Test Statistic Of One Way Model Under Order Restriction

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Abstract

We have studied different procedures for testing equality of fixed effects against the alternative that there are order restriction type , one way model for the score test , general score test and Likelihood Ratio test .Tows cases have been considered with know and unknown variances .To get the critical points, the simulation technique was used.

Introduction

Many situations occur in statistical inference where the prior information of an ordinal kind exists, i.e. when the data are arranged in ordered groups, the mean value of a random variable is assured to change monotonically with the ordering of the groups. For example, in a dosage response experiment, the probability of response is usually an increasing function of dose level. It is then reasonable to take account of the order restrictions in making inferences about the group means, such as point or interval estimations or significance tests.

It is possible to make better estimates or perform more powerful tests when the information of the prior knowledge is fully utilized than when it is ignored. Taking shapes or order restrictions into account can improve the efficiency of statistical analysis by reducing the error or expected error estimates or by increasing the power of test procedures, provided that the hypothesized order restriction actually holds.

The purpose of the theses

The main purpose of the theses is to derive the test statistics for Likelihood Ratio test , The score test and general score test for one way model. Tows cases have been considered with know and unknown variances .To get the critical points, the simulation technique was used.

The Models and Assumptions

The classical general linear is given by

$$Y = M\beta + \varepsilon \qquad (1)$$

With the following six assumptions about the model:

1- $E(Y) = b\beta$ Where M is an $N \times k$ design matrix of known (i.e., observable

And non-random) quantities and β is a $k \times 1$ vector of unknown parameters

Ranging over a k-dimensional Euclidean space.

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2- $Cov(Y) = \Delta \in D$ **Where** D **has a rich linear structure .That is,**

$D = [\Delta_0 : \theta \in \theta]$ **Where** $\Delta_0 = \sum_{i=0}^{k} \theta_i V_i$ **and** θ **contains a nonempty open set**

G of k+1 dimensional Euclidean space.

3- $V_0 = I \in D$

4- Δ_0 **Is a positive definite for all** $\theta \in \Theta$ **.**

5- β **And** θ **are functionally independent.**

6- The linear structured in (2) is commutative.

7- The range of the matrix M is an invariant subspace of Δ_0 **for all** θ **.**

8- ε **Is a** $N \times 1$ **vector of the error, which are uncorrelated random variables**

With expected vector of values 0 and variance-covariance matrix V.

Often the components of the vector error are assumed to be independent and normally distributed. Having observed the value of M and Y, the vector β **and the components of V are estimated [Rady (1991)].**

The one way Model

In this model, we assume that we have k independent samples from the normal population with distinct unknown means μ_i **and variance** σ^2**, and that we have** n_i **observations on the** i^{th} **population.**

Let M **in eq(1) equal to**

$$\begin{bmatrix} I_{n_1} & 0 & \dots & 0 \\ 0 & I_{n_2} & \dots & 0 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & \dots & I_{n_k} \end{bmatrix}, \beta = \begin{bmatrix} \mu_1 \\ \mu_2 \\ . \\ . \\ . \\ \mu_k \end{bmatrix} \text{ and } V_0 = I \ , \Delta = \sigma^1 I$$

Letting y_{ij} **denote the** j^{th} **observation on the** i^{th} **population. The model in details can written as**

$$y_{ij} = \mu_i + \varepsilon_{ij} \ , \text{j=1,...,n}_i \ \ \text{i=1,...,k} \ , \text{N=}\sum_{i=1}^{k} n_i \qquad \textbf{(2)}$$

Where μ_i **is unknown parameter,** ε_{ij} **is the random error, with mean 0 and variance** σ^2**,** y_{ij} **is random variable with mean** μ_i **and variance** σ^2**.**

The hypothesis of equality is :

$H_0: \ \mu_1 = \mu_2 = \dots = \mu_k = \mu$ **(Say)**

The hypothesis of the means satisfy the order restriction can be written as:

$H_1: \ \mu$ **has an order restriction**

The Model and Its Estimation

The parameter of the model (2) was estimated as follows:

Let $\mu' = [\mu_1, \mu_2, \dots, \mu_k]$**, the likelihood function was given as:**

$$L(Y / \mu,\sigma^2) = (2\pi\sigma^2)^{-N/2} \prod_{i=1}^{k} \alpha_i^{-n/2} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{k} \frac{1}{\alpha_i} \sum_{j=1}^{n_i} (y_{ij} - \mu_i)^2 \right\} \quad (3)$$

$$LogL(Y / \mu,\sigma^2) = -\frac{N}{2}\log(2\pi) - \frac{N}{2}\log\sigma^2 - \frac{1}{2}\sum_{i=1}^{k} n_i \log\alpha_i - \left\{ \frac{1}{2\sigma^2} \sum_{i=1}^{k} \frac{1}{\alpha_i} \sum_{j=1}^{n_i} (y_{ij} - \mu_i)^2 \right\}$$

Under H_0

$$LogL(Y / \mu,\sigma^2) = -\frac{N}{2}\log(2\pi) - \frac{N}{2}\log\sigma^2 - \frac{1}{2}\sum_{i=1}^{k} n_i \log\alpha_i - \left\{ \frac{1}{2\sigma^2} \sum_{i=1}^{k} \frac{1}{\alpha_i} \sum_{j=1}^{n_i} (y_{ij} - \mu)^2 \right\}$$

$$\frac{\partial \log L(Y / \mu,\sigma)}{\partial \mu} = -\left\{ \frac{1}{2\sigma^2} \sum_{i=1}^{k} \frac{1}{\alpha_i} \sum_{j=1}^{n_i} (y_{ij} - \mu) x - 2 \right\}$$

By setting the derivatives to zero we got

$$\frac{1}{\sigma^2} \sum_{i=1}^{k} \frac{1}{\alpha_i} \sum_{j=1}^{n_i} y_{ij} = \frac{1}{\sigma^2} \sum_{i=1}^{k} \frac{1}{\alpha_i} n_i \hat{\mu}$$

$$\hat{\mu} = \frac{\sum_{i=1}^{k} \frac{1}{\alpha_i} \sum_{j=1}^{n_i} y_{ij}}{\sum_{i=1}^{k} \frac{n_i}{\alpha_i}} \quad \textbf{Then} \quad \hat{\mu} = \frac{\sum_{i=1}^{k} w_i y_i}{\sum_{i=1}^{k} w_i} \quad (4)$$

Where $\bar{y}_i = \frac{\sum_{j=1}^{n_i} y_{ij}}{n_i}$ and $w_i = \frac{n_i}{\alpha_i}$

To obtain the maximum value without the restrictions under H_0**, We got**

$$\frac{\partial \log L(Y / \mu,\sigma^2)}{\partial \mu_i} = -\frac{1}{\sigma^2 \alpha_i} \left\{ \sum_{j=1}^{n_i} y_{ij} - n_i \hat{\mu}_i \right\}$$

And by setting the derivatives to zero yields that

$$\hat{\mu}_i = \frac{\sum_{j=1}^{n_i} y_{ij}}{n_i} = \bar{y}_i$$

Under H_1**, the MLE of** μ_i is $\hat{\mu}_i^*$ **the isotonic regression of the vector** $\bar{Y} = (\bar{y}_1, \bar{y}_2, ..., \bar{y}_k)$ **with weights vector** $W = (w_1, w_2, ..., w_k)$ **and the quasi-order** $\leq$ **which determine** H_1**.**

$$\frac{\partial \log L(Y / \mu,\sigma^2)}{\partial \sigma^2} = -\frac{N}{2\sigma^2} + \left[\frac{2}{4(\sigma^2)^2} \sum_{i=1}^{k} \frac{1}{\alpha_i} \sum_{j=1}^{n_i} (y_{ij} - \hat{\mu})^2 \right]$$

Setting the derivatives to zero it yielded that

$$\frac{N}{\sigma^2} = \frac{1}{(\sigma^2)^2} \sum_{i=1}^{k} \frac{1}{\alpha_i} \sum_{j=1}^{n_i} (y_{ij} - \hat{\mu})^2 \ , \ \hat{\sigma}_0^2 = \frac{\sum_{i=1}^{k} \frac{1}{\alpha_i} \sum_{j=1}^{n_i} (y_{ij} - \hat{\mu})^2}{N} \quad (5)$$

And under H_1 the maximum likelihood estimates were $\hat{\mu}_i^*$ and

$$\hat{\sigma}^{*2} = \frac{\sum_{i=1}^{k} \frac{1}{\alpha_i} \sum_{j=1}^{n_i} (y_{ij} - \hat{\mu}_i^*)^2}{N} \qquad (6)$$

Tests of Hypothesis

Many of the methods of statistical inference are derived from the problem of comparing several normal populations. It is often desirable to test the null hypothesis that the means are equal, and the alternative is either unrestricted or has very stringent restrictions placed upon it. In application, a researcher may believe a prior that their ordering is known, or more generally that the means are isotonic with respect to a known quasi-order on the index set.

The significant level (α) represents the probability of type I error where

$$P_r(U \geq C / H_0) = \alpha \qquad (7)$$

Where U is a real value test statistic computed from data when testing the null hypothesis H_0 against the alternative H_1 , and C is the critical value.

The power value of the test is defined as:

$$P_r(U \geq C / H_1) = 1 - \beta \qquad (8)$$

Where U is a real valued test statistic computed from data under the alternative Hypothesis H_1, and β is the probability of type II error.

The most popular test considered in literature is the likelihood ratio test (LRT). Likelihood method is the primary approach used in isotonic inference. Although LRT is usually satisfactory in the unrestricted case, this is not always the case for restricted case. For restrictions due to a simple order, likelihood inference is satisfactory. However, for other restrictions such that the simple tree order type, umbrella order, or stochastic order, the likelihood method seems to have shortcomings, Barlow, et al. (1972), Shi (1988), Robertson et al. (1988), Cohen and Sackrowitz (1996), Robertson (1998), Pan and Khattree (1999), Cohen et al. (2000).

There are other tests that can be used, such as, the score test (ST) Kotz et al. (1981), and silvapuile and sirvapulre (1995)] and the generar score test (GST) Robertson, et al. (l988), and silvapulle and silvapulle (1995).

The Score Test (ST)

The score test (ST) which was originated from Silvery's Lagrange multiplier approach is used to test $H_0 : \sum(\gamma) = 0$, where γ is the vector of unknown parameter ($(\mu_1, \mu_2, ..., \mu_k \ , \sigma_1^2, \sigma_2^2, ..., \sigma_k^2)$ against any alternative hypothesis . The score test takes the from:

$$T = (S)' \Omega^{-1} (S) \qquad (9)$$

Where: $S = \frac{\partial \log L(y_1, y_2, ..., y_k / \gamma)}{\partial \gamma_i}$ **is the score vector under the null hypothesis:** Ω^{-1} **: denoted the inverse of the fisher information matrix of** γ **under** H_0 **i.e. for a likelihood function that is twice differentiable with respect to** γ**, define the observed information matrix** $\Lambda(\gamma)$ **to be the matrix with** $(i, j)^{th}$ **entry .**

$$\Lambda_{i,j}(\gamma) = \frac{-\partial^2 \log L(y_1, y_2, ..., y_k / \gamma)}{\partial \gamma_i \partial \gamma_j}$$

And assuming the $\Lambda_{i,j}(\gamma)$ **have finite expectation, define the information matrix to be the matrix with** $(i, j)^{th}$ **entry** $\Omega_{i,j}(\gamma) = E[\Lambda_{i,j}(\gamma)]$**.**

We derived score test (ST) under the two assumptions, known and unknown variance.

By applying the score test define in equation (9) we got the test statistic for the case of known variance as:

$$T = \sum_{i=1}^{k} w_i \{\bar{y}_i - \hat{\mu}\}^2 \qquad \textbf{(10)}$$

<u>Proof :</u>

$$\frac{\partial \log L(Y / \mu)}{\partial \mu_i} = \frac{1}{\sigma_i^2}\left\{\sum y_{ij} - \mathrm{n}_i \; \mu_i\right\}$$

$$\frac{\partial^2 \log L(Y / \mu)}{\partial \mu_i \partial \mu_i} = -\frac{n_i}{\sigma_i^2}$$

$$\frac{\partial^2 \log L(Y / \mu)}{\partial \mu_i \partial \mu_i} = 0$$

$$E\left(-\frac{\partial^2 \log L(Y / \mu)}{\partial \mu_i \partial \mu_i}\right) = \frac{n_i}{\sigma_i^2}$$

$$T = \left[\frac{1}{\sigma_1^2}\left\{\sum_{j=1}^{n_1} y_{1j} - n_1\mu_1\right\} \frac{1}{\sigma_2^2}\left\{\sum_{j=1}^{n_2} y_{2j} - n_2\mu_2\right\} \cdots \frac{1}{\sigma_k^2}\left\{\sum_{j=1}^{n_k} y_{kj} - n_k\mu_k\right\}\right]$$

$$\begin{bmatrix} \frac{\sigma_1^2}{n_1} & 0 & \cdots & 0 \\ 0 & \frac{\sigma_2^2}{n_2} & \cdots & 0 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & . & \frac{\sigma_k^2}{n_k} \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_1^2}\left\{\sum_{j=1}^{n_1} y_{1j} - n_1\mu_1\right\} \\ \frac{1}{\sigma_2^2}\left\{\sum_{j=1}^{n_2} y_{2j} - n_2\mu_2\right\} \\ . \\ . \\ . \\ \frac{1}{\sigma_k^2}\left\{\sum_{j=1}^{n_k} y_{kj} - n_k\mu_k\right\} \end{bmatrix}$$

$$T=\sum_{i=1}^{k}\frac{1}{n_i\sigma_i^2}\left\{\sum_{j=1}^{n_i}y_{ij}-n_i\hat{\mu}_i\right\}^2=\sum_{i=1}^{k}\frac{1}{n_i\sigma_i^2}\{n_i\bar{y}_i-n_i\hat{\mu}_i\}^2$$

$$=\sum_{i=1}^{k}\frac{n_i}{\sigma_i^2}\{\bar{y}_i-\hat{\mu}_i\}^2=\sum_{i=1}^{k}w(\{\bar{y}_i-\hat{\mu}_i\}^2$$

To obtain the power of function of score test, we got the critical point under the alternative hypotheses. The power function for the score test had the form:

$$1-\beta=\sum_{i=1}^{k}w_i\left\{\bar{y}_i-\hat{\mu}_i^*\right\}^2 \quad (11)$$

Case of unknown variance

The test statistic was obtained as:

$$T=\sum_{i=-1}^{k}\frac{n_i}{\alpha_i\hat{\sigma}_0^2}[\bar{y}_i-\hat{\mu}]^2 \quad (12)$$

Proof :

(A) The Score Vector :

$$\partial\log L(Y/\mu,\sigma^2)/\partial\mu_i=\frac{1}{\alpha_i\sigma^2}\sum_{j=1}^{n_i}(y_{ij}-\mu_i)=b_i$$

$$\partial\log L(Y/\mu,\sigma^2)/\partial\sigma^2=-\frac{N}{2\sigma^2}+\frac{1}{2(\sigma^2)^2}\sum_{i=1}^{k}\frac{1}{\alpha_i}(y_{ij}-\mu_i)^2$$

$$=\frac{1}{2(\sigma^2)^2}[\sum_{i=1}^{k}\frac{1}{\alpha_i}\sum_{j=1}^{n_i}(y_{ij}-\mu_i)^2-\sigma^2N]=c$$

B)The Information Matrix:

$$\partial^2\log L(Y/\mu,\sigma^2)/\partial\mu_i\partial\mu_i=\frac{-n}{\alpha_i\sigma^2}$$

$$\partial^2\log L(Y/\mu,\sigma^2)/\partial\sigma_i\partial\sigma_i=0$$

$$E(-\partial^2\log L(Y/\mu,\sigma^2)/\partial\mu_i\partial\mu_i=\frac{n_i}{\alpha_i\sigma^2}$$

$$\partial^2\log L(Y/\mu,\sigma^2)/(\partial\sigma^2)^2=\frac{N}{2(\sigma^2)^2}-\frac{1}{(\sigma^2)^3}\sum_{i=1}^{k}\frac{1}{\alpha_i}\sum_{j=1}^{n_i}(y_{ij}-\mu_i)^2$$

$$E(-\partial^2\log L(Y/\mu,\sigma^2)/(\partial\sigma^2)^2)=E[\frac{1}{2(\sigma^2)^2}\sum_{i=1}^{k}\frac{1}{\alpha_i}\sum_{j=1}^{n_i}(y_{ij}-\mu_i)^2-\frac{N}{2(\sigma^2)^2}]$$

$$=\frac{1}{(\sigma^2)^3}\sum_{i=1}^{k}\frac{1}{\alpha_i}E\sum_{j=1}^{n_i}(y_{ij}-\mu_i)^2-\frac{N}{2(\sigma^2)^2}$$

$$=\frac{1}{(\sigma^2)^3}\sum_{i=1}^{k}\frac{1}{\alpha_i}(n_i\alpha_i\sigma^2)-\frac{N}{2(\sigma^2)^2}$$

$$=\frac{N}{(\sigma^2)^2}-\frac{N}{2(\sigma^2)^2}=\frac{N}{2(\sigma^2)^2}$$

$$\partial^2 \log L(Y \,/\, \mu,\sigma^2)/\partial\sigma^2\partial\mu_i = -\frac{1}{2}\frac{1}{(\sigma^2)^3}2\sum_{j=1}^{n_i}\frac{1}{\alpha_i}(y_{ij}-\mu_i)$$

$$=-\frac{1}{(\sigma^2)^2}\sum_{i=1}^{k}\frac{1}{\alpha_i}\sum_{j=1}^{n_i}(y_{ij}-\mu_i)$$

$$E(-\partial^2 \log L(Y \,/\, \mu,\sigma^2)/\partial\sigma^2\partial\mu_i = E[\frac{1}{(\sigma^2)^2}\sum_{i=1}^{k}\frac{1}{\alpha_i}(y_{ij}-\mu_i)]$$

$$=[\frac{1}{(\sigma^2)^2}\sum_{i=1}^{k}\frac{1}{\alpha_i}\sum_{j=1}^{n_i}E(y_{ij}-\mu_i)=0$$

Then $S=[b_1,b_2,...,b_k,c]$

$$\Omega^{-1}=dig[\frac{\alpha_1\hat{\sigma}_0^2}{n_1}\,,\,\frac{\alpha_2\hat{\sigma}_0^2}{n_2}\,,...,\frac{\alpha_k\hat{\sigma}_0^2}{n_k}\,,\,\frac{2(\hat{\sigma}_0^2)^2}{N}]$$

The explicit form of the test statistic was:

$$T=\sum_{i=1}^{k}(b_i)^2\frac{\alpha_i\hat{\sigma}_0^2}{n_i}+\frac{2(\hat{\sigma}_0^2)^2}{N}c^2$$

$$=\sum_{i=1}^{k}\left\{\frac{1}{\alpha_i\hat{\sigma}_0^2}\sum_{j=1}^{n_i}(y_{ij}-\mu_i)\right\}^2\frac{\alpha_i\hat{\sigma}_0^2}{n_i}+\frac{1}{2(\sigma_0^2)^2N}[\sum_{i=1}^{k}\frac{1}{\alpha_i}\sum_{j=1}^{n_i}(y_{ij}-\hat{\mu})^2-\hat{\sigma}_0^2N]^2$$

$$=\sum_{i=1}^{k}\frac{1}{n_i\alpha_i\hat{\sigma}_0^2}[\sum_{j=1}^{n_i}(y_{ij}-\hat{\mu})]^2+\frac{1}{2(\sigma_0^2)^2N}[\hat{\sigma}_0^2N-\hat{\sigma}_0^2N]^2$$

$$=\sum_{i=1}^{k}\frac{1}{n_i\alpha_i\hat{\sigma}_0^2}[\sum_{j=1}^{n_i}(y_{ij}-\hat{\mu})]^2+\quad 0$$

$$=\sum_{i=1}^{k}\frac{n_i}{\alpha_i\hat{\sigma}_0^2}[\bar{y}_i-\hat{\mu}]^2$$

The power function for the score test under the assumptions of unknown variance had the form:

$$1-\beta=\sum_{i=1}^{k}\frac{n_i}{\alpha_i\hat{\sigma}^{*2}}\left\{\bar{y}_i-\hat{\mu}_i^*\right\}^2 \qquad \textbf{(13)}$$

The general score test (GST)

The general score test (GST) takes the form:

$$G=(S-\tilde{S})'\,\Omega^{-1}\,(S-\tilde{S}) \qquad \textbf{(14)}$$

Where: $\tilde{S}$ **:is the score vector evaluated at the maximum likelihood estimator under the alternative hypothesis . Using the equation (14) we got test statistic for the case of known variance:**

$$G=\sum_{i=1}^{k}w_i(\hat{\mu}_i^*-\hat{\mu})^2 \qquad \textbf{(15)}$$

<u>Proof:</u>

$$S_i-\tilde{S}_i=(\frac{1}{\sigma_i^2}\{\sum_{j=1}^{n_i}y_{ij}-n_i\hat{\mu}\}-\frac{1}{\sigma_i^2}\{\sum_{j=1}^{n_i}y_{ij}-n_i\hat{\mu}_i^*\})=\frac{n_i}{\sigma_i^2}\{\hat{\mu}_i^*-\hat{\mu}\}$$

$$G=[\frac{n_1}{\sigma_1^2}\{\hat{\mu}_1^*-\hat{\mu}\}\ \frac{n_2}{\sigma_2^2}\{\hat{\mu}_2^*-\hat{\mu}\}\ \ldots\ \frac{n_3}{\sigma_3^2}\{\hat{\mu}_k^*-\hat{\mu}\}]$$

$$\begin{bmatrix} \frac{\sigma_1^2}{n_1} & 0 & \ldots & 0 \\ 0 & \frac{\sigma_2^2}{n_2} & \ldots & 0 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & . & \frac{\sigma_k^2}{n_k} \end{bmatrix} \begin{bmatrix} \frac{n_1}{\sigma_1^2}\{\hat{\mu}_1^*-\hat{\mu}\} \\ \frac{n_2}{\sigma_2^2}\left\{\hat{\mu}_2^*-\hat{\mu}\right\} \\ . \\ . \\ . \\ \frac{n_k}{\sigma_k^2}\left\{\hat{\mu}_k^*-\hat{\mu}\right\} \end{bmatrix}$$

$$G=\sum_{i=1}^{k}\frac{n_1}{\sigma_1^2}\{\hat{\mu}_1^*-\hat{\mu}\}^2=\sum_{i=1}^{k}w_i\{\hat{\mu}_2^*-\hat{\mu}\}^2$$

Case of unknown variance

The test statistic for general score is obtained as :

$$G=\sum_{i=1}^{k}\frac{1}{\alpha_i\hat{\sigma}_1^2}n_i\{\bar{y}-\hat{\mu}\}^2+\sum_{i=1}^{k}\frac{\hat{\sigma}_0^2}{\alpha_i(\sigma_2^{*2})^2}\mathrm{n}_i\{\bar{y}_{i}-\hat{\mu}_i^*\}^2-2\sum_{i=1}^{k}\alpha_i\frac{n_i}{\hat{\sigma}^{*2}}\{\bar{y}_i-\hat{\mu}\}\{\bar{y}_i-\hat{\mu}_i^*\}] \quad \textbf{(16)}$$

Proof:

$$(S-\tilde{S})'=[\{\hat{b}_1-\hat{b}_1^*\}\ ,\{\hat{\mathrm{b}}_2-\hat{b}_2^*\}\ ,\ \ldots\ ,\{\hat{\mathrm{b}}_k-\hat{b}_k^*\},\{\hat{c}-\hat{c}^*\}]$$

Where $$\{\hat{b}_i-\hat{b}_i^*\}=\frac{1}{\alpha_i\hat{\sigma}_0^2}\sum_{j=1}^{n_i}(y_{ij}-\hat{\mu})-\frac{1}{\alpha_i\hat{\sigma}_0^{*2}}\sum_{j=1}^{n_i}(y_{ij}-\hat{\mu}_i^*)$$

$$\{\hat{c}-\hat{c}*\}=[\frac{1}{2(\hat{\sigma}_0^2)^2}\sum_{i=1}^{k}\frac{1}{\alpha_i}\sum_{j=1}^{n_i}(y_{ij}-\hat{\mu})^2-\frac{N}{2\hat{\sigma}_0^2}]-[\frac{1}{2(\hat{\sigma}_0^{*2})^2}\sum_{i=1}^{k}\frac{1}{\alpha_i}\sum_{j=1}^{n_i}(y_{ij}-\hat{\mu}_1^*)^2]$$

$$=[\frac{1}{2(\hat{\sigma}_0^2)^2}\sum_{i=1}^{k}\frac{1}{\alpha_i}\sum_{j=1}^{n_i}(y_{ij}-\hat{\mu})^2-\frac{1}{2(\hat{\sigma}_0^{*2})^2}\sum_{i=1}^{k}\frac{1}{\alpha_i}\sum_{j=1}^{n_i}(y_{ij}-\hat{\mu}_1^*)^2+\frac{N(\hat{\sigma}_0^2-\hat{\sigma}_0^{*2})}{2\hat{\sigma}_0^{*2}\hat{\sigma}_0^2}]$$

$$\Omega^{-1}=dig[\frac{\alpha_1\hat{\sigma}_0^2}{n_1}\ ,\ \frac{\alpha_2\hat{\sigma}_0^2}{n_2}\ ,\ldots,\frac{\alpha_k\hat{\sigma}_0^2}{n_k}\ ,\ \frac{2(\hat{\sigma}_0^2)^2}{N}]$$

The explicit form of the test statistic was

$$G=\sum_{i=1}^{k}\frac{\alpha_i\hat{\sigma}_0^2}{n_i}\{\hat{b}_i-\hat{b}_i^*\}^2+\frac{2(\hat{\sigma}_0^2)^2}{N}\{\hat{c}-\hat{c}^*\}$$

$$=\sum_{i=1}^{k}\frac{\alpha_i\hat{\sigma}_0^2}{n_i}[\frac{1}{\alpha_i\hat{\sigma}_0^2}n_i\{\bar{y}_i-\hat{\mu}\}-\frac{1}{\alpha_i\hat{\sigma}^{*2}}n_i\{\bar{y}_i-\hat{\mu}_i^*\}]^2+0$$

$$=\sum_{i=1}^{k}\frac{\alpha_i\hat{\sigma}_0^2}{n_i}[\frac{1}{\alpha_i\hat{\sigma}_0^2}n_i\{\bar{y}_i-\hat{\mu}\}]^2+\sum_{i=1}^{k}\frac{\alpha_i\hat{\sigma}_0^2}{n_i}[\frac{1}{\alpha_i\hat{\sigma}_0^2}n_i\{\bar{y}_i-\hat{\mu}_i^*\}]^2$$

$$-2\sum_{i=1}^{k}\frac{\alpha_i\hat{\sigma}_0^2}{n_i}[\frac{1}{\alpha_i\hat{\sigma}_0^2}n_i\{\bar{y}_i-\hat{\mu}\}\frac{1}{\alpha_i\hat{\sigma}^{*2}}n_i\{\bar{y}_i-\hat{\mu}_i^*\}]$$

$$=\sum_{i=1}^{k}\frac{1}{\alpha_i\hat{\sigma}_0^2}n_i\{\bar{y}_i-\hat{\mu}\}^2+\sum_{i=1}^{k}\frac{\hat{\sigma}_0^2}{\alpha_i(\hat{\sigma}_0^{*2})^2}n_i\{\bar{y}_i-\hat{\mu}_i^*\}^2-2\sum_{i=1}^{k}\frac{n_i}{\alpha_i\hat{\sigma}_0^{*2}}\{\bar{y}_i-\hat{\mu}\}\{\bar{y}_i-\hat{\mu}_i^*\}$$

Result and discussion

The exact distribution for tests statistic were difficult obtained exactly theoretically. The simulation technique was used to determine the critical point for the classes of combinations of number of treatment levels (k), the variance (σ^2) and the samples size (n). A simulation study was used to generate data which had the properties of the different cases,

The value of the means were used from Robertson et al.(1988). For k = 3, we used the means {63.9, 58.2, 62.3}. For k = 4, the means were {63.9, 58.2, 62.3, 75.4}, at k =5, the means values were {63.9, 58.2, 62.3, 75.4, 68.5}. For k = 6, the means values were,{63.9 , 58.2, 62.3,75.4, 68.5, 70} and for k=7, the means values were{57, 63.9, 58.2, 62.3, 75.4, 68.5, 70), Different values of variances σ^2= {2,16,25}, and different sample sizes n : {5, 20, 50} were used to obtain the critical values for the tests. For each test five values of treatment levels(k) and three different sample sizes (n) and variances (σ^2) were used. We had (5) (3) (3) =45 different combinations. The same generated data were used for the tests.

One thousand replications were carried out for each combination of the previous test statistics.

To get values for the critical points, the1000 values of the test statistics
Were sorted. We used the observations number 10, 50, 100 as the values for the significant levels 0.01, 0.05, and 0.1 respectively since the alternative hypothesis had the form $\{\mu_1\le\mu_2\le\mu_3\le\mu_4\le\mu_5\}$ in the simple order.

The regression equations for the three tests under different cases obtained under the two assumptions, known and unknown variances.

Data from Appendix were used to get the regression equations under the case of known and unknown variance

1-Score tests

i-case of known variance:

$L\,n\ y = 3.759 + 0.071\ \text{In}\alpha - 1.109\ \ln\sigma^2 + 1.089\ \ln n + 0.896\ \ln k$

t-value (40.87) (6.83) (-116.79) (96.92) (18.25)

R-SQ = 99.4%

Where the first line represented the regression equation, the second represented
The t-value of each coefficient and the three was the R-square for equation.

ii-case of unknown variance:

$L\,n\ y = 0.247 + 0.034\ \text{In}\alpha - 0.237\ \ln\sigma^2 + 1.026\ \ln n + 0.956\ \ln k$

t-value (4.60) (5.58) (-42.74) (156.22) (33.33)

R-q = 99.5%

2- The likelihood Ratio Test

i- case of known variance

A- Simple Order

$L\,n\ y = 3.851 + 0.082\ \text{In}\alpha - 1.137\ \ln\sigma^2 + 1.24\ \ln n + 0.956\ \ln k$

t-value (25.67) (4.83) (-73.42) (61.29) (8.64)

R-q = 98.4%

B- Umbrella Order

$L\,n\ y = 1.286 + 0.085\ \text{In}\alpha - 1.074\ \ln\sigma^2 + 1.083\ \ln n + 1.942\ \ln k$

t-value (8.63) (5.08) (-69.78) (59.40) (24.40)

R-q = 98.4%

ii case of unknown variance

A- Simple order

$$Ln\ y = 0.314 + 0.053\ \ln\alpha - 0.267\ \ln\sigma^2 + 0.073\ \ln n - 0.249\ \ln k$$

t-value (2.91) (4.38) (-24.04) (5.53) (-4.34)

R-q = 82.1%

B- Umbrella Order

$$Ln\ y = -0.1048 + 0.0069\ \ln\alpha - 0.075\ \ln\sigma^2 - 0.0136\ \ln n + 0.301\ \ln k$$

t-value (-2.85) (1.52) (-20.02) (-2.81) (13.49)

R-q = 79.5%

3- General score test

i-case of known variance :

$$Ln\ y = 3.851 + 0.082\ \ln\alpha - 1.137\ \ln\sigma^2 + 1.124\ \ln n + 0.692\ \ln k$$

t-value (25.67) (4.83) (-73.42) (61.29) (8.64)

R-Sq = 98.4%

ii-case of unknown variance :

A- Simple order

$$Ln\ y = 22.63 + 1.723\ \ln\alpha - 10.972\ \ln\sigma^2 + 0.447\ \ln n + 15.60\ \ln k$$

t-value (1.25) (1.57) (-10.81) (0.37) (0.181)

R-Sq = 52.2%

B- Umbrella Order

$$Ln\ y = -33.597 + 0.6772\ \ln\alpha - 6.1169\ \ln\sigma^2 + 1.4324\ \ln n + 36.885\ \ln k$$

t-value (-3.82) (1.15) (-12.30) (2.19) (6.13)

R-Sq = 50.1%

4- The power values

The power values of the score test according to some critical values was Obtained. The power $(1-\beta)$ could be calculated according to a specific Known value of critical values $\{C_0\}$ using the following equation to

Compute the power value: $1-\beta=\frac{1}{1000}\sum_{i=1}^{1000} z\,(p \ge C_0)$ **where Z is the**

Indicator function defines as: $z\,(p \ge C)=1$ **and** $z\,(p < C)=0$

5- The score test and the general score test for critical points did not depend
On the alternative hypothesis, while the test statistic depended on the
Estimators under the alternative hypothesis.

6- In k =3, we noticed that the value of the power function was not high since
The value of the test statistic was approximately similar to the value of the
Power function.

7- The simulated power values were increased with the increasing of the
Sample size (n), while they decreased with the increasing of variance (σ^2) **,**
But it was not affected by the number of treatment levels (k).

Table (1): the critical points for score test for different Values of α and σ^2 at k = 3 and n = 5

α / σ^2	Known σ^2			unknown σ^2		
	0.01	0.05	0.10	0.01	0.05	0.10
2	18.841	24.730	28.319	7.811	9.183	9.915
16	0.214	0.999	1.687	0.183	1.139	1.877
25	0.084	0.378	0.828	0.086	0.468	0.960

Table (2)
the critical points for score test for different Values of α and σ^2 at k = 4 and n = 20

α / σ^2	Known σ^2			unknown σ^2		
	0.01	0.05	0.10	0.01	0.05	0.10
2	1448.87	1501.33	1524.16	74.827	75.378	75.626
16	140.64	160.74	169.37	49.884	52.752	54.154
25	78.475	96.092	104.118	39.684	43.167	45.275

Table (3)
the critical points for score test for different
Values of α and σ^2 at k = 7 and n= 50

α / σ^2	Known σ^2			unknown σ^2		
	0.01	0.05	0.10	0.01	0.05	0.10
2	6219.05	6336.24	640.03	329.426	330.546	331.032
16	706.133	738.352	762.233	231.948	235.292	238.317
25	433.299	460,848	475.912	191.088	197.602	201.024

Table (4)
the critical points for general score test for different
Values of α and σ^2 at k = 3 and n = 50

α / σ^2	Simple Order			Umbrella Order		
	0.01	0.05	0.10	0.01	0.05	0.10
2	2.177	3.350	4.151	1.819	2.695	3.261
16	0.000	0.018	0.239	0.001	0071	0.269
25	0.000	0.000	0.023	0.001	0.026	0.097

Table (5)
the critical points for general score test for different
Values of α and σ^2 at k = 5 and n = 20

α / σ^2	Simple Order			Umbrella Order		
	0.01	0.05	0.10	0.01	0.05	0.10
2	184.057	197.788	204.764	49.589	52.232	54.304
16	51.667	57.637	61.23	24.254	27.398	29.131
25	36.501	40.324	43.723	18.553	21.522	23.154

Table (6)
the critical points for general score test for different Values of α and σ^2 at k = 7 and n= 50

α / σ^2	Simple Order			Umbrella Order		
	0.01	0.05	0.10	0.01	0.05	0.10
2	979.821	1018.54	1035.26	402.774	419.105	424.709
16	245.279	258.778	265.797	165.797	178.192	183.826
25	175.586	189.222	193.774	124.789	135.828	140.335

Table (7)
the critical points for the likelihood ratio test for different values of α and σ^2 at k = 3 and n = 20

α / σ^2	Simple Order						Umbrella Order					
	Known σ^2			Un Known σ^2			Known σ^2			Un Known σ^2		
	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
2	0.709	2.542	4.007	0.003	0.011	0.018	1.063	3.812	6.011	0.005	0.016	0.026
16	0.000	0.000	0.000	0.00	0.000	0.000	0.001	0.021	0.078	0.000	0.000	0.001
25	0.000	0.000	0.000	0.00	0.000	0.000	0.000	0.016	0.059	0.000	0.000	0.001

Table (8)
the critical points for the likelihood ratio test for different values of α and σ^2 at k = 5 and n = 20

α / σ^2	Simple Order						Umbrella Order					
	Known σ^2			Un Known σ^2			Known σ^2			Un Known σ^2		
	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
2	1165.4	1206.51	1232.1	0.681	0.694	0.701	586.36	619.31	64.27	0.337	0.349	0.359
16	113.71	126.73	134.87	0.409	0.448	0.469	59.886	68.443	72.501	0.210	0.234	0.246
25	66.869	76.353	81.493	0.319	0.349	0.379	36.879	42.872	46.101	0.1691	0.196	0.208

Table (9)
the critical points for the likelihood ratio test for different values of α and σ^2 at k = 6 and n = 50

α / σ^2	Simple Order						Umbrella Order					
	Known σ^2			Un Known σ^2			Known σ^2			Un Known σ^2		
	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
2	3378.2	3450.05	3504.89	0.698	0.709	0.713	3799.2	3875.3	3920.7	21.316	21.845	22.03
16	357.36	379.55	402.175	0.444	0.466	0.476	412.16	440.17	456.3	0.609	14.339	14.77
25	222.676	238.35	249.915	0.358	0.379	0.391	245.21	271.56	283.13	0.478	0.529	12.02

Table (10)
the power values for the score test for different values of α and σ^2 at k = 3 and n = 20

α σ^2	Simple Order						Umbrella Order					
	Known σ^2			Un Known σ^2			Known σ^2			Un Known σ^2		
	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
2	0.022	2.090	4.161	0.017	0.075	0.140	0.005	3.024	6.035	0.005	0.013	0.030
16	0.029	0.088	0.167	0.027	0.095	0.160	0.019	0.063	0.110	0.017	0.068	0.111
25	0.047	0.100	0.166	0.040	0.100	0.160	0.034	0.067	0.124	0.033	0.073	0.112

Table (11)
the power values for the score test for different values of α and σ^2 at k = 5 and n = 50

α σ^2	Simple Order						Umbrella Order					
	Known σ^2			Un Known σ^2			Known σ^2			Un Known σ^2		
	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
2	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
16	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000
25	1.000	1.000	1.000	1.000	1.000	1.000	0.097	0.901	0.922	0.798	0.901	0.918

Table (12)
the power values for the score test for different values of α and σ^2 at k = 6 and n = 50

α σ^2	Simple Order						Umbrella Order					
	Known σ^2			Un Known σ^2			Known σ^2			Un Known σ^2		
	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
2	1.000	1.000	1.000	1.000	1.000	1.000	0.841	0.983	0.977	1.000	1.000	1.000
16	1.000	1.000	1.000	1.000	1.000	1.000	0.822	0.948	0.948	1.000	1.000	1.000
25	1.000	1.000	1.000	0.999	1.000	1.000	0.960	0.930	0.900	1.000	1.000	1.000

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