H_2-Optimal Control Synthesis using State Derivative Feedback

Hazem I. Ali
Control and Systems Engineering Dept.
University of Technology
Baghdad-Iraq
hazemcontrol2001@yahoo.com

Abstract

In this paper, the derivation of H_2 optimal control using state derivative feedback to obtain a new control approach is presented. A control approach similar to linear quadratic regulator (LQR) is applied to find the optimal gain matrices that achieve the desired performance. The effectiveness and robustness of the proposed controller can be shown using the uncertain and under-actuated overhead crane system. The results show that the proposed controller can robustly stabilize the system in the presence of system parameters uncertainty. Further, more desirable time response specifications can be obtained using state derivative feedback H_2 control in comparison to the state feedback H_2 control.

Keywords: H_2 control, robust control, optimal control, state derivative feedback, overhead crane

1. Introduction

The state derivative feedback is very useful and necessary for satisfying the desired performance specifications in some control problems. In some practical cases such as suppression of vibration in mechanical systems, where the main sensors of vibration are accelerometers, the signals of states derivative are easier to obtain than the state signals [1, 2]. Moreover, several promising techniques that uses state derivative feedback for linear control systems have been developed by many researchers [1-6]. There are many approaches based on the state feedback theory have been extended to the area of state derivative feedback. Abdelaziz [4] presented the pole assignment control problem by using state derivative feedback for SISO LTI systems. Further, the state derivative feedback approach has been used in the control design of different systems, *e.g.* control of overhead cranes [6]; car wheel suspension systems [7] and control of cantilever beam [8]. Abdelaziz and Valasek [3] have used the state derivative feedback for the direct solution of the pole placement problem for single input linear systems. Abdelaziz [5] presented an approach to design a robust state derivative feedback controller for LTI multivariable systems. The sensitivity the closed loop system to uncertainty in the system and gain matrix was minimized.

On the other hand, the H_2 optimal control is used in the design of state feedback control by minimizing a quadratic performance index of the system and attenuating the disturbances [9]. In addition, the H_2 optimal control has been used to control various types of systems, *e.g.* two floor building [10], two wheeled inverted pendulum [11] and twin rotor system [12]. In this work, a new design of H_2 optimal control using state derivative feedback is presented. A comparison between state feedback H_2 control and state derivative feedback H_2 control is given for overhead crane system.

The structure of the paper is as follows: Section 2 presents the problem formulation. In section 3, the synthesis of the full state derivative feedback H_2 control is provided. To show the effectiveness of the proposed controller, an illustrative example is given in section 4. Finally, the conclusion is presented in section 5.

2. Problem Formulation

Consider the linear time invariant control system expressed by [13]:

$$\dot{x}(t) = Ax(t) + B_1 d(t) + B_2 u(t)$$

$$e(t) = C_1 \dot{x}(t) + D_{12} u(t)$$

$$z(t) = \dot{x}(t) \tag{1}$$

where $x(t) \in R^n$ represents vector of states, $e(t) \in R^h$ represents the controlled output vector, $z(t) \in R^r$ represents the output vector, $u(t) \in R^m$ represents the control vector and $d(t) \in R^l$ represents the exogenous input vector.

The following assumptions are made [5]:

1. The system matrix A is of full rank
2. (A, B_1) and (A, B_2) are stabilizable.
3. (C_1, A) is detectable.
4. All state derivative measurements are possible.

The objective of this work is to obtain the scalar state derivative feedback control law described by:

$$u(t) = -K\dot{x}(t) \tag{2}$$

This control law assigns the eigenvalues for the closed loop system that stabilize the system and achieve the desirable specifications.

3. Full State Derivative Feedback H_2 Control Synthesis

Equation (1) can be formulated as an H_2 problem as shown in Figure 1.

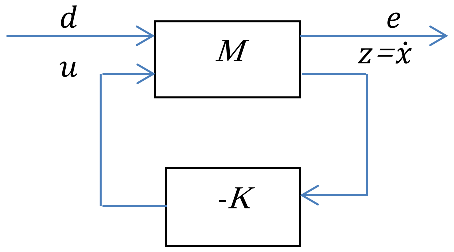


Figure 1: Block diagram of H_2 state derivative feedback [13].

where

$$M = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ I & 0 & 0 \end{bmatrix} \tag{3}$$

Assuming that *d*(*t*) is the white noise vector with unit intensity, then [13]:

$$||T_{ed}||^2_{H_2} = E(e^T(t)e(t)) \tag{4}$$

where T_{ed} represents the overall transfer function from *d*(*t*) to *e*(*t*).

Substituting *e*(*t*) from equation (1) in equation (4), yields:

$$e^T e = \dot{x}^T C_1^T C_1 \dot{x} + 2\dot{x}^T C_1^T D_{12} u + u^T D_{12}^T D_{12} u \tag{5}$$

The minimization of $||T_{ed}||^2_{H_2}$ is equivalent to the solution of the stochastic regulator problem by setting: $Q = C_1^T C_1, N = C_1^T D_{12}$ *and* $R = D_{12}^T D_{12}$ then,

$$E(e^T(t)e(t)) = J\big(\dot{x}(t), u(t)\big) = \int_0^\infty (\dot{x}^T(t)Q\dot{x}(t) + u^T(t)Ru(t) + 2\dot{x}^T(t)Nu(t))dt \tag{6}$$

where $J\big(\dot{x}(t), u(t)\big)$ represents the cost function to be minimized, $Q \in Q^{n\times n}$ represents symmetric positive semidefinite state weighting matrix, and $R \in R^{m\times m}$ is a symmetric positive definite control weighting matrix.

To achieve the stabilizing control with the desired dynamic behavior, the objective function in equation (6) is to be minimized. Since equation (6) is similar to the LQR objective function except the performance depends on state derivative instead of state, it can be rewritten as:

$$J(\dot{x}(t), v(t)) = \int_0^\infty (\dot{x}^T(t)Q_m\dot{x}(t) + v^T(t)Rv(t))dt \tag{7}$$

where

$$Q_m = Q - NR^{-1}N^T \tag{8}$$

$$v(t) = u(t) + R^{-1}N^T\dot{x}(t) \tag{9}$$

Consequently, the system in equation (1) will be rewritten as:

$$\dot{x}(t) = A_m x(t) + B_2 v(t) + B_1 d(t) \tag{10}$$

where $A_m = A - B_2R^{-1}N^T$

In term of *v*(*t*) and from equation (9), the control law is:

$$v(t) = -K_m\dot{x}(t) \tag{11}$$

where $K_m = K - R^{-1}N^T$

By substituting equation (11) in equation (10), the system equation will be:

$$\dot{x}(t) = A_m x(t) - B_2K_m\dot{x}(t) + B_1 d(t) = A_n x(t) + B_1 d(t) \tag{12}$$

where

$A_n = (I + B_2K_m)^{-1}A_m$, I is $n \times n$ identity matrix and the matrix $(I + B_2K)$ is assumed of full rank.

By substituting equation (11) in equation (7), the objective function will be:

$$J\big(\dot{x}(t), v(t)\big) = \int_0^\infty \big(\dot{x}^T(t)(Q_m + K_m^T RK_m)\dot{x}(t)\big)dt \tag{13}$$

Suppose that a constant positive semidefinite symmetric matrix *P* that satisfy equation (13) can be obtained,

$$\dot{x}^T(t)(Q_m + K_m^T RK_m)\dot{x}(t) = -\frac{d}{dt}\big(x^T(t)Px(t)\big) = -\dot{x}^T(t)Px(t) - x^T(t)P\dot{x}(t) \tag{14}$$

Therefore, the performance index will be obtained as:

$$J\big(\dot{x}(t), v(t)\big) = \int_0^\infty \Big(\dot{x}^T(t)Q_m\dot{x}(t) + \big(K_m\dot{x}(t)\big)^T R\big(K_m\dot{x}(t)\big)\Big)dt = -x^T(t)Px(t)|_0^\infty = -x^T(\infty)Px(\infty) + x^T(0)Px(0) \tag{15}$$

Suppose that the closed loop system is asymptotically stable, then $x(\infty) \to 0$. Therefore the performance index is determined in terms of initial conditions and matrix P as:

$$J = x^T(0)Px(0) \tag{16}$$

From equation (12), the following relationship can be obtained:

$$x(t) = A_n^{-1}\dot{x}(t) + B_1 d(t),\ A_n^{-1} = A_m^{-1}(I + B_2 K_m) \tag{17}$$

then equation (14) can be rewritten as:

$$\dot{x}^T(t)(Q_m + K_m^T R K_m)\dot{x}(t) = -\dot{x}^T(t)(PA_n^{-1} + A_n^{-T}P)\dot{x}(t) \tag{18}$$

By comparing the two sides of equation (18), we obtain:

$$PA_n^{-1} + A_n^{-1}P + K_m^T R K_m + Q_m = 0 \tag{19}$$

According to the second method of Lyapunov, if A_n^{-1} is stable matrix, there exists a positive definite matrix P that satisfies equation (19). Substituting equation (17) in equation (19), one can obtain:

$$P[A_m^{-1}(I + B_2 K_m)] + [A_m^{-1}(I + B_2 K_m)]^T P + K_m^T R K_m + Q_m = 0 \tag{20}$$

$$PA_m^{-1} + A_m^{-T}P + PA_m^{-1}B_2 K_m + K_m^T B_2^T A_m^{-T} P + K_m^T R K_m + Q_m = 0 \tag{21}$$

Since R is positive definite symmetric matrix, then $R = T^T T$, where T is nonsingular matrix. Hence equation (21) will be rewritten as:

$$PA_m^{-1} + A_m^{-T}P + PA_m^{-1}B_2 K_m + K_m^T B_2^T A_m^{-T} P + K_m^T T^T T K_m + Q_m = 0 \tag{22}$$

By reformulating equation (22), the following equation can be obtained:

$$PA_m^{-1} + A_m^{-T}P + (TK_m + T^{-T}B_2^T A_m^{-T}P)^T (TK_m + T^{-T}B_2^T A_m^{-T}P) - PA_m^{-1}B_2 R^{-1} B_2^T A_m^{-T} P + Q_m = 0 \tag{23}$$

The minimization of the objective function requires the minimization of the following term with respect to K_m:

$$\dot{x}^T(t)(TK_m + T^{-T}B_2^T A_m^{-T}P)^T (TK_m + T^{-T}B_2^T A_m^{-T}P)\dot{x}(t) \tag{24}$$

The minimum of equation (24) occurs when it equals to zero, then

$$TK_m + T^{-T}B_2^T A_m^{-T}P = 0 \tag{25}$$

The optimal gain matrix K_m is:

$$K_m = -T^{-1}T^{-T}B_2^T A^{-T}P = -R^{-1}B_2^T A_m^{-T}P \tag{26}$$

Finally, the optimal stabilizing control is:

$$v(t) = -K_m\dot{x}(t) = R^{-1}B_2^T A_m^{-T}P\dot{x}(t) \tag{27}$$

Substituting equation (27) in equation (9) yields:

$$u(t) = R^{-1}B_2^T A_m^{-T}P\dot{x}(t) - R^{-1}N^T\dot{x}(t) \tag{28}$$

then

$$K = -R^{-1}[B_2^T(A - B_2 R^{-1}N^T)^{-T}P - N^T] \tag{29}$$

For closed loop system, the state derivative feedback H_2 control is:

$$\dot{x}(t) = A_c x(t) + B_c r(t) + B_1 d(t)$$

$$y(t) = Cx(t) \tag{30}$$

where

$$A_c = (I + B_2 K)^{-1}(A - B_2 C),$$

$B_c = (I + B_2 K)^{-1}B_2$ and C represents the output matrix.

The steps for the design can be summarized as follows:

1. Check the system rank (The system matrix A must be of full rank).
2. Check the system stabilizablity and detectability.
3. Check the possibility of state derivative measurements.
4. Apply equation (29) to find the gain matrix.
5. Find the system response using equation (30).

4. Illustrative Example

To show the effectiveness of the proposed controller, the overhead crane system shown in Figure 2 is considered. This system is inherently nonlinear, unstable, non-minimum phase and an under-actuated mechanical system (two degrees of freedom must be controlled by only one control signal) [14, 15].

The crane system is described by the following state space model:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\\ \dot{x}_4(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & \frac{-b_1-\mu_1}{m_t} & \frac{gm_l}{m_t} & \frac{b_2+\mu_2}{lm_t}\\ 0 & 0 & 0 & 1\\ 0 & \frac{b_1+\mu_1}{lm_t} & -\left(\frac{g}{l}+\frac{gm_l}{lm_t}\right) & -\left(\frac{b_2+\mu_2}{m_l l^2}\right)-\left(\frac{b_2+\mu_2}{m_t l^2}\right)\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{k_u}{m_t}\\ 0\\ -\frac{k_u}{lm_t}\end{bmatrix}u$$

$$y(t) = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix} \quad (31)$$

where $x_1(t) = x$ (trolley position), $x_2(t) = \dot{x}$ (trolley velocity), $x_3(t) = \theta(t)$(swing angle) and $x_4(t) = \dot{\theta}(t)$(swing velocity).

The nominal system parameters are taken as follows [15]: g (gravitational acceleration)= $10\,\frac{m}{s^2}$, *l* (pendulum length)=0.4 *m*, m_t (trolley mass)=2.4 *kg*, m_l (load mass)=0.23 *kg*, b_1 (cart friction coefficient)=0.05 *Ns/m*, b_2 (load reaction friction coefficient)=0.005 *Ns/m*, μ_1 (cart coulomb friction force)=1.6 *N*, μ_2 (load reaction coulomb torque)=0.1 *Nm*, k_u (motor constant)=8 *N/volt*. Figure 3 shows the system states trajectories when state feedback H_2 control and state derivative feedback H_2 control are applied. Suppose that the system initial conditions are *x*(0)=[1 0.5 0.2 0.1]. It shows that the response obtained using state derivative feedback H_2 control is fast with small oscillation amplitudes in comparison to that obtained using state feedback H_2 control. At the same time, a low control effort can be obtained using state derivative H_2 control as shown in Figure 4. The resulting feedback gain matrices in cases, state feedback H_2 control and state derivative feedback H_2 control respectively are:

K= [1.4076 0.7387 0.0073 -0.0140] and,

K= [1.1489 1.0707 0.2806 -0.0718].

On the other hand, to show the effectiveness of H_2 state feedback controller and the proposed state derivative H_2 control in tracking control, a step input has been applied to the system. Figure 5 shows the time response for the system with H_2 state feedback controller. It shows that the trolley position reaches the steady state in about 4 sec.. and the swing angle deviates between -0.18 and 0.12 degree. Further, a low control effort has been obtained. In case of H_2 state derivative feedback controller, it is shown that the trolley position reaches the steady state in about 4 sec.. with a deviation in the swing angle between -0.08 and 0.02 degree as shown in Figure 6. This means that a small deviation was obtained using state derivative H_2 controller. Also, a low control effort can be obtained.

To validate the robustness of the proposed control, a 50% variation in system parameters *l* (pendulum length), m_t (trolley mass) and m_l (load mass)) has been considered. Figure 7 shows the system time response when the system parameters are changed. It is clear that the proposed control can robustly stabilize the system.

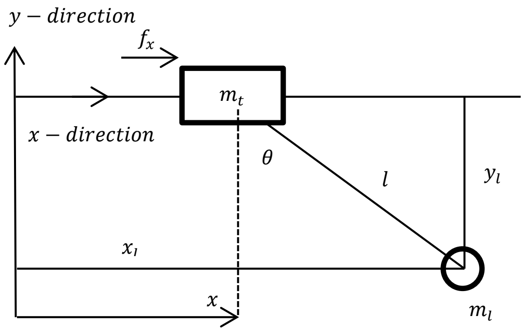
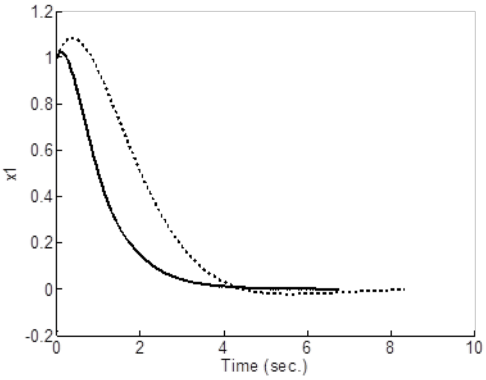


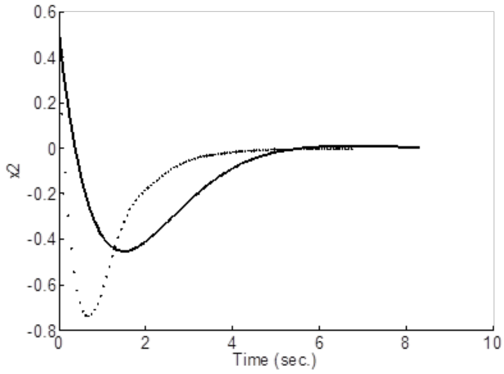
Figure 2: Overhead Crane equivalent [14].

5. Conclusion

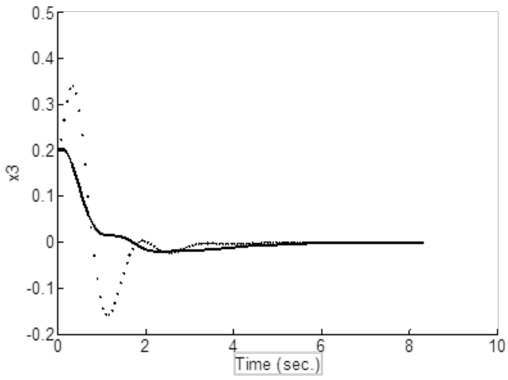
The H_2 optimal control using state derivative feedback has been derived in this work. To validate the effectiveness of the proposed control, an overhead crane which is an underactuated and uncertain mechanical system was considered. It was shown that the proposed control could robustly stabilize the system in the presence of system parameters uncertainty. Moreover, a more desirable time response specification has been achieved with low control effort in comparison to the state feedback H_2 control.



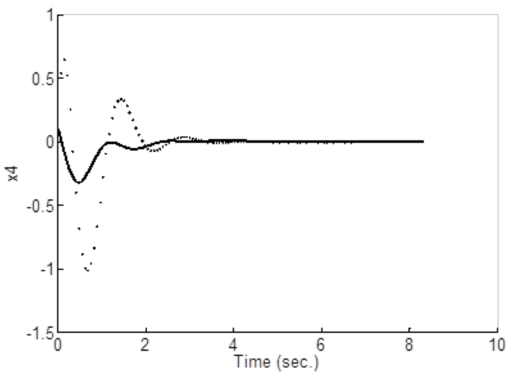
(a) $x_1(t)$



(b) $x_2(t)$



(c) $x_3(t)$



(d) $x_4(t)$

Figure 3: System state trajectories using state feedback H_2 control (dotted line) and state derivative feedback H_2 control (solid line).

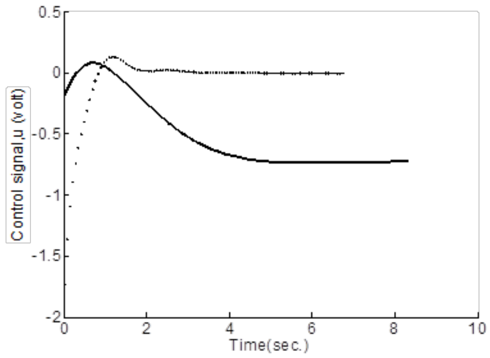
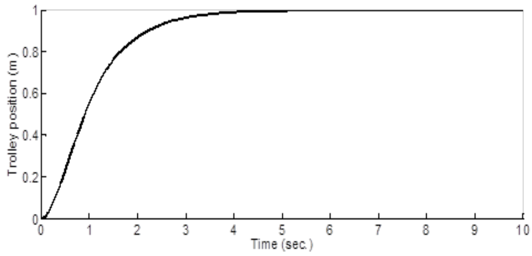
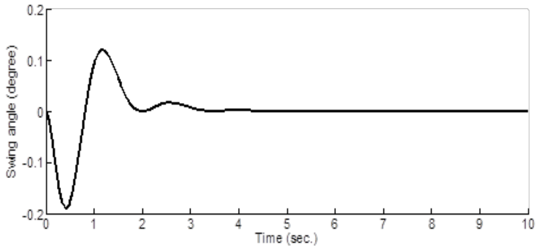


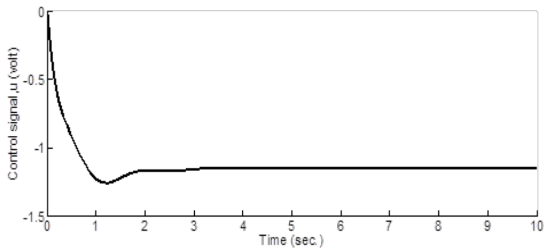
Figure 4: System control signal with state feedback H_2 control (dotted line) and state derivative feedback H_2 control (solid line)



(a) Trolley position

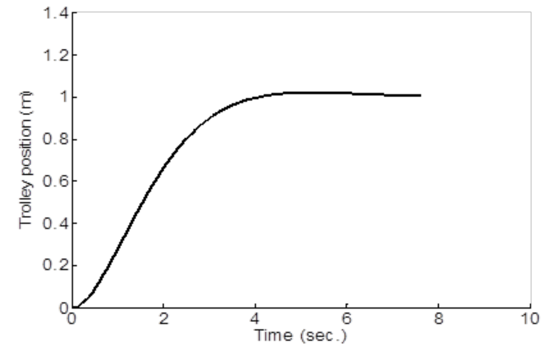


(b) Swing angle

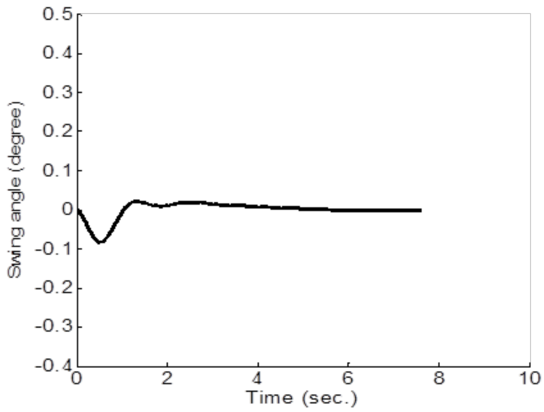


(c) Control signal

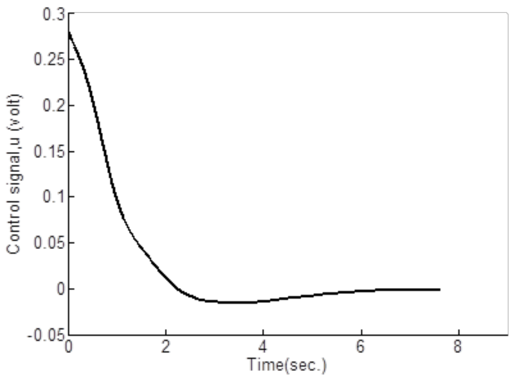
Figure 5: System time response with state feedback H_2 control.



(a) Trolley position

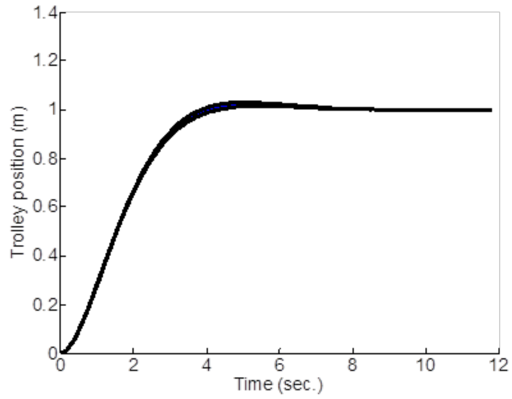


(b) Swing angle

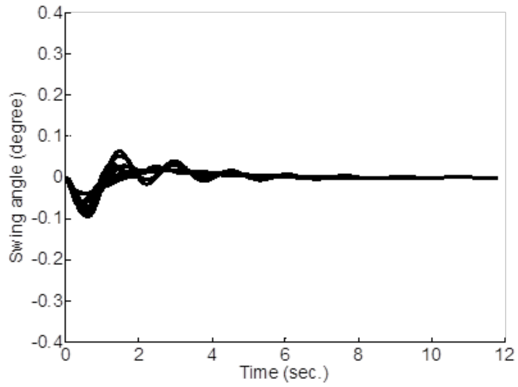


(c) Control signal

Figure 6: System time response with state derivative feedback H_2 control.



(a) Trolley position



(b) Swing angle

Figure 7: System time response with 50% variation in system parameters using state derivative feedback H_2 control.

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تصميم المسيطر الأمثل H2 باستخدام مشتقة حالات التغذية الراجعة

ا.م.د. حازم ابراهيم علي
قسم هندسة السيطرة والنظم
الجامعة التكنولوجية
العراق - بغداد

خلاصة

هذا البحث يقدم تصميم للسيطرة المثلى (H_2) باستخدام (state derivative feedback). طريقة السيطرة مشابهه الى (LQR) المستخدم لايجاد القيم المثلى للمصفوفات والتي تحقق المواصفات المطلوبة. لاظهار فعالية ومتانة المسيطر المقترح تم تطبيقه على نظام المرفاع ذو المعاملات المتغيرة. بينت النتائج فعالية المسيطر المقترح في السيطرة المتينة على استقرارية النظام بوجود التغيرات في معاملات النظام. بالاضافة الى ذلك يمك الحصول على مواصفات افضل باستخدام (derivative feedback H_2 control) بالمقارنة مع (state feedback H_2 control)