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Models of Anomalous Diffusion and its analysis

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Abstract

The anomalous diffusion is characterized by deviation from Gaussian statistics and the absence of a linear time dependency in the mean square displacement. This study investigates anomalous diffusion processes that exhibit a power-law growth in mean square displacement as time progresses.

The first stage is to create the model using random methods, that is, by using random walks. The following statement describes a continuous-time random walk model represented by a series of convolution-type integral equations depicting probability density functions. Fractional differential equations for time and space are derived from the master equations by choosing probability density functions with infinite first and/or second moments.

The obtained model equations are analyzed with respect to elementary boundary value problems in constrained fields. The main focus is on studying elementary boundary value problems related to the generalized fractional time diffusion equation, especially using the fractional Caputo derivative.

This equation applies a well-established maximum principle to stochastic partial differential equations of the elliptic and parabolic type (SPDEs). This concept is used to make preliminary estimates of the answer before using it to prove the uniqueness of the solution. To prove the existence of a solution, a clearly defined generalized solution is first generated using the spectroscopic method. Under certain additional circumstances, a comprehensive solution may be considered a solution in the traditional sense.

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Introduction

Anomalous transport processes provide a natural means of describing the structural and dynamic features of complex systems, which are characterized by a wide range of elementary particles participating in the transport processes. This thesis will mostly examine anomalous diffusion processes[1]. We define anomalous diffusion processes as those processes that no longer follow Gaussian statistics for long periods of time. In particular[2], a deviation from the linear time dependence of the mean squared displacement (MSD) of the particle participating in the anomalous diffusion process can be observed, i.e. the relationship.

$$\overline{X_t^2} \approx \ Kt \tag{1}$$

It is important to observe that the relationship (1) can be understood as the primary characteristic of (gBm)[10], and furthermore, it is a direct consequence of the central limit theorem and the Markovian property of the underlying random process. Contrary to (1), the anomalous diffusion exhibits a non-linear increase in the mean square displacement (MSD) as time progresses[3], as depicted by the following relationship.

$$\overline{X_t^2} \approx K_\alpha t^\alpha \tag{2}$$

The equation suggests that the mean squared displacement does not follow a linear time dependence, as is typically observed in normal diffusion processes that follow Gaussian statistics. instead, in anomalous diffusion processes, the mean squared displacement exhibits a power-law growth, indicating a deviation from the expected behavior. This equation is derived from a stochastic formulation of the model using random walk processes and is based on the analysis of probability density functions. Understanding the behavior of the mean squared displacement in anomalous diffusion processes is important for characterizing and modeling these processes accurately. For cases of anomalous diffusion, the central limit theorem is invalidated and must be substituted by the generalized Levy-Gnedeuko central limit theory.The relationship (2) distinguishes states for the anomalous diffusion. Following [1-15]

- Sub diffusion ($0 < \alpha < 1$)
- Normal diffusion ($\alpha = 1$)
- Super diffusion ($1 < \alpha < 2$)

We investigate the generalized fractional time diffusion equation and its significant variations, namely the polynomial equation and the distributed order equation. The time fractional diffusion equation is mathematically equivalent to the continuous time random walk model, specifically the mean squared displacement (MSD). Where the interval of time between two consecutive jumps varies. However, the magnitude of the variance in jump length remains minimal and directly related to t^α (refer to the provided source).

A derivative of the distributed order is defined as the average value of fractional derivatives inside a specified interval (such as [0; 1]), where each derivative is weighted by a positive weight function $w(t)$. The multi-term time-fractional diffusion equation is a significant and specific instance of the time-fractional diffusion equation of dispersed order. In this scenario, the weight function is expressed as a finite linear combination of $Dirac\ \delta - functions$ with positive weight coefficients.

Continuous-time random walk

The continuous-time random walk (CTRW) is a stochastic process characterized by jumps and used to represent both standard and anomalous diffusion. It is applicable when the time spent at a particular location is significantly longer than the time required to go to a new position, meaning that leaps can be treated as instantaneous occurrences. Montroll and Weiss[2] introduced the Continuous Time Random Walk (CTRW) in the field of physics. Shlesinger [11] recently produced a review that helped increase the popularity of the CTRW. Theoretical, numerical, and empirical investigations on the CTRW have been reviewed by Weiss, Metzler, and Klafter[9].

Within the context of the established random walk model for grey Brownian motion, the random walker makes jumps at each time step $t = 0, t_1 + \ t_2, \dots, t_{n-1} + t_n$, and so on, in a direction chosen randomly. This results in the walker covering a distance of Δx, which represents the lattice constant.

Let $f(x,t)\Delta x$ represent the probability of a random walk from x and $x \ + \ \Delta x$ at time t. The primary equation.

$$f(x, t + \Delta t) = \ \frac{1}{2}\ f(x + \Delta x, t) + \frac{1}{2}\ f(x - \Delta x, t) \tag{3}$$

It can be derived for one-dimensional grey-Brownian motion (gBm), the Taylor expansions

$$f(x,t+\Delta t) = f(x,t) + \Delta t \frac{\partial f}{\partial t} + Q((\Delta t)^2)$$

The equation states that the updated value of $f(x,t + \Delta t)$ is equal to the current value of $f(x,t)$ plus the product of the time step Δt and the partial derivative of f with respect to time $\left(\frac{\partial f}{\partial t}\right)$, the equation is commonly used in numerical methods for solving partial differential equations, where it is known as the forward Euler method, the equation allows for the estimation of the value of f at a later time based on its current value and the rate of change at that point in time.

$$f(x \mp \Delta x, t) = f(x,t) \mp \Delta x \frac{\partial f}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 f}{\partial x^2} + Q((\Delta x)^3)$$

The equation is an approximation that involves the function values at two neighboring points, x and $x + \Delta x$, and their derivatives, the equation is commonly used in numerical methods to approximate the behavior of a function $f(x,t)$ at neighboring points in space. It provides an estimate of the function value at a point $x + \Delta x$ based on the function value and its derivatives at the point x. Lead to

$$\frac{\partial f}{\partial t} = \frac{(\Delta x)^2}{2\Delta t} \frac{\partial^2 f}{\partial x^2} + \Delta x \, Q\left(\frac{(\Delta x)^2}{\Delta t}\right) + Q(\Delta t) \tag{4}$$

The equation that represents a stochastic partial differential equation (SPDE) for a function f with respect to time t and space x and this equation is derived from a stochastic formulation of a model for anomalous diffusion processes, the equation captures the behavior of anomalous diffusion, which is characterized by deviations from Gaussian statistics and a non-linear time dependence of the mean squared displacement the equation is used to model anomalous diffusion processes that exhibit a power-law growth of the mean squared displacement over time its derived from a stochastic random walk model and is solved in the Fourier-Laplace domain the equation is a special case of a time-fractional diffusion equation with the Caputo fractional derivative.

The equation is important for understanding and analyzing anomalous diffusion processes and can be used to study initial-boundary-value problems in bounded domains.

In the continuum limit $\Delta t \to 0$ and $\Delta x \to 0$, this equation becomes the diffusion equation.

$$\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2} \tag{5}$$

provided that the diffusion coefficient

$$D = \lim_{\Delta t \to 0, \Delta x \to 0} \frac{(\Delta x)^2}{2\Delta t}$$

This is a linear diffusion equation. However, if we introduce a non-linear term in the diffusion coefficient, such as $D = k_\alpha (\frac{\partial^2 f}{\partial x^2})^\alpha$, we get a non-linear diffusion equation:

$$\frac{\partial f}{\partial t} = k_\alpha \left(\frac{\partial^2 f}{\partial x^2}\right)^\alpha \tag{6}$$

This equation shows a non-linear dependence of the diffusion coefficient on the second spatial derivative.

In contrast to the random walk model of Brownian motion. Continuous time random walk is based on the idea that there is a certain jump length. In addition, the waiting time elapsed between two successive jumps is governed by a joint probability density function (pdf), which we will refer to as jump pdf. For $\psi(x,t)$, the jump length pdf.

$$\gamma(x) = \int_0^\infty \psi(x,t)\, dt \tag{7}$$

and the waiting time pdf

$$w(t) = \int_{-\infty}^\infty \psi(x,t)\, dx \tag{8}$$

Can be inferred.
The primary attributes of CTRW procedures are the distinctive waiting time.

$$T = \int_0^\infty w(t)t\, dt \tag{9}$$

and the jump length variance

$$\Sigma^2 = \int_{-\infty}^\infty \gamma(x)\, x^2 dx \tag{10}$$

The distinction between CTRW processes lies on their finite or infinite nature. Typically, the following distinct cases are identified[11]]:

- Both T and Σ^2 are finite: Brownian motion (diffusion equation as a deterministic model)
- T divergent, Σ^2 is finite: Sub-diffusion (time-fractional diffusion equation as a deterministic model)
- T is finite, Σ^2 divergent: Levy flight (space-fraction equation as a deterministic model)
- Both T and Σ^2 are infinite: Levy flight (time-space-fraction equation as a deterministic model)

The CTRW model can be characterized by the primary equation in the form of integral equations of the twisted type [8]. Here, we demonstrate the procedure for constructing these equations.
Let $\eta(x,t)$ represent the probability density function (pdf) of a particle arriving at place x during time t.

$$\eta(x,t) = \int_{-\infty}^\infty dx' \int_0^t \eta(x', t')\ \psi(x - x', t - t')dt' + \delta(x)\delta(t) \tag{11}$$

The probability density function, denoted as $f(x;\ t)$, represents the likelihood of being at position x at time t.

$$f(x;\ t) = \int_0^t \eta(x, t')\ \psi(t - t')dt' \tag{12}$$

Where

$$\psi(t) = 1 - \int_0^t w(t')dt' \tag{13}$$

The equation represents a mathematical model for atypical diffusion processes. The function $f(x,t)$ can be evaluated using the probability density function η and the kernel function ψ, which are essential for analyzing anomalous diffusion models and their solutions. The kernel function is employed in formulating the continuous time random walk (CTRW) model within the paper's framework [9]. The selection of the kernel function dictates the characteristics of the random walk process and therefore influences the attributes of the final diffusion model. Distinct kernel functions can result in various forms of anomalous diffusion, such as subdiffusion or superdiffusion, which are distinguished by diverse growth patterns of the mean squared displacement as time progresses. The characteristics of the kernel function, such as its moments, can also impact the mathematical analysis of the resulting model equations, including the determination of whether solutions exist and are unique. Hence, the choice of a suitable kernel function is crucial for accurately representing and studying anomalous diffusion processes[9,10,11]. The equation enables the examination and investigation of anomalous diffusion processes, which do not adhere to Gaussian statistics and demonstrate a departure from the linear time dependence of the mean squared displacement. Comprehending and resolving this equation can yield valuable knowledge on the behavior and characteristics of abnormal diffusion phenomena, which have utility in several domains including finance, economic, and biology.

Now, we convert these equations into the frequency domain by utilizing the Fourier and Laplace transforms. By utilizing the established product property of the Fourier and Laplace transforms, we can determine the Fourier-Laplace transform of the jump probability density function $f(x;\ t)$ from equations above.

$$\hat{\tilde{f}}(k,s) = \frac{1-\tilde{w}(s)}{s}\frac{\hat{f}_0(k)}{1-\hat{\tilde{\psi}}(k,s)} \tag{14}$$

where $\hat{f}_0(k)$ denotes the Fourier transform of the initial condition $f_0(x)$. It is worth to note that a purely probabilistic proof of this equation is given in [7].

let's examine the Continuous Time Random Walk (CTRW), in which the waiting time between jumps becomes infinitely large, while the variance of the distance covered in each jump remains finite. In order to achieve this objective, a specific probability density function (pdf) with a long-tailed waiting time and its asymptotic behavior are considered.

$w(t) \approx A_\alpha(\tau/t)^{1+\alpha} \quad ,t \to \infty \quad ,0 < \alpha < 1$

The formula states that as t approaches positive infinity, the function $w(t)$ can be approximate by A_α times $(\tau/t)^{1+\alpha}$, where$0 < \alpha < 1$.

Here, A_α is a constant and τ is a parameter. The formula suggests that the behavior of $w(t)$ follows a power-law growth with respect to time. The exponent $1 + \alpha$ in the formula indicates the rate of growth, with larger values of α leading to faster growth. Its asymptotics in the Laplace domain can be easily determined by the so-called Tauberian theorem and is as follows[8]:

The Tauberian theorem provides a connection between the asymptotic behavior of a function in the time domain and its behavior in the Laplace domain. In the case of the waiting time PDF $w(t)$ given by $w(t) \approx A_\alpha(\tau/t)^{1+\alpha}$ as $t \to \infty$, we can use the Tauberian theorem to find its Laplace domain asymptotics.

The Laplace transform of a function $w(t)$ is denoted by $\tilde{w}(s)$ and is defined as:

$$\tilde{w}(s) = \int_0^\infty w(t)\, e^{-st} dt$$

Using the given form of $w(t)$, we can compute its Laplace transform:

$$\tilde{w}(s) = \int_0^\infty A_\alpha(\tau/t)^{1+\alpha}\, e^{-st} dt$$

This integral may not have a closed-form solution, but we're interested in its behavior as $s \to 0$, where we can approximate $e^{-st} \approx 1$ for t in a neighborhood of 0. Using this approximation, we have:

$$\tilde{w}(s) = A_\alpha \int_0^\infty (\tau/t)^{1+\alpha}\, dt$$

Now, we evaluate this integral. We can perform a change of variables to simplify the integral:

$$\int_0^\infty (\tau/t)^{1+\alpha}\, dt = \tau^{1+\alpha} \int_0^\infty t^{-(1+\alpha)} dt$$

The integral on the right-hand side is a well-known integral and converges for $0 < \alpha < 1$. Its value is $\Gamma(1-\alpha)$ where Γ is the gamma function.

Therefore, we have:

$\tilde{w}(s) = A_\alpha \tau^{1+\alpha}\Gamma(1-\alpha)$

As $s \to 0$, we can use the small argument approximation for the gamma function

$\Gamma(1-\alpha) \approx \frac{1}{1-\alpha}$, which gives:

$$\tilde{w}(s) = A_\alpha \tau^{1+\alpha}\frac{1}{1-\alpha}$$

Finally, we rewrite this expression in a more convenient form:

$\tilde{w}(s) \approx 1 - (s\tau)^\alpha \quad ,s \to 0$

This is the Laplace domain asymptotic behavior of the waiting time PDF $w(t)$ as $s \to 0$. It shows that the Laplace transform of $w(t)$ exhibits a power-law behavior in s with exponent α as $s \to 0$. This can be assumed without any loss of generality. The Laplace transform of the function w can be easily determined.

$$\tilde{w}(s) \approx \frac{1}{1+\ s^{\alpha}} \tag{15}$$

Together with the Gaussian jump length pdf, the Fourier transform in the form

$$\hat{\gamma}(k) \approx 1 - \sigma^2 k\ , \quad k \to 0 \tag{16}$$

By applying Tauber's theorems to Laplace and Fourier transforms, the final equation can be transformed into a time fractional partial differential equation for large values of t and $|x|$. To derive the asymptotic expression $\hat{\tilde{f}}(k,s)$from the asymptotic expression

$$\hat{\tilde{f}}(k,s) = \frac{1-\ \tilde{w}(s)}{s}\,\frac{\widehat{f_0}(k)}{1-\ \hat{\tilde{\psi}}(k,s)}\ , \qquad as\ s \to 0\ and\ k \to 0 \tag{17}$$

Applying the partial differentiation operator D t to equation (17) transforms the continuous random walk (CTRW) model into an initial value problem for big t and $|x|$. $f(x,0) = f_0(x)$. Time fractional diffusion equation

$$D_t^{\alpha} =\ K_{\alpha}\,\frac{\partial^2 f}{\partial x^2} \tag{18}$$

Practical application

We gave a specific definition of the anomalous diffusion equation driven by CTRW in Eq. (35) and a wide range of random integrals in Eq. (36) in the previous section. These equations can be effectively achieved through simulation with respect to stochastic partial differential equations driven by CTRWs as described in Section II.

The marginal distributions of hops and waiting periods are described in Section II. These conditions may seem strict, but they can be illustrated by [11,14,15]. A random number can be generated from a symmetric stable Lévy probability density using the Laplace transform method developed by Chambers et al [10]. And implemented by McCulloch [8]. The results were obtained using the R programming language and visualized in a 3D diagram. The simulation involved modeling the motion of dust particles at high temperatures using the anomalous dispersion equation.

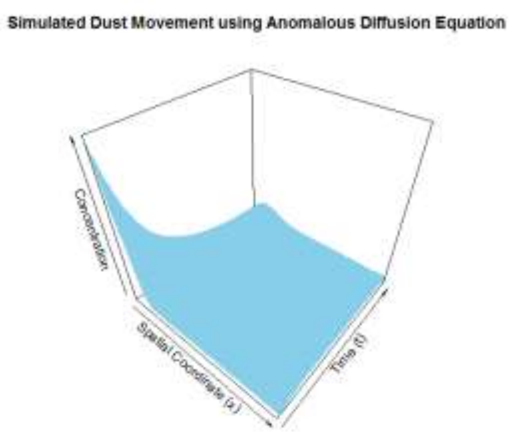

Figure 1: $\boldsymbol{when\ \alpha = -0.5\ , D = 3}$

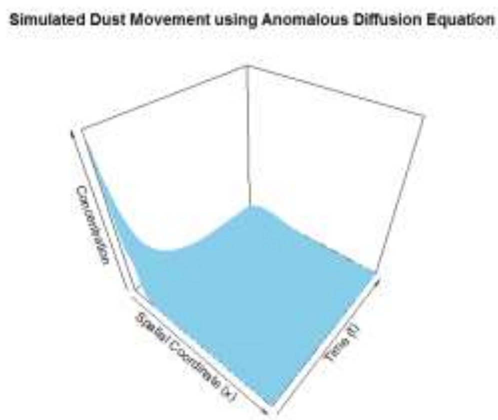

Figure 2: $\boldsymbol{when\ \alpha = 0.3\ , D = 2}$

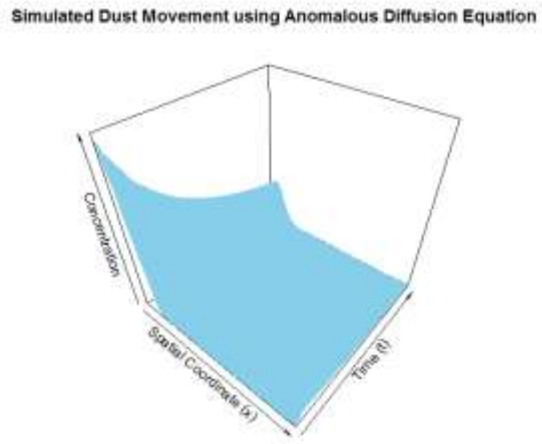

Figure 3: $\boldsymbol{when\ \alpha = 0.5\ , D = 3}$

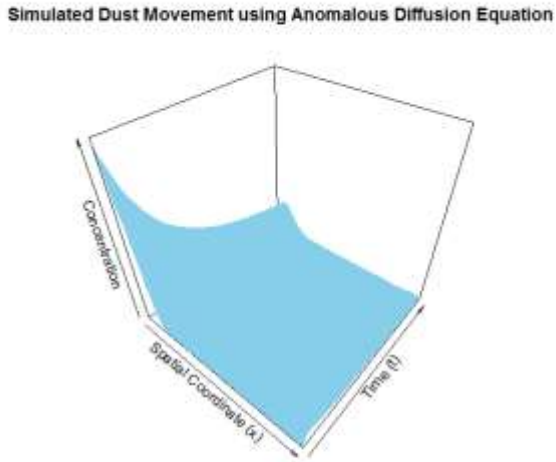

Figure 4: $\boldsymbol{when\ \alpha = 0.6\ , D = 1}$

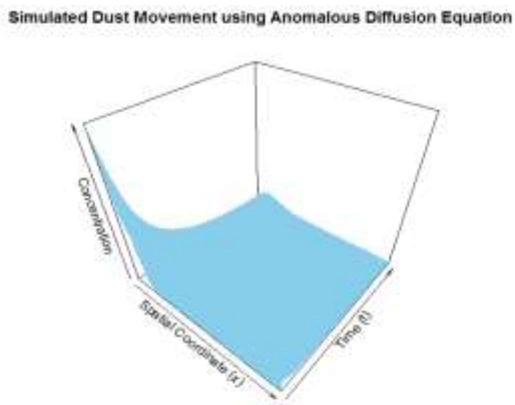

Figure 5: $\boldsymbol{when\ \alpha = 1.9\ , D = 2}$

Summary and conclusions

This brief paper outlines a basic analysis of a stochastic process defined as the product of ggBm, an independent random variable, and a power law with respect to time. Various analytical and numerical observations are being discussed.

The key characteristic of the suggested stochastic process is its ability to bridge the gap between ggBm and CTRW, producing outcomes consistent with experimental observations of air-dust motion.

We view the suggested stochastic technique as a promising model for the complex phenomenon of abnormal diffusion . Future research will focus on systematically studying diffusion processes in financial sciences.

References

[1] W. Wang and E. Barkai, "Fractional advection diffusion asymmetry equation, derivation, solution and application," *Journal of Physics A: Mathematical and Theoretical*, vol. 57, no. 3, p. 035203, Jan. 2024, doi: 10.1088/1751-8121/ad1844.

[2] L. Beghin, L. Cristofaro, and J. Gajda, "Non-Gaussian Measures in Infinite Dimensional Spaces: the Gamma-Grey Noise," *Potential Analysis*, Aug. 2023, **Published**, doi: 10.1007/s11118-023-10099-0.

[3] G. Germano, M. Politi, E. Scalas, and R. L. Schilling, "Stochastic calculus for uncoupled continuous-time random walks," *Physical Review E*, vol. 79, no. 6, Jun. 2009, doi: 10.1103/physreve.79.066102.

[4] Yu. Luchko, "A New Fractional Calculus Model for the Two-dimensional Anomalous Diffusion and its Analysis," *Mathematical Modelling of Natural Phenomena*, vol. 11, no. 3, pp. 1–17, 2016, doi: 10.1051/mmnp/201611301.

[5] W. Bock, S. Desmettre, and J. L. da Silva, "Integral representation of generalized grey Brownian motion," *Stochastics*, vol. 92, no. 4, pp. 552–565, Jul. 2019, doi: 10.1080/17442508.2019.1641093.

[6] R. Metzler, "Superstatistics and non-Gaussian diffusion," *The European Physical Journal Special Topics*, vol. 229, no. 5, pp. 711–728, Mar. 2020, doi: 10.1140/epjst/e2020-900210-x.

[7] M. Grothaus and F. Jahnert, "Mittag-Leffler analysis II: Application to the fractional heat equation," *Journal of Functional Analysis*, vol. 270, no. 7, pp. 2732–2768, Apr. 2016, doi: 10.1016/j.jfa.2016.01.018.

[8] G. Germano, M. Politi, E. Scalas, and R. L. Schilling, "Stochastic calculus for uncoupled continuous-time random walks," *Physical Review E*, vol. 79, no. 6, Jun. 2009, doi: 10.1103/physreve.79.066102.

[9] M. Woringer, I. Izeddin, C. Favard, and H. Berry, "Anomalous Subdiffusion in Living Cells: Bridging the Gap Between Experiments and Realistic Models Through Collaborative Challenges," *Frontiers in Physics*, vol. 8, May 2020, doi: 10.3389/fphy.2020.00134.

[10] F. Di Tullio, P. Paradisi, R. Spigler, and G. Pagnini, "Gaussian Processes in Complex Media: New Vistas on Anomalous Diffusion," *Frontiers in Physics*, vol. 7, Sep. 2019, doi: 10.3389/fphy.2019.00123.

[11] L. Decreusefond and A. S. Üstünel, "Fractional Brownian motion: theory and applications," *ESAIM: Proceedings*, vol. 5, pp. 75–86, 1998, doi: 10.1051/proc:1998014.

[12] M. A. F. dos Santos and L. M. Junior, "Random diffusivity models for scaled Brownian motion," *Chaos, Solitons & Fractals*, vol. 144, p. 110634, Mar. 2021, doi: 10.1016/j.chaos.2020.110634.

[13] R. Ben-Zvi, A. Nissan, H. Scher, and B. Berkowitz, "A continuous time random walk (CTRW) integro-differential equation with chemical interaction," *The European Physical Journal B*, vol. 91, no. 1, Jan. 2018, doi: 10.1140/epjb/e2017-80417-8.

[14] M. J. Saxton, "A Biological Interpretation of Transient Anomalous Subdiffusion. I. Qualitative Model," *Biophysical Journal*, vol. 92, no. 4, pp. 1178–1191, Feb. 2007, doi: 10.1529/biophysj.106.092619.

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نماذج الانتشار الشاذ وتحليله

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المستخلص

يتميز الانتشار الشاذ بالانحراف عن الإحصائيات الغوسية وغياب التبعية الزمنية الخطية في متوسط الإزاحة المربعة. تبحث هذه الدراسة في عمليات الانتشار الشاذة التي تظهر نموًا في قانون القوة في متوسط الإزاحة المربعة مع تقدم الوقت.

المرحلة الأولى هي إنشاء النموذج باستخدام الطرق العشوائية، أي باستخدام مسارات عشوائية. يصف البيان التالي نموذج المشي العشوائي في الوقت المستمر الذي يمثله سلسلة من المعادلات التكاملية من النوع الملتوي التي تصور وظائف الكثافة الاحتمالية. يتم اشتقاق المعادلات التفاضلية الكسرية للزمان والمكان من المعادلات الرئيسية عن طريق اختيار دوال الكثافة الاحتمالية ذات العزوم الأولى و/أو الثانية اللانهائية.

يتم تحليل معادلات النموذج التي تم الحصول عليها فيما يتعلق بمشاكل قيمة الحدود الأولية في المجالات المقيدة. ينصب التركيز الرئيسي على دراسة مسائل القيمة الحدية الأولية المتعلقة بمعادلة انتشار الزمن الكسرية المعممة، وخاصة باستخدام مشتق كابوتو الكسرية.

تطبق هذه المعادلة مبدأ الحد الأقصى الراسخ على المعادلات التفاضلية الجزئية العشوائية من النوع الإهليلجي والمكافئ (SPDEs). ويستخدم هذا المفهوم لإجراء تقديرات أولية للإجابة قبل استخدامها لإثبات تفرد الحل. لإثبات وجود الحل، يتم أولاً إنشاء حل معمم محدد بوضوح باستخدام الطريقة الطيفية. وفي ظل ظروف إضافية معينة، يمكن اعتبار الحل الشامل حلاً بالمعنى التقليدي.

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