ,if V is vectors spaces of dimension two define on the field GF(q). then any subset from V which are meet for all prime from V counting at least n(q-1)+1points[Hirschfeld, J.W.P. (1979)].

A blocking set in a projective plane is a set B of points, such that every line contains at least one point of B. If B contains a line, it is called trivial. If no proper subset of B is a blocking set it is called minimal [Hirschfeld, J.W.P. and Storme, L. update(2001)] .Let B be a non-trivial minimal blocking set, and let l be a line containing $l < q + 1$ points of B. Then it follows immediately that $|B| \geq q + l$, by considering the lines through a point P of L not belonging to the blocking set. If we have equality, then every line through P different from L contains precisely one point of B .Blocking sets of this kind are called of Rédei- type and were studied in [Bruen,A.A. and Thas,J.A. (1977)] and in [Blokhuis,A. A.and Brouwer, E.and . Sz″onyi, T. (1995)].We call B of Rédei- type if there exists a line l such that $|B \setminus l| = q$ (the line l is called a Rédei line of B. [Dipaola,J.(1969)]made idea about projective triangle which are an example of a blocking set of size 3(q+1)/2 in the Desarguesian planes of odd orders.

The Elation α in the projective plane PG(2,q) is bijection fixed the points of ***l*** , and reverse the lines passing through p on ***l*** [Innamorate and Storme, 2004]**.**

(1-1)Theorem:[Barát ,.and Innamorati ,2003**]**

Let B be a blocking set of size b in the projective plane PG(2,q)then:

1. $\sum_{i=0}^{q+1} r_i = q^2 + q + 1,$

2. $\sum_{i=1}^{q+1} i r_i = b(q+1),$

3. $\sum_{i=2}^{q+1} i(i-1) r_i = b(b-1),$

4. $\sum_{i=1}^{q+1} \nu_i = q+1,$

5. $\sum_{i=2}^{q+1} (i-1)\nu_i = b-1,$

6. $\sum_{i=0}^{q} \upsilon_i = q+1,$

7. $\sum_{i=1}^{q} i\upsilon_i = b,$

where r_i : denote the total number of i-secant to B.

: denote the total number of i-secant through a point P belongs to B.

ν_i υ_i : denote the total number of i-secant through a point Q belongs to PG(2,q)\ B.

(1-2) Theorem:[Hirschfeld, 1979]

In PG(2,q),where q is odd number, every q-arc lies on a conic ,and the number of a conic is one or four if q $\neq$ 3 or q = 3 respectively.

(1-3)Definition(Companion Matrix)[Hirschfeld, 1979]:

Let $f(x)=x^{n+1}-a^n x^n-\ldots-a_0$ be any monic polynomial over GF(q)then its Companion Matrix ,C(f) is given by the (n+1)×(n+1)matrix:

$$C(f) = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 \\ a_0 & a_1 & a_2 & . & . & . & a_n \end{bmatrix}$$

2. Minimal Blocking Sets in the Projective Plane PG(2,17)

In this section we study minimal blocking sets of size 27 of Rédei-type in PG(2,1٧).

(2-1) Cyclic projectivety on GF(17):

Respecting to the definition(1-3)we getting $f(x)=x^3-8x^2-1$ be a monic polynomial over GF(17), the companion matrix of f(x) is:

C(f)= $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 8 \end{bmatrix}$ is cyclic projectivety on PG(2,1٧).

The number of point in the PG(2,17) has 307 points and 307lines and every line passes throw18 points.

Let p_1 be the point p_1 =(1,0,0),then $P_i=P_{i-1}T, \forall_i$ =2,...,307,are the 307 points of PG(2,17).see Table(1)

Table(1)

i	p_i
1	1 0 0
2	0 1 0
3	0 0 1
4	1 0 8
⋮	
305	1 13 9
306	1 2 0
307	0 1 2

Let L_1 be the line which contains the points 1, 2, 10 ,16,87,110,120,152, 176, 180,192, 211,233, 254, 259, 272,279, 306, then $L_i=L_{i-1}T$, $\forall_i$ =2,...,307, are the lines of PG(2,17), the 307 lines L_i are given by the rows in Table (2).

Table (2)

L_1	1 2 10 16 87 110 120 152 176 180 192 211 233 254 259 272 279 306
L_2	2 3 11 17 88 111 121 153 177 181 193 212 234 255 260 273 280 307
L_3	3 4 12 18 89 112 122 154 178 182 194 213 235 256 261 274 281 1
⋮	
L_{307}	307 1 9 15 86 109 119 151 175 179 191 210 232 253 258 271 278 305

(2-2) Theorem:[Innamorati S. and Maturo, A., (1991)**]**

In PG(2,q),q≥4,there exists a blocking k- sets for every k with $2q-1 \le k \le 3q-3$

(2-3)Definition:[Hirschfeld, J.W.P. and Storme, L. update(2001)]

In PG(2,q),q odd, the projective triangle are a blocking k-set points projectively equivalent to the set $\{(1,0,-c)\,.\})q($GFsquare of the c is: $(0,-c,1)$, $(-c,1,0)$,

(2-4) Theorem:[Hirschfeld, J.W.P. and Storme, L. update(2001)]

In PG(2,q),q odd ,there exists a minimal blocking k-sets of Rédei-type ,the projective triangle of cardinality $\frac{3(q+1)}{2}$.

(2-5)Example:

Let q=17, the number square of GF(17) are 0,1,2,4,8,9,13,15,16 which mean values c. and 0,16,15,13,9,8,4,2,1 which mean values –c, respectively. This points respect to the definition (2-3)are:

− C =0	− C =1	− C =2	− C =4	− C =8	− C =9	− C =13	− C =15	− C =16
(٠،٠،١)	(٠،١،١)	(٠،٢،١)	(٠،٤،١)	(٠،٨،١)	(٠،٩،١)	(٠،١٣،١)	(٠،١٥،١)	(٠،١٦،١)
(٠،١،٠)	(١،١،٠)	(٢،١،٠)	(٤،١،٠)	(٨،١،٠)	(٩،١،٠)	(١٣،١،٠)	(١٥،١،٠)	(١٦،١،٠)
(١،٠،٠)	(١،٠،١)	(١،٠،٢)	(١،٠،٤)	(١،٠،٨)	(١،٠،٩)	(١،٠،١٣)	(١،٠،١٥)	(١،٠،١٦)

change this to equivalent points in the PG(2,17) are

− C =0	− C =1	− C =2	− C =4	− C =8	− C =9	− C =13	− C =15	− C =16
(٠،٠،١)	(٠،١،١)	(٠،٩،١)	(٠،١٣،١)	(٠،١٥،١)	(٠،١،٢)	(٠،٤،١)	(٠،٨،١)	(٠،١٦،١)
(٠،١،٠)	(١،١،٠)	(٩،١،٠)	(١٣،١،٠)	(١٥،١،٠)	(١،٢،٠)	(٤،١،٠)	(٨،١،٠)	(١٦،١،٠)
(١،٠،٠)	(١،٠،١)	(١،٠،٩)	(١،٠،١٣)	(١،٠،١٥)	(٢،٠،١)	(١،٠،٤)	(١،٠،٨)	(١،٠،١٦)

Which is minimal blocking sets of Rédei-type of size 27 which are representing projective triangle .

(2-6)Theorem:

The projective triangles are minimal blocking sets of size 27 in PG(2,17) that are of Rédei-type.

The following lemmas prove that minimal blocking set of size 31 exist in PG(2,17). From now on ,let B be a minimal blocking set of size 31 in PG(2,17).

(2-7) Lemma: Any point of B lies on at least four tangents.

proof . Let P $\in$ B and let l be a tangent line to B at P. Consider PG(2,17)\ l and call this AG(2,17).Then a set B\\$l$ of size 30 remains. A minimal blocking set in AG(2,17)contains at least 33 points. This means that we have to add at least three points to B\\$l$ to get a blocking set in AG(2,17). The external lines to B\\$l$ in AG(2,17) are the tangents to B at P, different from l. Hence P lies on at least four tangents to B.

(2-8) Lemma: There exists at least one 4-secant in B.

proof. Suppose there are only 1-, 2- and 3-secants. Let the number of them be denoted by a, b and c respectively. Then the following equations from theorem(1-1) must hold by standard counting arguments.

a+b+c=307 (1)

a+2b+3c=496 (2)

2b+6c=930 (3)

From these equations, b = −363, which is a contradiction.

(2-9) Lemma: The number of all tangents for B lies on at least 124.

proof. from lemma (2-7)there exists 4 tangents passing through every point in B. Since the number of the points of B are 31.Hence the number of all tangents for B lies on at least 124.

(2-10) Lemma: Let L be a secant of B in n points then $\cdot 1 \le n \le 14$

Proof .Let L be a secant of B in n=15,and have p be a point lies outside B. Hence |B| $\geq$ 17*1+15=32,which is a contradiction to the size of B.

3.The Search of Minimal Blocking 31- Sets of Rédei-type in The Projective Plane PG(2,17)

In this section we study existence of minimal blocking sets of size 31of Rédei-type in PG(2,17).

(3-1)Theorem:

There exist minimal blocking sets of size 31 of Rédei-type in PG(2,17)which contain 14-secant and not contain i-secant;14<i<q.

Proof. let (x,y,z)be denote the coordinates of a projective point. let l :z=0 be the Rédei-line and let $p_i=(x_i,y_i,1)\equiv(x_i,y_i)$,i=1,…17,be the affine points of the blocking set B.

Let $Q_1=(0,1,0)$, $Q_2=(1,0,0)$, $Q_3=(1,16,0)$, $Q_4=(1,3,0)$be the four points of l \B are getting from group acts $S_3 \cong <T_1,T_2>$ which are:

$$T_1 : (x, y, z) \to (y + z, x + 16z, 0)$$

$$T_2 : (x, y, z) \to (16y + z, x + y, 0)$$

Which divide the points of Rédei-line into four orbits:

Orbit one: $\{(1,0,0)(0,1,0)(1,16,0)\}$

Orbit two: $\{(1,1,0),(1,15,0),(1,8,0)\}$

Orbit three: $\{(1,9,0),(1,2,0),(1,7,0),(1,11,0)(1,14,0)(1,5,0)\}$

Orbit four: $\{(1,4,0),(1,13,0),(1,3,0),(1,10,0)(1,6,0)(1,12,0)\}$

Fixing Orbit one $\{Q_1, Q_2, Q_3\}$ with point $Q_4 = (1,3,0)$ of orbit four outside B. So all affine lines x=az through Q_1 and y=bz through Q_2, contain exactly one affine point of the blocking set , we have

$x_i \neq x_j$ and $y_i \neq y_j$, if $i \neq j$

since Q_3 does not belong to B ,all affine lines x+y+az=0 contain exactly one point of B .Hence $x_i+y_i \neq x_j+y_j$, if $i \neq j$.

since $Q_4=(1,3,0)$ does not belong to B,all affine lines y-3x=dz contain exactly one point of B. Hence $y_i-3x_i \neq y_j-3x_j$, , if $i \neq j$.

These four conditions will be used in the search for a minimal blocking set of Rédei-type of size 31. we now select the first one affine points.

The 17 affine points of B can not form a 17-arc,i.e.,a set of 17 points, no three collinear. if $\{p_1,\dots,p_{17}\}$ were an 17-arc.

Then $\{p_1,\dots,p_{17}, Q_1\}$,$\{p_1,\dots,p_{17}, Q_2\}$,$\{p_1,\dots,p_{17}, Q_3\}$,$\{p_1,\dots,p_{17}, Q_4\}$ would be four 18-arcs.this contradicts that an 17-arc is uniquely extendable to 18-arc,see theorem(1-2).thus some three points of $p_1,\dots,p_{17}$ are collinear. Assume p_1, p_2 are collinear with a third affine point of B without loss of generality,

Let $p_1=(0,0)$. using the Elation α: $(x,y,z) \to (x+z, y+16z, z)$,with axis l ,and center (1,15,0),fixing points Rédei-line ,we can assume that points line passes through (0,0),(1,15) in the one orbit which are point (1,15,0) is the point in the orbit two. When we choose point p_1 then the values $x_i=0, y_i=0, x_i+y_i=0$, $y_i-3x_i=0$ and remain values $x_i=\{1,2,\dots,16\}$, $y_i=\{1,2,\dots,16\}$, $x_i+y_i=\{1,2,\dots,16\}$, $y_i-3x_i=\{1,2,\dots,16\}$.Choosing p_2 would be the following possibility

$\{(1,1),,(1,2),(1,3),(1,4),(1,5),(1,6),(1,8),(1,9),(1,10),(1,11),(1,12),(1,13),(1,14),(1,15)\}$

and choose $p_2=(1,1)$ would be remain values $x_i=\{2,\dots,16\}$, $y_i=\{2,\dots,16\}$, $x_i+y_i=\{1,3,4,\dots,16\}$, $y_i-3x_i=\{1,2,3,\dots,14,16\}$,and for choose p_3 then possibility are

$\{(2,2),(2,3),(2,5),(2,7),(2,8),(2,9),(2,10),(2,11),(2,12),(2,13),(2,14),(2,16)\}$

and choose $p_3=(2,2)$ would be remain values $x_i=\{3,\dots,16\}$, $y_i=\{3,\dots,16\}$, $x_i+y_i=\{1,3,5,6,\dots,16\}$, $y_i-3x_i=\{1,2,\dots,12,14,16\}$.

Choosing p_4 would be the following possibility

$\{(3,3),(3,4),(3,6),(3,8),(3,10),(3,11),(3,12),(3,13),(3,15)\}$

and choose $p_4=(3,3)$ would be remain values $x_i=\{4,\dots,16\}$, $y_i=\{4,\dots,16\}$, $x_i+y_i=\{1,3,5,7,8,\dots,16\}$, $y_i-3x_i=\{1,2,\dots,10,12,14,16\}$.

Choosing p_5 would be the following possibility

$\{(4,4),(4,5),(4,7),(4,9),(4,11),(4,14),(4,16)\}$

and choose $p_5=(4,4)$ would be remain values $x_i=\{5,\dots,16\}$, $y_i=\{5,\dots,16\}$, $x_i+y_i=\{1,3,5,7,9,\dots,16\}$, $y_i-3x_i=\{1,2,\dots,8,10,12,14,16\}$.

Choosing p_6 would be the following possibility

$\{(5,5),(5,6),(5,8),(5,10),(5,12),(5,14),(5,16)\}$

and choose $p_6=(5,5)$ would be remain values $x_i=\{6,\dots,16\}$, $y_i=\{6,\dots,16\}$, $x_i+y_i=\{1,3,5,7,9,11,12,13,14,15,16\}$, $y_i-3x_i=\{1,2,\dots,6,8,10,12,14,16\}$.

Choosing p_7 would be one of the following possibility

$\{(6,6),(6,7),(6,9),(6,11),(6,13),(6,15)\}$

and choose $p_7=(6,6)$ would be remain values $x_i=\{7,\dots,16\}$, $y_i=\{7,\dots,16\}$, $x_i+y_i=\{1,3,5,7,9,11,13,14,15,16\}$,

$y_i-3x_i=\{1,2,3,4,6,8,10,12,14,16\}$.
Choosing p_8 would be one of the following possibility $\{(7,7),(7,8)\}$
and choose $p_8=(7,8)$would be remain values
$x_i=\{8,\dots,16\}$,$y_i=\{7,9,\dots,16\}$,$x_i+y_i=\{1,3,5,7,9,11,13,14,16\}$,
$y_i-3x_i=\{1,2,3,6,8,10,12,14,16\}$. Remain one possibility for choose point nine is $p_9=(8,10)$.
would be remain values
$x_i=\{9,\dots,16\}$,$y_i=\{7,9,11,\dots,16\}$,$x_i+y_i=\{3,5,7,9,11,13,14,16\}$,
$y_i-3x_i=\{1,2,6,8,10,12,14,16\}$. Remain one possibility for choose point ten is $p_{10}=(9,7)$.
would be remain values
$x_i=\{10,\dots,16\}$,$y_i=\{9,11,\dots,16\}$,$x_i+y_i=\{3,5,7,9,11,13,14\}$,
$y_i-3x_i=\{1,2,6,8,10,12,16\}$.
For point eleven remain values two possibility are $p_{11}=\{(10,12),(10,14)\}$
choose point $p_{11}=(10,12)$ would be remain values
$x_i=\{11,\dots,16\}$,$y_i=\{9,11,13,14,15,16\}$,$x_i+y_i=\{3,7,9,11,13,14\}$,
$y_i-3x_i=\{1,2,6,8,10,12\}$.choose possibility point for the four condition would be
$p_{12}=(11,9)$ remain values $x_i=\{12,\dots,16\}$,$y_i=\{11,13,14,15,16\}$,
$x_i+y_i=\{7,9,11,13,14\}$,$y_i-3x_i=\{1,2,6,8,12\}$.
Choose only possibility point for the four condition would be $p_{13}=(12,14)$ remain values $x_i=\{13,\dots,16\}$,$y_i=\{11,13,15,16\}$,$x_i+y_i=\{7,11,13,14\}$, $y_i-3x_i=\{1,2,6,8\}$.
Choose only possibility point for the four condition would be $p_{14}=(13,11)$ remain values $x_i=\{14,15,16\}$,$y_i=\{13,15,16\}$,$x_i+y_i=\{11,13,14\}$, $y_i-3x_i=\{1,2,8\}$.
Choose only possibility point for the four condition would be $p_{15}=(14,16)$ remain values $x_i=\{15,16\}$,$y_i=\{13,15\}$,$x_i+y_i=\{11,14\}$, $y_i-3x_i=\{1,2\}$.
Choose only possibility point for the four condition would be $p_{16}=(15,13)$ remain values $x_i=\{16\}$,$y_i=\{15\}$,$x_i+y_i=\{14\}$, $y_i-3x_i=\{1\}$.
Choose only possibility point for the four condition would be $p_{17}=(16,15)$
Would be the following affine points are:

$$(1,1)\ (2,2)\ (3,3)\ (4,4)\ (5,5)\ (6,6)\ (7,8)\ (8,10)(9,7)(10,12)\ (0,0)\ (11,9)$$
$$(12,14)\ (13,11)\ (14,16)\ (15,13)\ (16,15)$$

transitively to the projective points such that in table (1):

$$(1,1,1)(1,1,9)(1,1,6)(1,1,13)(1,1,7)(1,1,3)(1,6,5)(1,14,15)(1,14,2)(1,8,12)$$
$$(0,0,1)(1,7,14)(1,4,10)(1,10,4)(1,6,11)(1,2,8)(1,2,16)$$

and add it to the points $B \cap L$ which are:

$$(1,1,0),(1,15,0),(1,8,0)(1,9,0)(1,2,0)(1,7,0),(1,11,0),(1,14,0),(1,5,0),(1,4,0),$$
$$(1,13,0),(1,10,0),(1,6,0),(1,12,0)\}$$

we getting 31 point form minimal blocking sets of size 31 .

The preceding theorems all lead to the following conclusion.

(3-2)Theorem. The projective plane PG(2,17) having minimal blocking sets of size 31.

The following important proposition about minimal blocking set of size 31 in the projective plane PG(2,17)in theorems which are:

(3-3) Theorem: There exist at most 17-secant passing through point not lies in B.
proof. Let L be 14-secant ,and P be a point lies on L\B. the number of the secants passing P are 17.Therefore

$$|B| \geq 1*14+17=31.$$

(3-4) Theorem: Every two 14-secant meeting in a point in B.
proof. suppose that ℓ_1, ℓ_2 be a 14-secant. and that ℓ_1, ℓ_2 meeting in outside of B. Hence $|B| \geq 2*14+16*1=34,$ which is a contradiction to the size of B.

(3-5)theorem: there exist at most one 13-secant passing through p ; $p \in L \cap B$
Proof.suppose that two 13-secant passing through p, we have
|B|≥1+13+2*12=38, which is a contradiction to the size of B.
(3-6) theorem: there exist at most one 12-secant passing through p ; $p \in L \cap B$
Proof.suppose that two 12-secant passing through p, we have
|B|≥1+13+2*11=36, which is a contradiction to the size of B.

(3-7) theorem: there exist at most one 11-secant passing through p ; $p \in L \cap B$
Proof.suppose that two 11-secant passing through p, we have
|B|≥1+13+2*10=34, which is a contradiction to the size of B.

(3-8) theorem: there exist at most one 10-secant passing through p ; $p \in L \cap B$
Proof.suppose that two 10-secant passing through p,we have
|B|≥1+13+2*9=32, which is a contradiction to the size of B.

(3-9) theorem: there exist at most two 9-secant passing through p ; $p \in L \cap B$
Proof.suppose that three 9-secant passing through p,we have
|B|≥1+13+3*8=38, which is a contradiction to the size of B.

(3-10) theorem: there exist at most two 8-secant passing through p ; $p \in L \cap B$
Proof.suppose that three 8-secant passing through p,we have
|B|≥1+13+3*7=35, which is a contradiction to the size of B.

(3-11) theorem: there exist at most three 7-secant passing through p ; $p \in L \cap B$
Proof.suppose that four 7-secant passing through p,we have
|B|≥1+13+4*6=38, which is a contradiction to the size of B.

(3-12) theorem: there exist at most three 6-secant passing through p ; $p \in L \cap B$
Proof.suppose that four 6-secant passing through p,we have
|B|≥1+13+4*5=34, which is a contradiction to the size of B.

(3-13) theorem: there exist at most four 5-secant passing through p ; $p \in L \cap B$
Proof.suppose that five 5-secant passing through p, we have
|B|≥1+13+5*4=34, which is a contradiction to the size of B.

(3-14) theorem: there exist at most five 4-secant passing through p ; $p \in L \cap B$
Proof.suppose that six 4-secant passing through p, we have
|B|≥1+13+6*3=32, which is a contradiction to the size of B.

(3-15) theorem: there exist at most eight 3-secant passing through p ; $p \in L \cap B$
Proof.suppose that nine 3-secant passing through p, we have
|B|≥1+13+9*2=32, which is a contradiction to the size of B.

(3-16) theorem: there exist at most seventeen 2-secant passing through p ; $p \in L \cap B$
Proof.suppose that eighteen 2-secant passing through p, we have

|B|≥1+13+18*1=32, which is a contradiction to the size of B.

(3-17) theorem: Let B be a minimal blocking set of size 31 having L be 14-secant and L` be i-secant such that 2≤i≤14.Then $L \cap L`$ be a point lies inside of B.
Proof.in case i=14 we have $L \cap L`$ be a point lies inside of B(see theorem(3-4)).
and in case $L \cap L`$ be a point lies outside of B for all 2≤i≤13.Then
|B|≥14+i+16>31, which is a contradiction to the size of B.
Therefore $L \cap L`$ be a point lies inside of B for all 2≤i≤14.

(3-18) theorem:
1-Every two 9-secant meeting outside of B.
2- Every two 8-secant meeting outside of B.
3- Every one 9-secant and one 8-secant meeting outside of B.
4- Every one 9-secantand one 7-secant meeting outside of B.

Proof.(1) Let L_1,L_2 be two 9-secant meeting $p \notin L \cap B$ .Then
|B|≥9+9+16*1=34, which is a contradiction to the size of B.
(2) Let L_1,L_2 be two 8-secant meeting $p \notin L \cap B$.Then
|B|≥8+8+16*1=32, which is a contradiction to the size of B.
(3) Let L_1 be 9-secant and L_2 be 8-secant meeting outside of B .Then
|B|≥9+8+16*1=33, which is a contradiction to the size of B.
(4) Let L_1 be 9-secant and L_2 be 7-secant meeting outside of B .Then
|B|≥9+7+16*1=32, which is a contradiction to the size of B.

(3-19) theorem:
1-Any four 5-secant must be crossing in a point of B.
2-Any three 6-secant must be crossing in a point of B.
3-Any three 7-secant must be crossing in a point of B.

Proof.(1) Let L_1,L_2, L_3,L_4 be four 5-secant in B .
Suppose that $L_1 \cap L_2 \cap L_3 \cap L_4 = \{p\}$ and $p \notin B$ .Then
|B|≥4*5+16*1=36, which is a contradiction to the size of B.
(2) Let L_1,L_2, L_3 be three 6-secant in B .
Suppose that $L_1 \cap L_2 \cap L_3 = \{p\}$, and $p \notin B$ .Then
|B|≥3*6+16*1=34, which is a contradiction to the size of B.
(3) Let L_1,L_2, L_3 be three 7-secant in B .
Suppose that $L_1 \cap L_2 \cap L_3 = \{p\}$, and $p \notin B$.Then
|B|≥3*7+16*1=38, which is a contradiction to the size of B.

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