Journal of Babylon University/Pure and Applied Sciences/ No.(1)/ Vol.(18): 2010

On Lipschitz Continuity Of Harmonic Quasiregular Maps On The Unit Ball In R^n

Ahmed Wady Shehab
Math Dep, Science Collage, Kufa University

Abstract

In this work, we study Lipschitz continuity of $\phi : S^{n-1} \to R^n$. Lipschitz continuity of $\phi : S^{n-1} \to R^n$ implies Lipschitz continuity of its harmonic extension $u = P[\phi] : B^n \to R^n$, provided u is a quasiregular map. The analogous statement is true for Holder continuity of $\phi : S^{n-1} \to R^n$ without the assumption of quasiregular of its harmonic extension $u = P[\phi] : B^n \to R^n$.

الخلاصة

في هذا العمل درسنا استمرارية ليبشتز الى $\phi : S^{n-1} \to R$. استمرارية ليبشتز الى $\phi : S^{n-1} \to R$ تؤدي الى استمرارية ليبشتز الى توسيعها التوافقي $u = P[\phi] : B^n \to R^n$ على شرط ان يكون u تطبيق شبه منتظم.استمرارية هولدر الى $\phi : S^{n-1} \to R$ تؤدي الى استمرارية هولدر الى توسيعها التوافقي $u = P[\phi] : B^n \to R^n$ بدون ان نفرض ان u هي تطبيق شبه منتظم .

1. Introduction and Preliminaries

Let G be an open connected subset of the Euclidean space R^n , $n \geq 2$,and let $f : G \to R^n$ be a continuous mapping of Sobolev class $W^{1,n}_{10c}(G)$.Then f is said to be quasiregular if there is a number $K \in [1, \infty)$ such that the inequality $\left|f^{/}(x)\right|^n \leq KJ_f(x)$ is satisfied almost everywhere (a.e)in G .Here , $f^{/}(x)$ denotes the formal derivative of f at x ,i.e the $(n \times n)$-matrix of partial derivatives of the coordinate functions f_j of $f = (f_1,..., f_n)$. Further, $\left|f^{/}(x)\right| = \max\{\left|f^{/}(x)h\right| : |h| = 1\}$ and $J_f(x)$ is the Jacobian determinant of f at x.The smallest $K \geq 1$ in the above inequality is called the outer dilataion of f and is denoted by $K_O(f)$.If f is quasiregular ,then the inner dilatation of $K_I(f)$ is the smallest constant $K \geq 1$ in the inequality $J_f(x) \leq K_I(f^{/}(x))^n$, $\left|f^{/}(x)\right| = \min\{\left|f^{/}(x)h\right| : |h| = 1\}$.The maximal dilatation is the number $K(f) = \max\{K_O(f), K_I(f)\}$, and one says that f is K-quasiregular if $K(f) \leq K$.

For the dimension $n = 2$ and $K = 1$,the class of K-quasiregular mappings agrees with that of the complex analytic functions.Injective quasiregular mappings in dimensions $n \geq 2$ are called quasiconformal .For $n = 2$ and $K \geq 1$,a K-quasiregular mappings $f : G \to R^2$ can be represented in the form $\varphi o w$,where $w : G \to G^{/}$ is a $K-$ quasiconformal homeomorphism and $\varphi : G^{/} \to R^2$ is analytic function .There is no such representation in dimension $n \geq 3$.Note that for all dimensions $n \geq 2$ and $K > 1$ the set of those points where a $K-$ quasiconformal mapping is not differentiable map non empty and the behavior of the mapping may be very curious at the points of this set ,which has also Lebesgue measure zero . Thus ,there is substantial difference between the two cases $K > 1$ and $K = 1$.For higher dimension

$n \geq 3$ the theory of quasiregular mappings is essentially different from the plane case $n = 2$ .There are several reasons for this :

a) there are neither general existence theorems nor counterpart of power series expansions in higher dimensions;

b) the usual methods of function theory are not applicable in the higher dimensional setup;

c) in the plane case the class of conformal mappings is very rich ,while in higher dimensions is very small;

d) for dimension $n \geq 3$ the branch set (i.e the set of those points at which the mapping fails to be local homeomorphism) is more complicated than in two – dimensional case ;for instance ,it dose not contain isolated points.

A restriction on the behavior of increase of a function. If for any points x and x' belonging to an interval $[a,b]$ the increase of a function satisfies the inequality $|f(x) - f(x')| \leq M|x - x'|^{\alpha}$,(*) where $0 < \alpha < 1$ and M is constant ,then one says that f satisfies Lipschitz condition of order α on $[a,b]$ and writes $f \in Lip\alpha, f \in LipM^{\alpha}$ or $f \in H^{\alpha}(M)$ .Every function that satisfies a Lipschitz condition with some $\alpha > 0$ on $[a,b]$ is uniformly continuous on $[a,b]$, and functions that satisfy a Lipschitz condition of order $\alpha = 1$ are absolutely continuous .A function that has abounded derivative on $[a,b]$ satisfies a Lipschitz condition on $[a,b]$ with any $\alpha \leq 1$.The Lipschitz condition (*) is equivalent to the condition $\omega(\delta, f) \leq M\delta^{\alpha}$ where $\omega(\delta, f)$ is the modulus of continuity of f on $[a,b]$.In the case $0 < \alpha < 1$ the condition also called Holder condition of order α .

L-space A space X of arbitrary elements for which ,associated with certain sequences $\{a_n\}_{n=1}^{\infty}$ in the space ,there is an $a \in X$, called the limit of the sequence and denoted $a = \lim_{n\to\infty} a_n$,such that :

1-If $\lim_{n\to\infty} a_n = a$ and $\{a_n\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}$,then $\lim_{k\to\infty} a_{nk} = a$ and

2-If $a_n = a$,for all n ,then $\lim_{n\to\infty} a_n = a$. $\{a_n\}$ is called a convergent sequence and is said to converge to a . L^* – space A space X which is an L-space and also satisfies: If $\{a_n\}_{n=1}^{\infty}$ dose not converge to a ,then there is a subsequence $\{a_{nk}\}_{k=1}^{\infty}$ for which no further subsequence converges to a .

Poisson kernel the family of functions P_r, given by the formula :

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2} = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} .$$

The values of a real harmonic function on the unit disk can be computed using convolution formula involving values of the harmonic function on the boundary of the unit disk and the Poisson kernel . Poisson kernels can be written for other domain :For example , the $S^O = \{x \in V^* : |x(s)| \leq 1, \text{for all } s \in S\}$,where V^* is the dual space of V .

It is known, even for $n = 2$, that Lipschitz continuity of $\phi : T \to C$ where $T = \{z \in C : |z| = 1\}$, does not imply Lipschitz continuity of $u = P[\phi]$. In fact $u = P[\phi]$ is Lipschitz continuous iff the Hilbert transform of $\frac{d}{d\theta}\phi(e^{i\theta})$ (which is define almost everywhere and bounded since ϕ is Lipschitz) is also in $L^{\infty}(T)$ (see Zygmund 1952).

Here , for any $n \geq 2$,

$$P[\phi](x) = \int_{S^{n-1}} P(x,\xi)d\sigma(\xi) \ , \ x \in B^n \ ,$$

Where $P(x,\xi) = \dfrac{1-|x|^2}{|x-\xi|^n}$ is the Poisson kernel for the unit ball $B^n = \{x \in R^n : |x| < 1\}$, $d\sigma$ is the normalized surface measure on the unit sphere S^{n-1} and $\phi : S^{n-1} \to R^n$ is a continuous mapping .

The situation is different for C^α (or Holder) continuous $\phi : S^{n-1} \to R^n$, $0 < \alpha < 1$, i.e , for ϕ satisfying $|\phi(\xi) - \phi(\eta)| \leq C|\xi - \eta|^\alpha$. In that case Holder continuity of ϕ implies Holder continuity of its harmonic extension $u = P[\phi]$, (see [Dyakonov1997,Rickman1993]). In the case $n = 2$ it's a classical result ,following from Privalovs theorem (see Zygmund 1952).

2. Main Result

In this section we state and prove the main result of this paper .The following theorem appears in (see Matelejevic. 2006) without proof we give the proof for completeness .

Theorem (2.1)

Assume $\phi : S^{n-1} \to R^n$ satisfies the Lipschitz condition

$$|\phi(\xi) - \phi(\eta)| \leq L|\xi - \eta|, \xi, \eta \in S^{n-1} \ ,$$

And assume $u = P[\phi] : B^n \to R^n$ is K- quasiregular .Then

$$|u(x) - u(y)| \leq C^{/}|x - y|, x, y \in B^n \ ,$$

Where $C^{/}$ depends on L,K and n only.

Proof: Before we give the proof we must know that Kalaj obtained a related result, but under additional assumption of $C^{1,\alpha}$ regularity of ϕ ,(see Kalaj 2007).

The main part of the proof is the estimate of the tangential derivatives of u , and in that part quasiregularity plays no rolc. We choose $x_0 = r\xi_0 \in B^n, r = |x|, \xi_0 \in S^{n-1}$ .Let $T = T_{X_0} rS^{n-1}$ be the $n-1$- dimensional tangent plane at x_0 to the sphere rS^{n-1}. We want to prove that

(1) $\quad \|D(u|T)(x_0)\| \leq C(n)L$.

Without loss of generality we can assume $\xi_0 = e_n$ and $x_0 = re_n$ .By a simple calculation

$$\frac{\partial}{\partial x_j} P(x,\xi) = \frac{-2x_j}{|x-\xi|^n} - n(1-|x|^2)\frac{x_j - \xi_j}{|x-\xi|^{n+2}} \quad .$$

Hence , for $1 \leq j < n$ we have

$$\frac{\partial}{\partial x_j} P(x_0,\xi) = n(1-|x_0|^2)\frac{\xi_j}{|x_0-\xi|^{n+2}} \ .$$

It is important to note that this kernel is odd in ξ (with respect to reflection $(\xi_1,...\xi_1,...\xi_n) \to (\xi_1,... -\xi_j,...\xi_n)$), a typical fact for kernels obtained by differentiation. This observation and differentiation under integral sign gives ,for any $1 \le j < n$,

$$\frac{\partial u}{\partial x_0}(x_0) = n(1-r^2)\int_{S^{n-1}} \frac{\xi_i}{|x_0-\xi|^{n+2}}\phi(\xi)d\sigma(\xi)$$

$$= n(1-r^2)\int_{S^{n-1}} \frac{\xi_i}{|x_0-\xi|^{n+2}}(\phi(\xi)-\phi(\xi_0))d\sigma(\xi) \ .$$

Using the elementary inequality $|\xi_i| \le |\xi-\xi_0|, (1 \le j < n, \xi \in S^{n-1})$ and Lipschtiz continuity of ϕ we get

$$\left|\frac{\partial u}{\partial x_i}(x_0)\right| \le Ln(1-r^2)\int_{S^{n-1}} \frac{|\xi_i||\xi-\xi_0|}{|x_0-\xi|^{n+2}}d\sigma(\xi) \le Ln(1-r^2)\int_{S^{n-1}} \frac{|\xi-\xi_0|^2}{|x_0-\xi|^{n+2}}d\sigma(\xi)$$

.

In order to estimate the integral, we split S^{n-1} into tow subsets $E = \{\xi \in S^{n-1} : |\xi-\xi_0| \le 1-r\}$ and $F = \{\xi \in S^{n-1} : |\xi-\xi_0| > 1-r\}$ .since $|\xi - x_0| \ge 1-|x_0|$ for all $\xi \in S^{n-1}$ we have

$$\int_E \frac{|\xi-\xi_0|^2}{|x_0-\xi|^{n+2}}d\sigma(\xi) \le (1-r^2)^{-n-2}\int_E |\xi-\xi_0|^2 d\sigma(\xi) \le (1-r^2)^{-n-2}\int_0^{1-r} \rho^2\rho^{n-2}d\rho \le \frac{2}{n+1}(1-r)^{-1}$$

On the other hand, $|\xi-\xi_0| \le C_n|\xi - x_0|$ for every $\xi \in F$,so

$$\int_F \frac{|\xi-\xi_0|^2}{|x_0-\xi|^{n+2}}d\sigma(\xi) \le C_n^{n+2}\int_F |\xi-\xi_0|^{-n} d\sigma(\xi) \le C_n^{/}\int_{1-r}^{2} \rho^{-n}\rho^{n-2}d\rho \le C_n^{/}(1-r)^{-1}$$

Combining those two estimation we get

$$\left|\frac{\partial u}{\partial x_i}(x_0)\right| \le LC(n)$$

For $1 \le j < n$.Due to rotational symmetry, the same estimate holds for every derivative in any tangential direction. This establishes estimate (1). Finally, *k*-quasiregularity gives

$$\|D_u(x)\| \le LKC(n) \ .$$

Now the mean value theorem gives Lipschitz continuity of u.

We conclude by noting that ,for each $n \ge 2$, there is a Lipschitz continuity map $\phi : S^{n-1} \to R^n$ such that $u = P[\phi]$ is not Lipschitz continuous .We first briefly recall a well known example in the plane .Let $f(z) = \sum_{n=1}^{\infty}\frac{z^2}{n^2}$ and let $u(z) = \mathrm{Re}(z)$.Then

$$u(z) = \sum_{n=1}^{\infty}\frac{r^n \cos n\theta}{n^2} \text{ and } zf^{/}(z) = \sum_{n=1}^{\infty}\frac{z^n}{n} = -\log(1-z) \ .$$

So , $zf^{/}(z) = -\log|1-z| - iv(z)$,where $\pi/2 < v(z) < \pi/2$. Hence $d_\theta u(z) = v(z)$ is a bounded harmonic function while its harmonic conjugate $rd_r u(z) = \mathrm{Re}\, zf^{/}(z)$ is not .This implies that $u(e^{i\theta})$ is Lipschitz continuous on the unit circle , while $u(z)$ is not Lipschitz continuous in the disc However ,u has the following weaker property :

(2) $\left|u(z') - u(z'')\right| \le C\left(\left|z' - z''\right| + \left\|z'\right| - \left|z''\right\| \log \frac{1}{\left\|z'\right| - \left|z''\right\|}\right)$.

This gives a counterexample in any dimension $n \ge 2$.

Example(2.2)

Set $\phi(x_1, x_2,.., x_n) = ((u(x_1 + ix_2,..., x_n), x \in S^{n-1}$ .Then ϕ is a Lipschitz continuous map on S^{n-1},while its harmonic extension $U = P[\phi]$ is not Lipschitz continuous on the unit ball.

It is clear that $U(x) = P[\phi](x) = (u(x_1 + ix_2), x_2,..., x_n), x \in B^n$,is not Lipschitz continuous since $u(x_1 + ix_2)$ is not Lipschitz continuous on the disc .Proving Lipschitz continuity of ϕ on S^{n-1} reduces to checking Lipschitz continuity of $u(x_1 + ix_2) = u(x_1, x_2)$ on S^{n-1} .Choose $x' = (z'', w'')$ on S^{n-1} where $z' = (x_1', x_2')$, $w' = (x_3',...x_n')$, $z'' = (x_3'',..., x_n'')$. Then, using (2),

$$\left|U(x') - U(x'')\right| = \left|u(z') - u(z'')\right| \le C(d + \delta \log \frac{1}{\delta})$$

Where $d = \left|z' - z''\right|$ and $\delta = \left\|z'\right| - \left|z''\right\|$. On the other hand, we have

$$\left|x' - x''\right|^2 = d^2 + \left|w' - w''\right|^2 \ge d^2 + \left(\left|w'\right| - \left|w''\right| \right)^2$$

But

$$\left(\left|w'\right| - \left|w''\right|\right)^2 = (\sqrt{1 - \left|z'\right|^2} - \sqrt{1 - \left|z''\right|^2})^2 \ge 2\delta - \delta^2 .$$

Therefore for small $\delta > 0, \left|x' - x''\right| \ge \sqrt{\delta}$. Since $\delta \log \frac{1}{\delta} = 0(\sqrt{\delta})$ and, obviously, $\left|x' - x''\right| \ge d$, we have proven Lipschitz continuity of $u(x_1, x_2)$on the unit sphere S^{n-1}.

References

Alexer ,S .Bourdon ,P., and Ramey ,W.1992 :Harmonic Function Theory .-Spring – Verlag, New York .

Anderson G.A, and M .Vuorinen .1997 .Conformal Invariant ,Inequalities and Quasiconformal Mappings .Wily , New York.

Dyakonov, K.M. 1997 :Equivalent Norms on Lipschitz –Type Spaces of Holomorfhic Functions .Acta .Math.143-167.

Kalaj ,D.2007:On Harmonic Quashiconformal Mapping –Ann. Acad. Sci. Fenn. Math. 33.

Matelejevic ,M 2006 :Distorion of Harmonic Function and Harmonic Qushiconformal Quashi-isometry .-Roumaine Math.Pures Appl.711-722

Nolder, C.A., And D.M. Oberlin .1988: Moduli of Continuity and a Hardy – Littlewood Theorm. In Complex Analysis ,Joensuu 1987,Lecture Notes in Math .1351, Springer ,Brline,265-272.

Rickman ,S. 1993: Quashiregular Mappings –Spriger Verlag ,Berline.

Rudin ,W.1980.Function Theory in the Unit Ball in C^n . Spriger Verlag ,Berline

Zygmund ,A. 1952:Trigonometrical Series. Chelsea Publishing Co., 2nd edition ,New York.