Piecewise Polynomial And Spline Approximation

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Abstract

Let $s \in C^{r-1}[-1,1]$ be a spline of order k, on the interval $[-1,1]$, and $v = 1,2,3$. We estimate the degree of spline approximation of s by peceiwise polynomial of degree $\leq n$. In our second result of this paper we show that inverse theorem interims of higher moduli of smoothness.

Key words : spline approximation , moduli of smoothness , algebraic polynomial approximation .

الخلاصة

لتكن s متعددة حدود ريشية من الدرجة k تمتلك $(v-1)$ من المشتقات المستمرة على الفترة $[-1,1]$ و $v = 1,2,3$. في هذا البحث سنجد درجة التقريب لمتعددة الحدود الريشية باستخدام متعددات الحدود المتقطعة من الدرجة n . في النتيجة الثانية لهذا البحث حصلنا على المتراجحة العكسية بدلالة أعلى مقياس نعومة.

1. Introduction

Kopotun (2003), introduced a paper on k-monotone polynomial and spline approximation in L_P, $0 < p < \infty$ quasi norm . Al-Muhja (2009), discussed the errors of approximation of k-monotone function by k-monotone interpolation . It turns out that any two k-monotone functions f and g, whose graphs intersect each other at certain (sufficiently many) points in $[a,b]$, have to be "close" to each other in the sense that $\|f-g\|_P$, has to be small .

Let $S_r(z_n)$ be the space of all piecewise polynomial function of degree r (order $r+1$) with the knots a partition $z_n = (z_i)_{i=0}^n$, $-1 = z_\circ < z_1 < \ldots < z_{n-1} < z_n = 1$ of the interval $[-1,1]$. In other words $s \in S_r(z_n)$ if , on each interval (z_i, z_{i+1}), $0 \leq i \leq n-1$, s is in Π_r, where Π_r denotes the space of algebraic polynomials of degree $\leq r$.

As usual , $L_p(J)$, $0 < p \leq \infty$, denotes the space of all measurable functions f on J, such that $\|f\|_{L_p(J)} < \infty$, where $\|f\|_{L_p(J)} = \left(\int_J |f(x)|^p \, dx \right)^{1/p}$ if $p < \infty$, and $\|f\|_{L_\infty(J)} = \sup_{x \in J} |f(x)|$. We also denote $\|f\|_p := \|f\|_{L_p[-1,1]}$. For $k = 1,2,3,\ldots$, let

$$\Delta_h^k(f,x,J) = \begin{cases} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x - kh/2 + ih), & if \;\; x \pm kh/2 \in J, \\ 0, & o.w, \end{cases}$$

be the kth symmetric difference , and $\Delta_h^k(f,x) = \Delta_h^k(f,x,[-1,1])$. The kth modulus of smoothness of a function $f \in L_p(J)$ is defined by

$\omega_k(f,J)_p = \omega_k(f,|J|,J)_p$, and $\omega_k(f,t)_p = \omega_k(f,t,[-1,1])_p$.

The Ditzian-Totik modulus of smoothness which is defined for such an f , as follows $\omega^{\phi}_{k,v}(f,t)_P = \sup_{0<h\le\delta}\left\|\phi(\cdot)^v \Delta^k_{h\phi(\cdot)}(f,\cdot)\right\|_P$, where $\phi(x) = (1-x^2)^{1/2}$.

Also , note that $\omega^{\phi}_{0,v}(f,t)_p = \left\|\phi^v f\right\|_p$. One of recent results on spline of approximation due to Zhou (2001), who proved the following theorem .

1.1 Theorem A.

Let $P_n \in \Pi_n[-1,1]$, $n \ge k \ge 1$. Then for any $t \in [0, n^{-2}]$

$\omega_r(P_n,t) \approx t\omega_{r-1}(P'_n,t) \approx ... \approx t^r \left\|P_n^{(r)}\right\|$, $0 \le t \le n^{-2}$, with the equivalence constant depending only on r .

Recently , Y. Hu. And Y. Liu (2005), showed that theorem A can be considerably improved . Their result is stated as follows .

1.2Theorem B.

Let $r \ge 1$, $n \ge 1$, $0 \le t \le n^{-1}$ and $0 < p \le \infty$. then for any $P_n \in \Pi_n$

$$\omega^{\phi}_r(P_n,t)_p \approx t\omega^{\phi}_{r-1}(P'_n,t)_{\phi,p} \approx t^2\omega^{\phi}_{r-2}(P''_n,t)_{\phi^2,p} \approx ... \approx t^r\left\|P_n^{(r)}\right\|_{\phi^r,p} \qquad (1.1)$$

with the equivalence constants depending only on r and q .

2. The Main Result.

Using theorems A, B for equivalence of moduli of smoothness in the L_p spaces , $0 < p \le \infty$, we shall prove the following theorems :

2.1Theorem I.

Let $n \in N$, $k \ge 1$, s be a spline of order k (i.e. , $s \in S_k(I)$), and $v = 1,2,3$. If $s^{(v-1)}$ is continuous , then for any $0 < p \le \infty$,

$$\left\|\phi^v s^{(v)}\right\|_p \le C(p,n)\omega^{\phi}_k(s,n^{-1},I)_p \qquad (2.1)$$

The following Lemma is due to Kopotun (2003) . It will be used to polynomial approximation of splines .

2.2Lemma C.

Let $r \in N$, $k \ge 1$, $0 < p \le \infty$, $I = (a,b)$, and $P \in \Pi_r$, and let s be a spline of order k, with at most $C(k)$ pieces in I . Then $\|P-s\|_p \ge C_{p,r,k}\omega_k(P)_p$.

2.3Theorem II.

Let $k \in N$ be a spline of order k (i.e. , $s \in S_k(I)$), and $I = [-1,1]$ with $c_\circ \ge S$, for some constant $c_\circ$. If s is continuous on I , then for any $0 < p \le \infty$,

$$\omega_{k+1}(s,n^{-1},I)_p \le C_{k,p}\ E_n(s')_p \ .$$

To prove theorem I, we need the following result :

2.4Lemma D. [Liu 2005]

Let $\mu, \xi \in N$ be such that $\mu \geq 7\xi$, and $1 \leq t \leq n-1$ be a fixed . Then there exists a polynomial T_t of degree $\leq 4\mu\, n$, such that the following inequalities hold for $x \in I$

$$\left|T_t^{(k)}(x)\right| \leq c(\mu)\, \psi_t^{\mu}(x) h_t^{-k}, \quad 1 \leq k \leq \xi \quad .$$

PROOF OF THEOREM I :

Let $P_n(x) = \begin{cases} |Q_n - \sigma_i| \breve{S}(x) + \sigma_i(x);\ x \in I \\ Q_n \qquad ;\ o.w. \end{cases}$ is piecewise polynomial of degree $\leq n$ (see Al-Muhja 2009), and we need the spline $s(x) = Q_n - \sigma_i(x) \in S_k(I) \bigcap C^{(v-1)}(I)$, so

$$\left\|\phi^v s^{(v)}\right\|_p = \left\|\phi^v (Q_n - \sigma_i)^{(v)}\right\|_p \leq \left\|\phi^v \left(|Q_n - \sigma_i| \breve{S}(x) - \sigma_i\right)^{(v)}\right\|_p$$

$$\leq C\, n^v |I|^v \sum_{v=0}^{3} \left\|Q_n^{(v)} - \sigma_i^{(v)}\right\|_\infty \left\|\breve{S}^{(3-v)}(x) - \sigma_i^{(v)}\right\|_\infty ,$$

not true $\breve{S}^{(v)}$ we write , we make use of Markov's inequality

$$\left\|\phi^v s^{(v)}\right\|_p \leq C(p,v,n) n^v |I|^{v-k-1} \sum_{v=0}^{3} \left\|Q_n - \sigma_i\right\|_p \left\|\breve{S}(x) - \sigma_i(x)\right\|_p$$

$$\leq C(p,v,n) n^3 |I|^{3-k-1} \left\|Q_n - \sigma_i\right\|_p \leq C_{p,v,n} \|s\|_p ,$$

we choose a constant K to be sufficiently large , such that

$$\left\|\phi^v s^{(v)}\right\|_p \leq K \left\|\Delta_{h,\phi(\cdot)}^k (s,\cdot)\right\|_p \leq C \sup_{0<h\leq n^{-1}} \left\|\Delta_{h,\phi(\cdot)}^k (s,\cdot)\right\|_p = C\, \omega_k^\phi \left(s, n^{-1}, [-1,1]\right)_p ,$$

The Remez inequality (1936) is given in the following theorem .

2.5 Theorem E.

For any $P_n \in \Pi_n$, any measurable $A \in [-1,1]$, with a Lebesgue measure $2 - a\, n^{-2}$ for some $0 \leq a \leq \frac{n^2}{2}$, and $0 < p \leq \infty$, we have

$$\|P_n\|_p \leq C \|P_n\|_{L_p(A)} \qquad (2.2)$$

Remark : If $A \subseteq [-1,1]$, with a Lebesgue measure . Note that , using possible theorem I, to satisfying norm $\|\cdot\|_{L_p(A)}$.

PROOF OF THEOREM II :

Let $k \in N$, $x \in I$, and $0 < h \leq |I|$ be such that $x \pm \frac{(k+1)h}{2} \in I$, and suppose that q_n peceiwise polynomial which was described in proof theorem I, such that $q(\xi) = s(\xi)$, for some $\xi \in I$.

Then , for any $x \in I$, we have

$$\omega_{k+1}(s, I)_p = \sup_{0<h\leq\delta} \left\|\Delta_h^{k+1}(s,\cdot,I)\right\|_p \leq 2^{k+1} \left\|\Delta_h^{k+1}(s,\cdot,I)\right\|_p$$

$$\leq \left\|\Delta_h^{k+1}(s-q_n,\cdot,I)\right\|_p \leq 2^{k+1}|I|\left(\int_{\xi}^{x}\left|(s'(u)-q_n'(u))\right|^p du\right)^{1/p}$$

$$\leq c(k,p)|I|\left\|s'-q_n{}'\right\|_p ,$$

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