The Best Fit Approach for the Muskingum Coefficients X and K and The Computer Programs for Muskingum Method

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Abstract

The maximum linear correlation coefficient is used to determine the Muskingum coefficients X and K instead of the traditional drawing method. Two computer programs, coded in Quick-Basic 71, are presented in this research for the application of Muskingum method. The first program is used to determine the expected outflow hydrograph from a river reach. The second program is for determining the Muskingum coefficients X and K. Through two illustrative examples, it is concluded that using the maximum linear correlation coefficient in determining the coefficients X and K is simpler than the traditional drawing method. Moreover, the outflow hydrograph results from a river reach are closer to the real results. By the use of the constructed programs in this research, the time is saved and the errors that may occur at any step of calculations are prevented.

الخلاصة:

تم استخدام اكبر معامل ارتباط خطي في إيجاد قيم معاملات مسكنكم X و K بدلا من طريقة الرسم التقليدية. تم في هذا البحث تقديم برنامجين، مكتوبين بلغة بيسك السريعة ٧١، لتطبيق طريقة مسكنكم. يستخدم البرنامج الأول في إيجاد الهيدروغراف الخارج المتوقع من قطعة من النهر. ويستخدم البرنامج الثاني لإيجاد قيم المعاملات X و K. ومن خلال مثالين محلولين تم التوصل إلى أن استخدام اكبر معامل ارتباط خطي في إيجاد قيم المعاملات X و K هو ابسط من طريقة الرسم التقليدية. بالإضافة إلى ذلك، فان نتائج الهيدروغراف الخارج من قطعة من النهر تكون قريبة من النتائج الحقيقية. وباستخدام البرنامجين المنشأين في هذا البحث فقد تم توفير الوقت وتجنب الأخطاء التي من الممكن أن تحدث عند أي مرحلة من الحسابات.

1. Introduction

The Muskingum method is a linear model for the hydrologic routing of flow in streams (Linsely & others 1975, Chow & others 1988). For known inflow hydrograph (I) to a river reach and the Muskingum coefficients X and K, the outflow hydrograph (Q) from this reach can be determined by using the following equation, which is known as the routing equation for Muskingum method (Linsely & others 1975):

$$Q_2 = C_1 I_2 + C_2 I_1 + C_3 Q_1 \quad (1)$$

where

$$C_1 = \frac{\Delta t - 2KX}{2K(1-X) + \Delta t} \quad (2\text{-a})$$

$$C_2 = \frac{\Delta t + 2KX}{2K(1-X) + \Delta t} \quad \text{b)-(2}$$

$$C_3 = \frac{2K(1-X) - \Delta t}{2K(1-X) + \Delta t} \quad \text{c)-(2}$$

$$C_1 + C_2 + C_3 = 1 \quad \text{d)-(2}$$

where X and K are known as Muskingum coefficients (Raghunath 2006), Δt is the routing period which is in the same time units as K (Linsley & others 1958, Linsely & others 1975). The routing period is obtained from the inflow hydrograph (Raghunath 2006). For hydrologic routing, the value of X and K are assumed to be specified and constant through the range of flow (Linsely & others 1975, Chow & others 1988).

The application of Muskingum method is illustrated as follows. Values of C_1, C_2, and C_3 are computed by substituting the known values of K, X, and Δt in equations (2-a), (2-b), and (2-c) respectively. Values of I are given, and the products C_1I_2 and C_2I_1 are computed. With an initial value of Q given or estimated, the product C_2Q_1 is calculated and the three products added to obtain Q_2. The computed value of Q_2 becomes Q_1 for the next routing period and another value of Q can be determined. The process continues as long as values of I are known.

These calculations are usually performed manually, which is time consuming, especially when the data is large. Additionally, error may occur at any step of calculations. Therefore, in this research a computer program, Program 1, is constructed to perform all these calculations. Thus, the time is saved and the error that may occur at any step of calculations is prevented.

The Muskingum coefficient, X, is a dimensionless constant for the reach of a river (Raghunath 2006). It indicates the relative importance of the inflow and outflow in determining storage (Linsley & others 1958, Linsely & others 1975, Linsley & Franzini 1979). The constant X varies from 0 to 0.5 (Linsley & others 1958, Linsley & Franzini 1979, Chow and others 1988). A value of zero indicates that the outflow alone determine storage, as in a reservoir (Linsley & others 1958, Linsely & others 1975, Linsley & Franzini 1979). When X = 0.5, inflow and outflow have equal influence on storage (Linsley & others 1958, Linsley & Franzini 1979). In natural channels X ranges from 0.1 to 0.3 (Linsley & Franzini 1979, Raghunath 2006).

The Muskingum coefficient, K, known as the storage constant, is the ratio of storage to discharge and has the dimension of time (Linsley & Franzini 1979, Raghunath 2006). The value of K approximates the travel time of the wave through the reach (Linsely & others 1975, Linsley & Franzini 1979). In the absence of good data, the value of K may be estimated as the observed time of travel of peak flow through the reach (Linsley & others 1958, Chow & others 1988).

If observed inflow and outflow hydrographs are not available for a river reach, the values of X and K may be estimated using Muskingum-Cunge method described in Chow and others 1988. This method is used for determining the values of K and X on the basis of channel characteristics and flow rate in the channel (Chow and others 1988).

When observed inflow and outflow hydrographs are available for a river reach, the values of X and K can be determined. Assuming various values of X and using known values of the inflow and outflow, successive values of the numerator and denominator of the following expression for K can be computed (Linsely & others 1975, Chow & others 1988):

$$K = \frac{\left(\bar{I} - \bar{Q}\right)\Delta t}{X\Delta I + \left(1 - X\right)\Delta Q} \qquad (3)$$

where and are the average inflow and outflow respectively. The computed values of the numerator and denominator are plotted. This usually produces a graph in the form of a loop (Linsley & others 1958, Linsely & others 1975, Linsley& Franzini 1979, Chow and others 1988, Raghunath 2006). The value of X that produces a loop closest to a single line is taken to be the correct value for the reach, and K, according to Eq. (3), is equal to the slope of the line.

This method for determining the values of X and K parameters, which may be called as drawing method, may be considered as time consuming and tedious method. Moreover, there will be no general agreement among the researchers on the values of X and K parameters. This is because the loop closest to a single line is determined by visual interpretation and thus is greatly subjective. In this research, the maximum linear correlation coefficient is used to determine the values of X and K parameters instead of the traditional drawing method. Program 2, presented in this study, is used to determine the values of X and K parameters. Thus, the time is saved, different values of X with small intervals can be used, and the outflow hydrograph results from a river reach will be close to the real results.

The two programs constructed in this research are listed in Appendix 1 and Appendix 2, instead of listing flow charts, in order to make the researchers benefit from them.

Two examples will be considered in this research, and the results are obtained by using the constructed programs.

2. The Best Fit Approach for the Muskingum Coefficients X and K

As said in the introduction, the drawing method for determining the values of Muskingum coefficients X and K may be considered as time consuming and tedious method. In this research, the maximum linear correlation coefficient is used to determine the values of X and K parameters instead of the traditional drawing method.

Observed inflow and outflow hydrographs for a river reach are assumed to be available. The variable z is assumed to represent the denominator of Eq. 3 and the variable y is assumed to represents the numerator of Eq. 3. That is,

$$Z = X\Delta I + (1 - X)\Delta Q \qquad (4)$$

$$y = (\bar{I} - \bar{Q})\Delta t \qquad (5)$$

For any value of the Muskingum coefficient X ($0 \le X \le 0.5$), the average values of inflow ( ), the average values of outflow ( ), the difference of inflow (ΔI), and the difference of outflow (ΔQ) are computed. Substituting these values into Eq.4 and Eq.5, N points (z, y) will be obtained, where n is the number of data and N (N = n - 1) is the number of points. For these points, the linear correlation coefficient is required to be determined. The linear correlation coefficient is computed from the following equation (Gerald 1984, McCUEN 1985, Johnson & Kuby 2004):

$$R = \frac{N\sum zy - \left(\sum z\right)\left(\sum y\right)}{\sqrt{\left[N\sum z^2 - \left(\sum z\right)^2\right]\left[N\sum y^2 - \left(\sum y\right)^2\right]}} \qquad (6)$$

Computing z and y values for every X ($0 \le X \le 0.5$) manually represents a cumbersome method. Therefore, the computer program, Program 2, is constructed in this research to compute the correlation coefficient for every X ($0 \le X \le 0.5$). The value of X that produces the maximum linear correlation coefficient is taken to be the correct value for the reach.

The Muskingum coefficient K is equal to the slope of the best fit line. If the best fit line equation is of the form:

$$y = a + bz \qquad (7)$$

then b represents the slope of the best fit line and is equal to the constant K. Both a and b can be found using the least squares method (Gerald 1984, McCUEN 1985), numerical methods, such as Gauss-Newton method (Burden & Faires 2001, Chapra & Canale 2006), optimization methods, such as Nelder-Mead method (Mathews & Fink 2004), or software, such as grapher, Statistical, or Curve Expert. However, the coefficient (a) in Eq.7 is not required, and the slope of the best fit line, b (= K), can be directly determined using the following equation (Johnson & Kuby 2004):

$$K = \frac{N\sum zy - \left(\sum z\right)\left(\sum y\right)}{\left[N\sum z^2 - \left(\sum z\right)^2\right]} \qquad (8)$$

Program 2 computes K for every value of R.

3. Program 1

Program 1 is presented in Appendix 1. For known inflow hydrograph to a river reach, the time interval Δt, and the values of X and K parameters, the outflow hydrograph from this reach can be determined using the following equation:

$$Q_{j+1} = C_1 I_{j+1} + C_2 I_j + C_3 Q_j \quad (9)$$

where C_1, C_2, and C_3 are defined by equations 2-a, 2-b, and 2-c respectively.

The input and output variables of Program 1 are as follows:

Variable	Description
X	Dimensionless constant
K	Storage constant
DT	The time interval
C_1, C_2, and C_3	Constants defined by equations 2-a, 2-b, and 2-c respectively
N	Number of data
IN	Inflow hydrograph
Z1	$C_1 I_{j+1}$
Z2	$C_2 I_j$
Z3	$C_3 Q_j$
OE	Expected outflow hydrograph (OE = Z1 + Z2 + Z3)

4. Program 2

Program 2 is presented in Appendix 2. Inflow and outflow hydrographs to a river reach are assumed to be known. For any value of the Muskingum coefficient X ($0 \le X \le 0.5$), the variables z and y, defined by the following equations, are computed:

$$Z = X(I_{j+1} - I_j) + (1 - X)(Q_{j+1} - Q_j) \quad (10)$$

$$y = 0.5\Delta t[(I_{j+1} + I_j) - (Q_{j+1} + Q_j)] \quad (11)$$

The linear correlation coefficient for the points (z, y) is computed by Eq.6, and the storage constant, K, is computed by Eq.8.

The input and output variables of Program 2 are as follows:

Variable	Description
N	Number of data
DT	The time interval
IN	Inflow hydrograph
O	outflow hydrograph
X	Dimensionless constant
z	variable, defined by Eq.10
y	variable, defined by Eq.11
R	The linear correlation coefficient
K	Storage constant

Example 1

The inflow hydrograph readings for a stream reach are given below for which the Muskingum coefficients of K=36 hr and X=0.15 apply. Route the flood through the reach and determine the outflow hydrograph. Assume Q_1 equal to I_1= 42 cumec (Raghunath 2006, Example 9.3, pp. 272).

Time (hr)	0	12	24	36	48	60	72	84	96	108	120
Inflow (cumec)	42	45	88	272	342	288	240	198	162	133	110

Time (hr)	132	144	156	168	180	192	204	216	228	240
Inflow (cumec)	90	79	68	61	56	54	51	48	45	42

Solution

The following Table shows the results obtained from Program 1, with that obtained by Raghunath:

Table 1: Stream flow routing by the Muskingum method
(Program 1 and Raghunath results)

Time (hr)	Inflow I (cumec)	C_1I_{j+1} (cumec)	C_2I_j (cumec)	C_3Q_j (cumec)	Outflow (Q) (cumec) (Program 1)	Outflow (Q) (cumec) (Raghunath)
0	42	0.00	0.0	0.0	42.0	42.0
12	45	0.74	13.1	28.2	42.0	42.1
24	88	1.44	14.0	28.3	43.7	44.0
36	272	4.46	27.4	29.4	61.3	62.2
48	342	5.61	84.7	41.2	131.5	132.8
60	288	4.72	106.5	88.4	199.6	200.7
72	240	3.93	89.7	134.2	227.8	233.0
84	198	3.25	74.8	153.1	231.1	234.0
96	162	2.66	61.7	155.3	219.7	221.6
108	133	2.18	50.5	147.6	200.3	201.0
120	110	1.80	41.4	134.6	177.8	178.9
132	90	1.48	34.3	119.5	155.3	155.7
144	79	1.30	28.0	104.4	133.7	133.5
156	68	1.11	24.6	89.9	115.6	115.3
163	61	1.00	21.2	77.7	99.9	99.7
180	56	0.92	19.0	67.1	87.0	86.8
192	54	0.89	17.4	58.5	76.8	76.7
204	51	0.84	16.8	51.6	69.3	69.1
216	48	0.79	15.9	46.6	63.2	63.1
228	45	0.74	15.0	42.5	58.2	58.0
240	42	0.69	14.0	39.1	53.8	53.6

The difference between the outflow results obtained by Program 1 and Raghunath belongs to the C_1, C_2, and C_3 values.

For Program 1: C1= 1.639344E-02 C2= 0.3114754 C3= 0.6721312

For Raghunath: C1= 0.02 C2= 0.31 C3= 0.67

Additionally, the products C_1I_2, C_2I_1, and C_2Q_1 are not approximated by Program 1. Thus, the outflow results obtained by Program 1 are different from that obtained by Raghunath.

Example 2

The inflow and outflow hydrographs for a reach of a river are given below. Determine the value of the Muskingum coefficients K and X for the reach (Raghunath 2006, Example 9.2, pp. 271).

Time (hr)	0	24	48	72	96	120	144	168	192	216
Inflow (cumec)	35	125	575	740	456	245	144	95	67	50
Outflow (cumec)	39	52	287	624	638	394	235	142	93	60

Solution

Program 2 results are X = 0.19 and k = 0.688 while Raghunath results are X = 0.25 and k = 0.7. Figures 1 and 2 show these results.

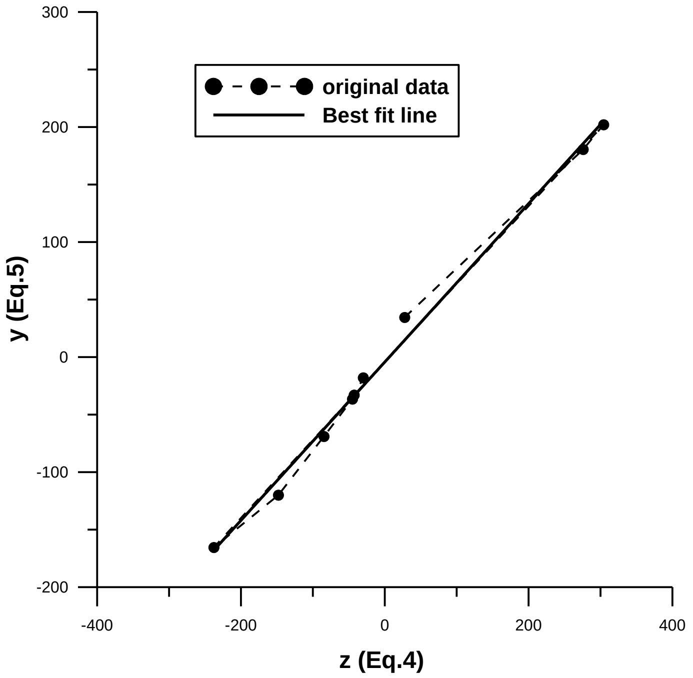

Fig.1: Program 2 results (X = 0.19, k = 0.688)

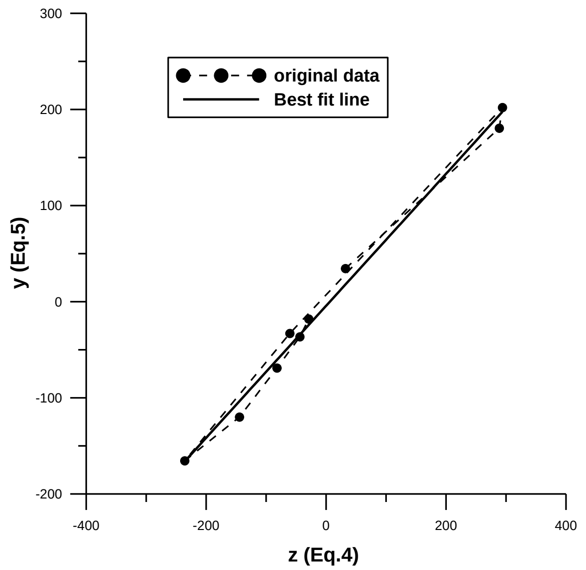

Fig.2: Raghunath results (X = 0.25, k = 0.7)

The linear correlation coefficients for the two cases are computed using Eq.6, and it is found that R= 0.9971 for Program 2 and R= .9958 for Raghunath. Thus, it can be concluded that Program 2 results are better than Raghunath results.

When another inflow and outflow hydrographs for the same river reach are known, the comparison between Program 2 results and Raghunath results will be further clear. The inflow data and X and k values are used to find the expected outflow. The X and k values that produce outflow results closer to the measured outflow will be the best values. Another inflow and outflow hydrographs for the same river reach are not known. Therefore, the residual sum of squares will be used. It is defined by the following equation:

$$E = \sum_{i=1}^{n} \left(MO_i - CO_i \right)^2 \qquad (12)$$

where

E = Residual sum of squares

MO = Measured outflow

CO = Computed outflow

Table 2 shows the computed and measured outflow hydrographs for the case $Q_1 = I_1 = 35$ cumec. The residual sum of squares results are as follows:

Case	E
X=0.19, K = 0.688	824.75
X=0.25, K = 0.7	1158.45

Thus, it can be concluded that Program 2 results are better than Raghunath results.

Table 2: Computed Outflow with Measured Outflow (Q_1 is assumed equal to $I_1 = 35$ cumec)

measured outflow	Computed Outflow X=0.19, K = 0.688	Computed Outflow X=0.25, K = 0.7
39	35	35
52	66.43	63.53
287	279.00	266.18
624	616.59	619.79
638	634.12	647.01
394	391.95	393.76
235	217.68	216.60
142	130.88	130.23
93	87.16	86.98
60	62.15	62.10

Table 3 shows the computed and measured outflow hydrographs for the case Q_1 equal to measured $Q_1 = 39$ cumec.

Table 3: Computed Outflow with Measured Outflow
(Q_1 is assumed equal to measured Q_1 = 39 cumec)

measured outflow	Computed Outflow X=0.19, K = 0.688	Computed Outflow X=0.25, K = 0.7
39	39	39
52	66.65	63.63
287	279.01	266.19
624	616.59	619.79
638	634.12	647.01
394	391.95	393.76
235	217.68	216.60
142	130.88	130.23
93	87.16	86.98
60	62.15	62.10

The residual sum of squares results are as follows:

Case	E
X=0.19, K = 0.688	814.99
X=0.25, K = 0.7	1144.55

Thus, it can be concluded that Program 2 results are better than Raghunath results.

5. Conclusions

The use of the maximum linear correlation coefficient in determining the coefficients X and K is simpler than the traditional drawing method. Moreover, the computed outflow hydrograph results from a river reach are closer to the real results. By the use of the constructed programs in this research, the time is saved and the errors that may occur at any step of calculations are prevented.

6. Recommendations

A. When the inflow and outflow hydrographs data for any reach in Iraqi rivers are available, application of Program 2 is recommended to determine the Muskingum coefficients X and K. Then, for known another inflow hydrograph to this reach, Program 1 is recommended to determine the expected outflow hydrograph from this reach.

B. For known inflow hydrograph to any reach in Iraqi rivers and the Muskingum coefficients X and K, Program 1 is recommended to determine the expected outflow hydrograph from this reach.

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Appendix1: Program 1

```
CLS
X = ?: K = ?: DT = ?
C = 2 * K * (1 - X) + DT
C1 = (DT - 2 * K * X) / C
C2 = (DT + 2 * K * X) / C
C3 = (2 * K * (1 - X) - DT) / C
N = ?
DIM IN(N), Z1(N), Z2(N), Z3(N), OE(N)
FOR I = 1 TO N: READ IN(I): NEXT I
DATA
OE(1) = ?
FOR I = 2 TO N
Z1(I) = C1 * IN(I)
Z2(I) = C2 * IN(I - 1)
Z3(I) = C3 * OE(I - 1)
OE(I) = Z1(I) + Z2(I) + Z3(I)
PRINT Z1(I), Z2(I), Z3(I), OE(I)
NEXT I
END
```

Appendix 2: Program 2

```
CLS
N =?: DT =?
DIM IN(N), O(N), SUIN(N - 1), DFIN(N - 1), SUO(N - 1), DFO(N - 1), z(N-1), y(N-1)
FOR I = 1 TO N: READ IN(I): NEXT I
DATA
FOR I = 1 TO N: READ O(I): NEXT
DATA
FOR X = 0 TO .5 STEP .01
FOR I = 1 TO N - 1
SUIN(I) = IN(I + 1) + IN(I): DFIN(I) = IN(I + 1) - IN(I): SUO(I) = O(I + 1) + O(I): DFO(I) =
O(I + 1) - O(I)
z(I) = X * DFIN(I) + (1 - X) * DFO(I): y(I) = .5 * DT * (SUIN(I) - SUO(I))
SUMz = SUMz + z(I): SUMz2 = SUMz2 + (z(I)) ^ 2: SUMy = SUMy + y(I):
SUMy2 = SUMy2 + (y(I)) ^ 2: SUMzy = SUMzy + z(I) * y(I)
NEXT I
R1 = (N - 1) * SUMzy - SUMz * SUMy: R2 = (N - 1) * SUMz2 - (SUMz) ^ 2
R3 = (N - 1) * SUMy2 - (SUMy) ^ 2: R4 = (R2 * R3) ^ .5: R = R1 / R4: K = R1 / R2
PRINT X, R, K
PRINT
SUMz = 0: SUMz2 = 0: SUMy = 0: SUMy2 = 0: SUMzy = 0
NEXT X
END
```