Convergence Gamma Distribution to Normal Distribution by Using Differential Geometry

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Abstract

In this research, we use the differential geometry to show that Gamma distribution converges to normal distribution by connecting between differential geometry and statistics.

We apply some formulas to calculate the Gaussian curvature (k) for Gamma distribution in the case of parametric lines are not orthogonal and show that if it is convergent to normal distribution by comparing the value of Gaussian curvature for Gamma distribution with the value of Gaussian curvature for the normal distribution.

الخلاصة

في هذا البحث، تم استخدام الهندسة التفاضلية لتبيان ان توزيع كاما يقترب الى التوزيع الطبيعي وذلك باستخدام الربط بين الهندسة التفاضلية والاحصاء.

تم تطبيق بعض الصيغ لحساب تقوس كاوس لتوزيع كاما في حالة الخطوط المعلمية غير متعامدة وتوضيح فيما اذا هذا التوزيع متقارب الى التوزيع الطبيعي بمقارنة القيمة لتقوس كاوس لتوزيع كاما مع القيمة لتقوس كاوس للتوزيع الطبيعي.

Introduction

This research present some interesting connections between statistics and differential geometry. Also, it contains the result of computing the Gaussian curvature of Gamma distribution by using different formulas. If the Gaussian curvature of Gamma distribution approaches to the Gaussian curvature of the normal distribution, we say that the distribution converges to normal distribution.

There are some researchers who worked in this field in end of twenty century and beginning of twenty one century, including (Kass, 1997) used the following formula

$$-\frac{1}{\sqrt{EG}}\left(\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)\right)$$

to compute the Gaussian curvature (K) of trinomial and t families, (Chen, 1999) used the formula

$$\frac{R_{1212}}{EG-F^2}=\frac{(12,12)}{EG-F^2} \text{ where } (12,12)=R_{1212}=\sum_{m=1}^{2}R^m_{121}g_{m2}$$

$$R^1_{ijk}=\frac{\partial}{\partial u_j}\Gamma^1_{ik}-\frac{\partial}{\partial u_i}\Gamma^1_{jk}+\Gamma^m_{ik}\Gamma^1_{mj}-\Gamma^m_{jk}\Gamma^1_{mi}\text{ , sum on } m,$$

to compute the Gaussian curvature (K) for the distributions (normal, Cauchy and t family). He showed that in normal distribution Gaussian curvature $K=-\frac{1}{2}$, and in Cauchy distribution $K=-2$, while in t family distribution with r degrees of freedom, he got $K=-\frac{r+3}{2r}$ and (Gruber, 2003) used the following formula

$$-\frac{1}{2\sqrt{EG-F^2}}\left[\frac{\partial}{\partial u}\frac{G_u-F_v}{\sqrt{EG-F^2}}-\frac{\partial}{\partial v}\frac{F_u-E_v}{\sqrt{EG-F^2}}\right]-\frac{1}{4(EG-F^2)^2}\begin{vmatrix}E & F & G\\ E_u & F_u & G_u\\ E_v & F_v & G_v\end{vmatrix}$$

to compute the Gaussian curvature of gamma family of distributions in the case of parametric lines are orthogonal *(F=0)* and showed that Gamma distribution converges to the normal distribution.

1. Metric Tensor for Gamma Distribution (Wasan, 2008)

In the case of parametric lines μ and v are not orthogonal ($F \neq 0$).

We know that

$$f(x;\mu,v)=\mu^v\frac{x^{v-1}}{\Gamma(v)}e^{-x\mu} \qquad x>0, \qquad \mu,v>0$$

be probability density function of Gamma distribution.

Then the logarithm of likelihood function of family Gamma, can be written as follows:

$$\ln f = v\ln\mu+(v-1)\ln x - x\mu - \ln\Gamma(v) \qquad (1)$$

From equation (1), we can derive the first and second partial derivatives with respect to the parametric lines μ and v.

$$\frac{\partial^2 \ln f}{\partial\mu^2}=-\frac{v}{\mu^2}$$

$$\frac{\partial^2 \ln f}{\partial\mu\partial v}=\frac{1}{\mu} \qquad (2)$$

$$\frac{\partial^2 \ln f}{\partial v^2}=-\left(\frac{\Gamma'(v)}{\Gamma(v)}\right)'$$

For the Gamma family of distributions, the coefficients of the first fundamental form:

$$E=-E\left(\frac{\partial^2 \ln f}{\partial\mu^2}\right)=\frac{v}{\mu^2}$$

$$F=-E\left(\frac{\partial^2 \ln f}{\partial\mu\partial v}\right)=-\frac{1}{\mu}$$

$$G=-E\left(\frac{\partial^2 \ln f}{\partial v^2}\right)=\psi'(v)$$

where $\psi(v)=\frac{\Gamma'(v)}{\Gamma(v)}$ is the digamma function.

$$\text{Metric Tensor}=\begin{pmatrix}\frac{v}{\mu^2} & -\frac{1}{\mu}\\ -\frac{1}{\mu} & \psi'(v)\end{pmatrix}$$

0r

$$ds^2=\frac{v}{\mu^2}(d\mu)^2-\frac{2}{\mu}d\mu dv+\psi'(v)(dv)^2 \qquad (3)$$

2.The Gaussian curvature of the Probability Distribution (Chen, 1999)

First, we define the six well known Christoffel symbols as:

$$\Gamma_{11}^{1}=\frac{GE_u-2FF_u+FE_v}{2(EG-F^2)},\qquad \Gamma_{12}^{2}=\frac{EG_u-FE_v}{2(EG-F^2)},$$
$$\Gamma_{11}^{2}=\frac{2EF_u-EE_v-FE_u}{2(EG-F^2)},\qquad \Gamma_{22}^{1}=\frac{2GF_v-GG_u-FG_v}{2(EG-F^2)},\qquad (4)$$
$$\Gamma_{12}^{1}=\frac{GE_v-FG_u}{2(EG-F^2)},\qquad \Gamma_{22}^{2}=\frac{EG_v-2FF_v+FG_u}{2(EG-F^2)}.$$

Since E,F and G are functions of parameters (u,v) and are continuously twice differentiable E_u, E_v, F_u, F_v, G_u and G_v all exists and are all well defined. Additionally, no assumption is made regarding $F=0$, and so the parametric lines are not necessarily orthogonal. However, if $F=0$, the six Christoffel symbols can be greatly simplified.

Now, we select four formulas that can be used to compute the Gaussian curvature of the distributions:

$$(A)\ K=\frac{1}{\sqrt{EG}}\left(\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)\right)$$

$$(B)\ K=-\frac{1}{2\sqrt{EG-F^2}}\left[\frac{\partial}{\partial u}\frac{G_u-F_v}{\sqrt{EG-F^2}}-\frac{\partial}{\partial v}\frac{F_u-E_v}{\sqrt{EG-F^2}}\right]-\frac{1}{4(EG-F^2)^2}\begin{vmatrix}E & F & G\\ E_u & F_u & G_u\\ E_v & F_v & G_v\end{vmatrix}$$

$$(C)\ K=\frac{1}{D}\left[\frac{\partial}{\partial v}\left(\frac{D}{E}\Gamma_{11}^{2}\right)-\frac{\partial}{\partial u}\left(\frac{D}{E}\Gamma_{12}^{2}\right)\right]=\frac{1}{D}\left[\frac{\partial}{\partial u}\left(\frac{D}{G}\Gamma_{22}^{1}\right)-\frac{\partial}{\partial v}\left(\frac{D}{G}\Gamma_{12}^{1}\right)\right]$$

$D^2=EG-F^2$ where

$$(D)\ K=\frac{R_{1212}}{EG-F^2}=\frac{(12,12)}{EG-F^2}$$

where $(12,12)=R_{1212}=\sum_{m=1}^{2}R_{121}^{m}\,g_{m2}$

, sum on m, $\quad R_{ijk}^{1}=\frac{\partial}{\partial u_j}\Gamma_{ik}^{1}-\frac{\partial}{\partial u_i}\Gamma_{jk}^{1}+\Gamma_{ik}^{m}\Gamma_{mj}^{1}-\Gamma_{jk}^{m}\Gamma_{mi}^{1}$

where the quantities of R_{ijk}^{1} are components of a tensor of the fourth order. This tensor is called the mixed Riemann curvature tensor. Notice that g_{11}, g_{12}, and g_{22} are simply tensor notation for E, F and G.

Clearly formula (A) is a special case that is valid only when the parametric lines are orthogonal.

Remark:

We can not use the formula (A) to compute the Gaussian curvature of Gamma distribution because we take the case that the parametric lines μ, v of this distribution are not orthogonal*($F\neq 0$)*.

3.The Gaussian Curvature for Gamma Distribution

3.1 We use the formula (B) to compute the Gaussian curvature of this distribution.

$$K=-\frac{1}{2\sqrt{EG-F^2}}\left[\frac{\partial}{\partial\mu}\frac{G_\mu-F_v}{\sqrt{EG-F^2}}-\frac{\partial}{\partial v}\frac{F_\mu-E_v}{\sqrt{EG-F^2}}\right]-\frac{1}{4(EG-F^2)^2}\begin{vmatrix}E & F & G\\ E_\mu & F_\mu & G_\mu\\ E_v & F_v & G_v\end{vmatrix}$$

where

$$\begin{vmatrix}E & F & G\\ E_\mu & F_\mu & G_\mu\\ E_v & F_v & G_v\end{vmatrix}=\begin{vmatrix}\frac{v}{\mu^2} & -\frac{1}{\mu} & \psi'(v)\\ -\frac{2v}{\mu^3} & \frac{1}{\mu^2} & 0\\ \frac{1}{\mu^2} & 0 & \psi''(v)\end{vmatrix}=\left[\frac{v\psi''(v)}{\mu^4}-\frac{2v\psi''(v)}{\mu^4}-\frac{\psi'(v)}{\mu^4}\right]$$

$$=-\frac{v\psi''(v)+\psi'(v)}{\mu^4}.$$

Now, we can say that

$$K=-\frac{1}{4(EG-F^2)^2}\begin{vmatrix}E & F & G\\ E_\mu & F_\mu & G_\mu\\ E_v & F_v & G_v\end{vmatrix}$$

$$=\frac{\mu^4}{4(v\psi'(v)-1)^2}\left[\frac{v\psi''(v)+\psi'(v)}{\mu^4}\right]$$

$$=\frac{v\psi''(v)+\psi'(v)}{4(v\psi'(v)-1)^2},$$

where $\psi(v)=\frac{\Gamma'(v)}{\Gamma(v)}$ is the digamma function

$$\lim_{v\to\infty}K(v)=\lim_{v\to\infty}\frac{-(1/2v^2+o(1/v^3))}{4((1/4v^2)+o(1/v^3))}=-\frac{1}{2}$$

$$\lim_{v\to 0} K(v) = -\frac{1}{4}$$

3.2 We use the formula (C) to compute the Gaussian curvature of this distribution. We can find that

$$\Gamma_{11}^{1} = \frac{GE_{\mu} - 2FF_{\mu} + FE_{v}}{2(EG - F^{2})}$$

$$= \frac{\frac{-2v\psi'(v)}{\mu^{3}} + \frac{2}{\mu^{3}} - \frac{1}{\mu^{3}}}{2(\frac{v\psi'(v)}{\mu^{2}} - \frac{1}{\mu^{2}})} = \frac{-2v\psi'(v)+1}{\mu^{3}} \frac{\mu^{2}}{2v\psi'(v)-2}$$

$$= \frac{1-2v\psi'(v)}{2\mu(v\psi'(v)-1)}$$

$$\Gamma_{12}^{1} = \frac{GE_{v} - FG_{\mu}}{2(EG - F^{2})} = \frac{\frac{\psi'(v)}{\mu^{2}}}{\frac{2v\psi'(v)-2}{\mu^{2}}}$$

$$= \frac{\psi'(v)}{\mu^{2}} \frac{\mu^{2}}{2v\psi'(v)-2} = \frac{\psi'(v)}{2(v\psi'(v)-1)}$$

$$\Gamma_{22}^{1} = \frac{2GF_{v} - GG_{\mu} - FG_{v}}{2(EG - F^{2})}$$

$$= \frac{\frac{\psi''(v)}{\mu}}{\frac{2(v\psi'(v)-1)}{\mu^{2}}} = \frac{\psi''(v)}{\mu} \frac{\mu^{2}}{2(v\psi'(v)-1)} = \frac{\mu\psi''(v)}{2(v\psi'(v)-1)}$$

$$\Gamma_{11}^{2} = \frac{2EF_{\mu} - EE_{v} - FE_{\mu}}{2(EG - F^{2})} = \frac{\frac{2v}{\mu^{4}} - \frac{v}{\mu^{4}} - \frac{2v}{\mu^{4}}}{\frac{2(v\psi'(v)-1)}{\mu^{2}}}$$

$$= -\frac{v}{\mu^{4}} \frac{\mu^{2}}{2(v\psi'(v)-1)} = \frac{v}{2\mu^{2}(1-v\psi'(v))}$$

$$\Gamma_{12}^{2} = \frac{EG_{\mu} - FE_{v}}{2(EG - F^{2})} = \frac{\frac{1}{\mu^{3}}}{\frac{2(v\psi'(v)-1)}{\mu^{2}}}$$

$$= \frac{1}{\mu^{3}} \frac{\mu^{2}}{2(v\psi'(v)-1)} = \frac{1}{2\mu(v\psi'(v)-1)}$$

$$\Gamma_{22}^{2} = \frac{EG_{v} - 2FF_{v} + FG_{\mu}}{2(EG - F^{2})}$$

$$= \frac{v\psi''(v)}{\mu^{2}} \frac{\mu^{2}}{2(v\psi'(v) - 1)} = \frac{v\psi''(v)}{2(v\psi'(v) - 1)}$$

We take the left hand side of formula (C)

$$K = \frac{1}{D}\left[\frac{\partial}{\partial v}\left(\frac{D}{E}\Gamma_{11}^{2}\right) - \frac{\partial}{\partial \mu}\left(\frac{D}{E}\Gamma_{12}^{2}\right)\right]$$

$$= \frac{1}{\sqrt{EG - F^{2}}}\left[\frac{\partial}{\partial v}\left(\frac{\sqrt{EG - F^{2}}}{E}\Gamma_{11}^{2}\right)\right]$$

$$= \frac{\mu}{\sqrt{v\psi'(v) - 1}}\left[\frac{\partial}{\partial v}\left(\frac{\sqrt{v\psi'(v) - 1}}{\mu}\frac{\mu^{2}}{v}\frac{-v}{2\mu^{2}(v\psi'(v)}\right)\right]$$

$$= -\frac{1}{2\sqrt{v\psi'(v) - 1}}\left[\frac{\partial}{\partial v}(v\psi'(v) - 1)^{-1/2}\right]$$

$$= \frac{v\psi''(v) + \psi'(v)}{4\sqrt{v\psi'(v) - 1}\,(v\psi'(v) - 1)^{3/2}} = \frac{v\psi''(v) + \psi'(v)}{4(v\psi'(v) - 1)^{2}}$$

$$\lim_{v \to \infty} K(v) = -\frac{1}{2}$$

$$\lim_{v \to 0} K(v) = -\frac{1}{4}$$

Now, we take the right hand side of formula (C)

$$K = \frac{1}{D}\left[\frac{\partial}{\partial \mu}\left(\frac{D}{G}\Gamma_{22}^{1}\right) - \frac{\partial}{\partial v}\left(\frac{D}{G}\Gamma_{12}^{1}\right)\right]$$

$$= \frac{1}{\sqrt{EG - F^{2}}}\left[-\frac{\partial}{\partial v}\left(\frac{\sqrt{EG - F^{2}}}{G}\Gamma_{12}^{1}\right)\right]$$

$$= -\frac{\mu}{\sqrt{v\psi'(v) - 1}}\left[\frac{\partial}{\partial v}\left(\frac{\sqrt{v\psi'(v) - 1}}{\mu}\frac{1}{\psi'(v)}\frac{\psi'(v)}{2(v\psi'(v) - 1)}\right)\right]$$

$$= -\frac{1}{2\sqrt{v\psi'(v) - 1}}\left[\frac{\partial}{\partial v}(v\psi'(v) - 1)^{-1/2}\right]$$

$$= \frac{v\psi''(v) + \psi'(v)}{4\sqrt{v\psi'(v) - 1}\,(v\psi'(v) - 1)^{3/2}} = \frac{v\psi''(v) + \psi'(v)}{4(v\psi'(v) - 1)^{2}}$$

$$\lim_{v \to \infty} K(v) = -\frac{1}{2} \qquad \lim_{v \to 0} K(v) = -\frac{1}{4}$$

3.3 We use the formula (*D*) to compute the Gaussian curvature of this distribution.

$$K = \frac{R_{1212}}{EG - F^2}$$

$$R_{1212} = R^1_{121} g_{12} + R^2_{121} g_{22},$$

where

$$R^1_{121} = \frac{\partial}{\partial v}\Gamma^1_{11} - \frac{\partial}{\partial \mu}\Gamma^1_{21} + \Gamma^m_{11}\Gamma^1_{m2} - \Gamma^m_{21}\Gamma^1_{m1}$$

$$= \frac{\partial}{\partial v}\Gamma^1_{11} - \frac{\partial}{\partial \mu}\Gamma^1_{21} + \Gamma^1_{11}\Gamma^1_{12} + \Gamma^2_{11}\Gamma^1_{22} - \Gamma^1_{21}\Gamma^1_{11} - \Gamma^2_{21}\Gamma^1_{21}$$

$$= \frac{v\psi''(v) + \psi'(v)}{2\mu(v\psi'(v)-1)^2} - \frac{v\psi''(v)}{4\mu(v\psi'(v)-1)^2} - \frac{\psi'(v)}{4\mu(v\psi'(v)-1)^2}$$

$$= \frac{2v\psi''(v) + 2\psi'(v) - v\psi''(v) - \psi'(v)}{4\mu(v\psi'(v)-1)^2}$$

$$= \frac{v\psi''(v) + \psi'(v)}{4\mu(v\psi'(v)-1)^2}$$

$$R^1_{121} g_{12} = -\frac{v\psi''(v) + \psi'(v)}{4\mu^2(v\psi'(v)-1)^2}$$

$$R^2_{121} = \frac{\partial}{\partial v}\Gamma^2_{11} - \frac{\partial}{\partial \mu}\Gamma^2_{21} + \Gamma^m_{11}\Gamma^2_{m2} - \Gamma^m_{21}\Gamma^2_{m1}$$

$$= \frac{\partial}{\partial v}\Gamma^2_{11} - \frac{\partial}{\partial \mu}\Gamma^2_{21} + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{21}\Gamma^2_{11} - \Gamma^2_{21}\Gamma^2_{21}$$

$$= \frac{1 + v^2\psi''(v)}{2\mu^2(v\psi'(v)-1)^2} + \frac{1}{2\mu^2(v\psi'(v)-1)} + \frac{1 - 2v\psi'(v)}{4\mu^2(v\psi'(v)-1)^2}$$

$$- \frac{v^2\psi''(v)}{4\mu^2(v\psi'(v)-1)^2} + \frac{v\psi'(v)}{4\mu^2(v\psi'(v)-1)^2} - \frac{1}{4\mu^2(v\psi'(v)-1)^2}$$

$$= \frac{2v^2\psi''(v) - v^2\psi''(v) + v\psi'(v)}{4\mu^2(v\psi'(v)-1)^2}$$

$$= \frac{v^2\psi''(v) + v\psi'(v)}{4\mu^2(v\psi'(v)-1)^2}$$

$$R^2_{121} g_{22} = \frac{v^2\psi''(v)\psi'(v) + v\psi'^2(v)}{4\mu^2(v\psi'(v)-1)^2}$$

$$R_{1212} = \frac{v^2\psi''(v)\psi'(v) + v\psi'^2(v) - v\psi''(v) - \psi'(v)}{4\mu^2(v\psi'(v)-1)^2}$$

$$K(v) = \frac{v^2\psi''(v)\psi'(v) + v\psi'^2(v) - v\psi''(v) - \psi'(v)}{4\mu^2(v\psi'(v)-1)^2} \; \frac{\mu^2}{v\psi'(v)-1}$$

$$= \frac{v\psi''(v)\left[v\psi'(v)-1\right]+\psi'(v)\left[v\psi'(v)-1\right]}{4(v\psi'(v)-1)^3}$$

$$= \frac{v\psi''(v)+\psi'(v)}{4(v\psi'(v)-1)^2}$$

$$\lim_{v\to\infty} K(v) = -\frac{1}{2} \qquad \lim_{v\to 0} K(v) = -\frac{1}{4}$$

As $v \to \infty$ the curvature of the Gamma family of distributions tends toward -1/2 the curvature of the normal family of distributions.

This result shows that Gamma distribution converges to the normal distribution.

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