On r- Compact Spaces

Hiyam H. Kadhem
Dept. of Math. , College of Mathematics and Computer Sciences , Al-Kufa University

Abstract

The main purpose of this paper is to introduce and study a new type of compact spaces which is r-compact space where a topological space (X,τ) is said to be an r-compact space if every regular open cover of X has a finite subfamily whose closures cover X . Several properties of an r-compact space are proved.

الخلاصة

الغرض الرئيسي من هذا البحث هو تقديم ودراسة نوع جديد من الفضاءات المرصوصة وهو الفضاء المرصوص−r حيث أن الفضاء التبولوجي (X,τ) يسمى فضاء مرصوص−r إذا كان لكل غطاء مفتوح بانتظام للـX له عائلة جزئيه منتهية والتي يكون انغلاقها يغطي X . برهنا العديد من الخصائص للفضاء المرصوص-r .

1.Introduction

Compactness occupies a very important place in topology and so do some of its weaker and stronger forms, one of these forms is H-closednes where the theory of such spaces was introduced by Alexandroff and Urysohn in 1929.In 1969, nearly -compact spaces was introduced by M. K. Singal and Asha Aathur . Another type of compact space which is S-compact space was introduced in 1976 by Travis Thompson. From time to time several other forms of compactness have been studied .In this work, we shall study a new weaker form of compact spaces , namely r-compact space.

Now,let (X ,τ) be a topological space ,and let $A\subseteq X$, we say that :

i) A is a regular open set in X if and only if, $A = \overline{A}^{o}$,(Dugundji , 1966).

ii) A is a regular closed set in X if and only if, $A = \overline{A^{o}}$,(Dugundji , 1966).

iii) A is a semi open set in X if and only if , there is an open set U such that $U\subseteq A\subseteq \overline{U}$,(Levine,1963).

Some time we use X to denote the topological space (X ,τ) and we will the symbol ✪ to indicate the end of the proof .

1.1 Remark:

i) The complement of every regular open(closed) set is a regular closed (open) set,(Dugundji , 1966).

ii) Every regular open set is an open set ,(Dugundji , 1966).

iii) Every open set is a semi open set,(Levine,1963).

iv) From (ii) and (iii), we can get that every regular open set is a semi open set .

2.Preliminaries

In this section , we introduce and recall the basic definitions needed in this work.

First, we state the following definition:

2.1 Definition(Willard ,1970): A space X is said to be compact if and only if, every open cover of X has a finite sub cover of X.

Now, we introduce the concept of r-compact in the following definition:

2.2 Definition: A space X is said to be r-compact if and only if, every regular open cover of X has a finite subfamily whose closures cover X .

Next , we recall the following definition which we needed in the following sections.

2.3 Definition(Willard ,1970)**:** A space X is said to be extremely disconnected if and only if , the closure of every open set in X is also open in X.

3.Relationship between r-compact and some types of compact spaces

At once, we recall the definition of some types of compact spaces, as follows:

3.1 Definition(Willard ,1970): A space X is said to be:

i) quasi H- closed if and only if , every open cover of X has a finite subfamily whose closures cover X,(Cameron ,1978).

ii) nearly-compact if and only if, every open cover of X has a finite subfamily , the interiors of the closures of which cover X,(Herrington,1974).

iii) S-closed if and only if, every semi open cover of X has a finite subfamily whose closures cover X,(Thompson,1976).

The above notations are related in the following diagram:

$$\begin{tikzcd} & \text{nearly -compact} \arrow[d, "1"] \arrow[dr, "6"] & \\ \text{compact} \arrow[ur, "3"] \arrow[r, "4"] \arrow[rr, bend right=60, "5"'] & \text{quasi H-closed} \arrow[r, "7"] & \text{r-compact} \\ & \text{S-closed} \arrow[u, "2"] \arrow[ur, "8"'] & \end{tikzcd}$$

It is clear that the implications 1,2,3 and 4 hold . Next, we prove 5,6,7and 8, respectively.

3.2 Theorem: Every compact space is an r-compact space .

Proof: Let X be a compact space and let $\{ V_\alpha \mid \alpha\in\Omega\}$ be a regular open cover of X .Since X is a compact space .So, there exist $V_{\alpha_1},\dots, V_{\alpha_n}$ such that $X=\bigcup_{i=1}^{n} V_{\alpha_i}$ .Since V_{α_i} is a regular open set in X, for each $\alpha_i\in\Omega$ and for each i=1,2,…,n .So, $V_{\alpha_i} = \overline{V_{\alpha_i}}^{\,o}$, for each $\alpha_i\in\Omega$ and for each i=1,2,…,n . Hence , $X=\bigcup_{i=1}^{n} V_{\alpha_i} = \bigcup_{i=1}^{n} \overline{V_{\alpha_i}}^{\,o} \subseteq \bigcup_{i=1}^{n} \overline{V_{\alpha_i}}$ …(*), since $\overline{V_{\alpha_i}} \subseteq X$ for each $\alpha_i\in\Omega$ and for each i=1,2,…,n. Then , $\bigcup_{i=1}^{n} \overline{V_{\alpha_i}} \subseteq X$ …(**). From(*) and(**), we obtain that $X=\bigcup_{i=1}^{n} \overline{V_{\alpha_i}}$ .Therefore, X is an r-compact space.۞

3.3 Theorem: Every nearly-compact space is an r-compact space .

Proof: Let X be a nearly-compact space and let $\{V_\alpha \mid \alpha\in\Omega\}$ be a regular open cover of X .From (1.1)part(ii), we obtain that $\{V_\alpha \mid \alpha\in\Omega\}$ is an open cover of X. Since X is a nearly- compact space . So, there exist $V_{\alpha_1},\ldots,V_{\alpha_n}$ such that $X=\bigcup_{i=1}^{n}\overset{o}{\overline{V_{\alpha_i}}}$. Since $\bigcup_{i=1}^{n}\overset{o}{\overline{V_{\alpha_i}}}\subseteq\bigcup_{i=1}^{n}\overline{V_{\alpha_i}}$ .So, $X\subseteq\bigcup_{i=1}^{n}\overline{V_{\alpha_i}}$ …(*), since $V_{\alpha_i}\subseteq X$ for each $\alpha_i\in\Omega$ and for each i=1,2,…,n. Then , $\bigcup_{i=1}^{n}\overline{V_{\alpha_i}}\subseteq X$ …(**). From(*) and(**), we obtain that $X=\bigcup_{i=1}^{n}\overline{V_{\alpha_i}}$ .Thus, X is an r-compact space.

3.4Theorem: Every quasi H-closed space is an r-compact space .

Proof: Let X be a quasi H-closed space and let $\{V_\alpha \mid \alpha\in\Omega\}$ be a regular open cover of X . From (1.1)part(ii), we conclude that $\{V_\alpha \mid \alpha\in\Omega\}$ is an open cover of X. Since X is quasi H-closed. So, there exist $V_{\alpha_1},\ldots,V_{\alpha_n}$ such that $X=\bigcup_{i=1}^{n}\overline{V_{\alpha_i}}$ .Thus, X is an r-compact space.

3.5 Theorem: Every S-closed space is an r-compact space .

Proof: Let X be an S-closed space and let $\{V_\alpha \mid \alpha\in\Omega\}$ be a regular open cover of X . From (1.1)part(iv), we can get that $\{V_\alpha \mid \alpha\in\Omega\}$ is a semi open cover of X. Since X is S-closed. So, there exist $V_{\alpha_1},\ldots,V_{\alpha_n}$ such that $X=\bigcup_{i=1}^{n}\overline{V_{\alpha_i}}$ .Hence, X is an r-compact space.

Directly from the definition of an extremely disconnected space , we can prove the following theorem:

3.6 Theorem: In an extremely disconnected space X , the following are equivalent:

i) X is r-compact.

ii) X is nearly-compact.

iii) X is quasi H-closed.

4.Main Results

In this section , we prove several properties of r-compact spaces. First , we prove the finite intersection property in the following theorem:

4.1 Theorem: If a space X is an r-compact and extremely disconnected space , then for every family $\{V_\alpha \mid \alpha\in\Omega\}$of regular closed sets in X satisfying $\bigcap_{\alpha\in\Omega}V_\alpha=\varphi$ there is a finite subfamily $V_{\alpha_1},\ldots,V_{\alpha_n}$ with $\bigcap_{i=1}^{n}V_{\alpha_i}=\varphi$.

Proof: Let $\{V_\alpha \mid \alpha\in\Omega\}$ be a family of regular closed sets in X satisfying $\bigcap_{\alpha\in\Omega}V_\alpha=\varphi$ .Then , $\{X-V_\alpha \mid \alpha\in\Omega\}$ is a regular open cover of X. Since X is r-compact .Then, there exist a finite subfamily $V_{\alpha_1},\ldots,V_{\alpha_n}$ such that :

$X=\bigcup_{i=1}^{n}\overline{X-V_{\alpha_i}}=\bigcup_{i=1}^{n}\overline{X\cap V_{\alpha_i}^{c}}=\bigcup_{i=1}^{n}\overline{V_{\alpha_i}^{c}}$. Since $V_{\alpha_i}^{c}$ is a regular open set in X, for each i=1,2,…,n and for each $\alpha_i\in\Omega$. So, $V_{\alpha_i}^{c}$ is an open set , for each i=1,2,…,n and for each $\alpha_i\in\Omega$.And, since X is an extremely disconnected space . So, $V_{\alpha_i}^{c}=\overset{o}{\overline{V_{\alpha_i}^{c}}}=\overline{V_{\alpha_i}^{c}}$, for each i=1,2,…,n and for each $\alpha_i\in\Omega$.That is, $V_{\alpha_i}^{c}=\overline{V_{\alpha_i}^{c}}$, for each i=1,2,…,n and for each $\alpha_i\in\Omega$. Hence, $X=\bigcup_{i=1}^{n}V_{\alpha_i}^{c}$. Therefore, $\bigcap_{i=1}^{n}V_{\alpha_i}=\varphi$.✪

Next, we study the heridatily property of r-compact .

4.2Theorem: Every regular closed subset of an r-compact is r-compact.

Proof: Let X be an r-compact space , F be a regular closed subset of X and let $\{V_\alpha \mid \alpha\in\Omega\}$ be a regular open cover of X ,since F is a regular closed subset of X .Then, F^c is a regular open subset of X . Thus, $\{V_\alpha\cup F^c \mid \alpha\in\Omega\}$ is a regular open cover of X . Since X is an r-compact space .So, there exist $V_{\alpha_1},\dots,V_{\alpha_n}$ such that $X=\bigcup_{i=1}^{n}\overline{V_{\alpha_i}\cup F^{c}}=\left(\bigcup_{i=1}^{n}\overline{V_{\alpha_i}}\right)\bigcup\overline{F^{c}}$. Then, F=F∩X = $F\cap\left[\left(\bigcup_{i=1}^{n}\overline{V_{\alpha_i}}\right)\bigcup\overline{F^{c}}\right]=\left[F\cap\left(\bigcup_{i=1}^{n}\overline{V_{\alpha_i}}\right)\right]\bigcup\left[F\cap\overline{F^{c}}\right]$.So, $F=F\cap\left(\bigcup_{i=1}^{n}\overline{V_{\alpha_i}}\right)$. Hence, $F\subseteq\left(\bigcup_{i=1}^{n}\overline{V_{\alpha_i}}\right)$.That is ,$F=\left(\bigcup_{i=1}^{n}\overline{V_{\alpha_i}}\right)$.So, F is an r-compact space. ✪

Now, to study the continuous property we need for the following lemma:

4.3Lemma: If $f:X\to Y$ is a continuous function of a space X into an extremely disconnected space Y , and if V is a regular open set in Y , then $f^{-1}(V)$is a regular open set in X.

Proof: Since V is a regular open set in Y .Then, from (1.1) part(ii), we obtain that V is an open set in Y, since f is continuous. So, $f^{-1}(V)$is an open set in X. That is , $[f^{-1}(V)]^o=f^{-1}(V)$ ...(*). Now, since Y is an extremely disconnected space and since V is a regular open set in Y .Thus, $V=\overset{o}{\overline{V}}=\overline{V}$ .That is , $V=\overline{V}$ ...(**). $\overline{[f^{-1}(V)]}\subseteq[f^{-1}(\overline{V})]$ because of f is continuous. So, $\overset{o}{\overline{[f^{-1}(V)]}}\subseteq[f^{-1}(\overline{V})]^{o}$. From (**) , we can get that $[f^{-1}(\overline{V})]^o=[f^{-1}(V)]^o$. From (*) we conclude that, $[f^{-1}(V)]^o=[f^{-1}(V)]$. Thus, $\overset{o}{\overline{[f^{-1}(V)]}}\subseteq f^{-1}(V)$ …….(1). In another hand, we have

$f^{-1}(V) \subseteq \overline{[f^{-1}(V)]}$. Then, $[\, f^{-1}(V)\,]^{o} \subseteq \overline{[f^{-1}(V)]}^{\,o}$. Thus, $[\, f^{-1}(V)\,] \subseteq \overline{[f^{-1}(V)]}^{\,o}$ …..(2). Therefore, from (1) and (2) we obtain that $[\, f^{-1}(V)\,] = \overline{[f^{-1}(V)]}^{\,o}$. So, $f^{-1}(V)$ is a regular open set in X. ❂

<u>4.4Theorem:</u> If f :X→Y is a continuous function of an r-compact space X onto an extremely disconnected space Y , then Y is r-compact .

<u>Proof:</u> Let $\{ V_\alpha \mid \alpha\in\Omega\}$ be a regular open cover of Y . Since f is a continuous function and Y is an extremely disconnected space , then from the above lemma we obtain that $\{ f^{-1}(V_\alpha) \mid \alpha\in\Omega\}$ is a regular open cover of X. Since X is r-compact . So, there exists a finite subfamily $f^{-1}(V_{\alpha_1}),\ldots, f^{-1}(V_{\alpha_n})$ such that $X = \bigcup_{i=1}^{n} \overline{f^{-1}(V_{\alpha_i})}$. Thus, $Y = f(\bigcup_{i=1}^{n} \overline{f^{-1}(V_{\alpha_i})}) \subseteq \bigcup_{i=1}^{n} \overline{V_{\alpha_i}}$. Hence, $Y = \bigcup_{i=1}^{n} \overline{V_{\alpha_i}}$. ❂

Directly , from (3.6) we can prove the following corollaries:

<u>4.5 Corollary(1):</u> If f :X→Y is a continuous function of an r-compact space X onto an extremely disconnected space Y, then Y is a quasi H-closed (nearly compact) space.

<u>4.6 Corollary(2):</u> If f :X→Y is a continuous function of a compact (or, nearly-compact, quasi H-closed, S-closed) space X onto an extremely disconnected space Y, then Y is an r-compact (quasi H-closed, nearly compact) space.

<u>4.7 Corollary(3):</u> If f :X→Y is a continuous function of an extremely disconnected and quasi H-closed (or, nearly compact) space X onto an extremely disconnected space Y , then Y is an r-compact (quasi H-closed , nearly compact) space.

Next, we study the converse continuous image of an r-compact space . So, we need the following lemma:

<u>4.8Lemma:</u> Let f :X→Y be an open continuous bijective function of an extremely disconnected space X into a space Y . If V is a regular open set in X , then f (V)is a regular open set in Y.

<u>Proof:</u> Since V is a regular open set in Y and since X is an extremely disconnected space. Then, $V = \overline{V}^{\,o} = \overline{V}$. That is, $V = \overline{V}$. Since f is a continuous function .So, $f(V) = f(\overline{V}) \subseteq \overline{[f(V)]}$. Since V is an open function .Thus, $f(V) = [f(V)]^{o} \subseteq \overline{[f(V)]}^{\,o}$. That is, $f(V) \subseteq \overline{[f(V)]}^{\,o}$…(1). Now, we have $\overline{[f(V)]}^{\,o} \subseteq \overline{[f(V)]}$.Since f is an open bijective function .So, f is a closed function . Hence, $\overline{[f(V)]} \subseteq f(\overline{V}) = f(V)$.Then, $\overline{[f(V)]}^{\,o} \subseteq f(V)$…(2).From (1) and (2) we can get that $\overline{[f(V)]}^{\,o} = f(V)$.That is f(V) is a regular open set in Y. ❂

<u>4.9Theorem:</u> If f :X→Y is an open continuous bijective function of an extremely disconnected space X onto an r-compact space Y , then X is r-compact .

<u>Proof:</u> Let $\{ V_\alpha \mid \alpha\in\Omega\}$ be a regular open cover of X . From (4.8), we conclude that $\{f(V_\alpha) \mid \alpha\in\Omega\}$ is a regular open cover of Y .Since Y is r-compact .Then , there exist $f(V_{\alpha_1}),\ldots, f(V_{\alpha_n})$ such that $Y=\bigcup_{i=1}^{n}\overline{f(V_{\alpha_i})}$. Then, $X=f^{-1}\left(\bigcup_{i=1}^{n}\overline{f(V_{\alpha_i})}\right)$.Since f is a closed function . Thus, $\overline{f(V_{\alpha_i})}=f\left(\overline{V_{\alpha_i}}\right)$ .Thus, $X=\bigcup_{i=1}^{n}f^{-1}\left(f\left(\overline{V_{\alpha_i}}\right)\right)=\bigcup_{i=1}^{n}\overline{V_{\alpha_i}}$ .Therefore, X is r-compact space. ✪

From the above theorem and theorems (3.2),(3.3), (3.4),and(3.5) we can get the following corollary:

<u>4.10 Corollary(1):</u> If f :X→Y is a homomorphism of an extremely disconnected space X onto a compact (or, nearly-compact, quasi H-closed, S-closed) space Y, then Y is an r-compact space.

Directly from theorems (4.4) and(4.9) we can prove the following corollary:

<u>4.11 Corollary(2):</u> An r-compact property is a topological property under an extremely disconnected spaces.

References

Cameron ; D. E. , (1978). Properties of S- Closed Spaces , Proc. Amer. Math. Soc. , 72(3), 581-586 .

Dugundji ;J. ,(1966). Topology, Allyn and Bacon, Boston .

Herrington; L. L. , (1974). Properties of Nearly - Compact Spaces , Proc. Amer. Math. Soc., 45, 431 – 436.

Levine; N. , (1963). Semi-Open Sets and Semi-Continuity in Topological Spaces , Amer. Math. Monthly, 70, 36-41.

Thompson;T., (1976). S-Closed Spaces , Proc. Amer. Math. Soc., 60, 335-338.

Willard ; S., (1970).General Topology , Addison- Wesley Publishing Company, Inc. .