On the Characters a Quaternion Group Q_{2n} Where $n=p^r$, $r\in Z^+$ and p is any Prime Number and $p\neq 2$

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Abstract

In this research, we present an important result on the characters of the quaternion group Q_{2n} where $n=p^r$ and $r\in Z^+$ and p is any prime number and not equal to 2, and the main result is the following proposition:-

Every rational valued characters of the quaternion group can be written as Z-linear combination of induced characters $1_{c_i}^{Q_{2n}}$, C_i runs over all cyclic subgroup of Q_{2n}.

الخلاصة

في هذا البحث توصلنا الى النتيجة الاتية حول شواخص الزمرة Q_{2n} عندما $n=p^r$ وان r عدد صحيح موجب و p عدد اولي اكبر من 2 وكانت النتيجة الرئيسية هي الاتية:-

كل شاخص من شواخص الزمرة Q_{2n} (في حالة $n=p^r$) يمكن التعبير عنه بصيغة تركيب خطي للشواخص المحثثة $1_{c_i}^{Q_{2n}}$حيث معاملاتها اعداد صحيحة وان C_i زمرة جزئية من الزمرة Q_{2n}.

Introduction

Let Q_{2n} be a quaternion group, and Q a rational field, two elements X and Y of Q_{2n} are called Q-conjugate if the cyclic group generated by X is conjugate to the cyclic group generate by Y. This defines an equivalence relation on Q_{2n}, their equivalence classes are called the Γ-classes of Q_{2n}, let these be

$1 = \Gamma_1, \Gamma_2, \dots \Gamma m$

Let $x_i \in \Gamma i$ be the representative of the class Γi and $C_i = \langle X_i \rangle$.

Due to Artin theorem asserts if G be a finite group and θ be a rational valued character of G then;

$$\theta = \sum_{i=1}^{m} a_i \ 1_{c_i}^{G} \qquad \text{where } a_i \in Q \qquad \dots(1)$$

So according to this theorem each valued character χ of Q_{2n} can be written as:

$$\chi = \sum_{i=1}^{m} a_i \ 1_{c_i}^{Q_{2n}} \qquad \text{where } a_i \in Q \qquad \dots(2)$$

In the present paper, we have proved that each rational valued character χ of Q_{2n}, which is expressible as in (1) with $a_i \in Z$, where $n=2^r$ such that r is positive integer number i.e.

$$\chi = \sum_{i=1}^{m} a_i \ 1_{c_i}^{Q_{2n}} \qquad \text{where } a_i \in Z \qquad \dots(3)$$

There is no doubt that the problem of constructing the rational valued characters table of Q_{2n}, where $n=2^r$ and $r\in Z^+$ and p is any prime number, and $p\neq 2$, would be rather simplified if we knew that each rational valued characters of Q_{2n} can be written as a Z-linear combination of the induced characters $1_{c_i}^{Q_{2n}}$.

On the group $K(Q_{2n})$

Definition (1) : For each positive integer n the general quaternion group of order 4n can be defined as follows:

$Q_{2n} = \{<X, Y> : X^{2n} = Y^4 = 1 , Y\times Y^{-1} = X^{-1}\}$

For more information see [Board *et al.,* 1973; Curtise, 1988)

Definition (2): Let Cf (G, Z) be the set of all Z-Valued class function of G which are constant on Q-Classes. Let R(Q, Z), be the intersection of Cf (G, Z) with R(G) the group of generalized characters of G.

R(Q, Z) is a finitely generated Z-module with bases Q-Characteristics of G.

$$\chi_i = \sum_{\sigma\in \mathrm{Gal}(Q(\chi_i)/Q)}^{m} \chi_i^{\sigma} \qquad , \chi_i \in \mathrm{I_{rr}}(C)$$

And rank m, we will denote by K(G) the factor group $\frac{Cf(G,Z)}{R(Q,Z)}$

Theorem (1):

For a finite group G, we have;

$$|K(G)| = \left(\frac{\prod_{i=1}^{m} n_i |N(C_i)|}{\phi(|C_i|)} \right)^{\frac{1}{2}}$$

For proof see (Kirdar, 1988)

Theorem (2): Let G be a finite group, $\{C_i = <X_i> , 1 \le I \le m\}$ set of non-conjugate cyclic subgroups of G, and n_i be the number of conjugate cyclic subgroup of G contained in the Q-Class Γi, then each rational valued character of G can be written as a Z-linear combination of $1_{C_i}^{C_n}$ if and only if:

$$\left(\prod_{i=1}^{m} \frac{n_i}{\phi(|C_i|)} \right)^{\frac{1}{2}} = \prod_{i=1}^{m} \frac{(|N(C_i)|)^{\frac{1}{2}}}{|C_i|}$$

for proof see (Kirdar, 1988)

Corollary 1:

The rational valued characters of a cyclic group C_n of order n can be written as a Z-linear combination of characters $1_{C_i}^{C_n}$, where C_i runs over the set of subgroup of C_n.

For proof see(Kirdar, 1988).

Lemma 1: Let m be the number of Q-classes Γ_i of Q_{2n}, where n is an even number, then $m=m_1+2$, where m_1 is the number of Q-classes Γ_i of the cyclic subgroup $C_{2n}=<X>$.

For proof see (Mohammed, 1995).

Corollary 2: We have,

$$|K(Q_{2n})| = |K(Q_2, p^h)| = |K(C_2 p^h) \oplus C_4|$$

The main result

In this article we shall prove the following

Proposition

Every rational valued character of the quaternion can be written as z-linear combination of induced characters $1_{C_i}^{C_n}$, Ci runs over all cycles subgroup of Q2n, where m is the number of Γ classes of Q_{2n}.

Let n=2^r and r is any positive integer number , by theorem (1) we get

$$|K(Q_{2n})| = |K(Q_{2p^r})| = \left(\prod_{i=1}^{m} \frac{n_i(|N(C_i)|)}{\phi(|C_i|)}\right)^{1/2} \qquad \ldots(1)$$

By corollary (2) we get;

$$|K(Q_{2n})| = |K(Q_{2p^r})| = |K(C_{2p^r})|.|C_4| = 4|K(Q_2 p^r)| \qquad \ldots(2)$$

Further more by equation (1) lemma (1) and corollary (2), we get;

$$|K(Q_{2n})| = |K(Q_{2p^r})| = \left(\prod_{i=1}^{m_1} \frac{n_i(|N(C_i)|)}{\phi(|C_i|)}\right)^{1/2} . \left(\frac{n_s |N(C_4)|^{1/2}}{\phi|(C_4)|}\right)$$

but;

$\phi(|C_4|) = \phi(4) = 2$

and $n_s = 2$

$$\therefore |K(Q_{2n})| = |K(Q_{2p^r})| = \left(\prod_{i=1}^{m_1} \frac{n_i(|N(C_i)|)}{\phi(|C_i|)}\right)^{1/2} . \left[\frac{2^{r+1}|N(C_4)|}{2}\right]^{\frac{1}{2}}$$

$$\therefore |K(Q_{2n})| = \left(\prod_{i=1}^{m_1} \frac{n_i(|N(C_i)|)}{\phi(|C_i|)}\right)^{1/2} . [|N(C_4)|]^{\frac{1}{2}} \qquad \ldots(3)$$

Also, by equation (2) and equation (3), we get;

$$4|K(Q_{2p^r})| = \left[\prod_{i=1}^{m_1} \frac{n_i(|N(C_i)|)}{\phi(|C_i|)}\right]^{1/2} . [|N(C_4)|]^{\frac{1}{2}}$$

So

$$|K(C_{2p^r})| = \left[\prod_{i=1}^{m_1} \frac{n_i(|N(C_i)|)}{\phi(|C_i|)}\right]^{1/2} ,$$

$[|N(C_4)|]^{\frac{1}{2}}$, it mean

$N(C_4) = 16$ …(4)

Hence

$$\prod_{i=1}^{m}\frac{(|N(C_i)|)^{1/2}}{(|C_i|)}=\prod_{i=1}^{m}\frac{(|N(C_i)|)^{1/2}}{(|C_i|)}\cdot\frac{(|N(C_4)|)^{1/2}}{|C_4|}$$

$$=\prod_{i=1}^{m_1}\frac{(|N(C_i)|)^{1/2}}{(|C_i|)}=\frac{4}{4} \qquad \ldots(5)$$

On the other hand

$$\therefore\prod_{i=1}^{m}\frac{(|N(C_i)|)^{1/2}}{(|C_i|)}=\prod_{i=1}^{m_1}\frac{(|N(C_i)|)^{1/2}}{(|C_i|)}\cdot\left(\frac{n_s}{\phi|C_4|}\right)$$

Since $n_s=2$, $\phi|C_i|=\phi(4)=2$,

So

$$\prod_{i=1}^{m}\frac{n_i}{\phi(|C_i|)}=\prod_{i=1}^{m_1}\frac{n_i}{\phi|C_i|} \qquad \ldots(6)$$

From (5) and (6) we get

$$\left[\prod_{i=1}^{m}\frac{n_i}{\phi(|C_i|)}\right]^{\frac{1}{2}}=\left[\prod_{i=1}^{m_1}\phi\frac{(|N(C_i)|)}{\varphi|C_i|)}\right]^{\frac{1}{2}}$$

Then by theorem (3) we have the rational valued characters of quaternion group Q_{2n}, where $n=p^r$ for all p is any prime number and r is a positive integer number, can be written as a Z-linear combination of the induced characters $1_{c_i}^{Q_{2n}}$, where C_i runs over the set of all subgroup of Q_{2n}.

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