On S_β – Dimension Theory

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Abstract

In this paper we introduce and define a new type of S_β − dimension theory and the concept of $indX, IndX$ *and* $dimX$, for a topological space X have been studied. In this work, these concepts will be extended by using $S_\beta - open$ sets.

Keywords: $S_\beta - open$, semi-open ,semi-pre-open (=βopen), $S_\beta - ind$, $S_\beta - Ind$, and $S_\beta - \dim$.

الخلاصة

في هذا البحث نقدم ونعرف نوع جديد حول نظرية البعد من النمط S_β لقد درسنا المفاهيم $indX$ و $IndX, dimX$. للفضاءات التبولوجية X و في هذا العمل سوف نوسع هذه المفاهيم بأستخدام المجموعات المفتوحة من النمط S_β

الكلمات المفتاحية: $S\beta$ المفتوحة، شبه مفتوحة، شبه نصف مفتوحة، β المفتوحة، $S_\beta - ind$ ، $S_\beta - Ind$ ، $S_\beta - \dim$

Introduction

Dimension theory starts with “dimension function” which is a function defined on the class of topological spaces such that $d(X)$ is an integer or∞, with the properties that $d(X) = d(Y)$ if X and Y are homeomorphic and $d(R^n) = n$ for each positive integer n. The dimension functions taking topological spaces to the set$\{-1,0,1,\dots\}$. The dimension functions $\text{ind}, \text{Ind}, \dim$, were investigated by [Pears ,1975]. Actually the dimension functions, $S - \text{ind}X, S - \text{Ind}X$ and $S - \dim X$ by using $S - \text{open}$ sets were studied in [Raad,1992], also the dimension functions, $b - \text{ind}X, b - \text{Ind}X$ and $b - \dim X$, by using $b - \text{open}$ sets were studied in [Sama Kadhim Gabar,2010], and the dimension functions, $f - \text{ind}X, f - \text{Ind}X$ and $f - \dim X$, by using $f - \text{open}$ sets were studied in [Nedaa, 2011]. In this chapter we recall the definitions of ind, Ind, dim, from [Pears ,1975], and the dimension functions, $N - \text{ind}, N - \text{Ind}, N - \dim$ are by using $N - \text{open}$ sets [Enas,2014] . Then the dimension functions, $S_\beta - \text{ind}$, $S_\beta - \text{Ind}$ and $S_\beta - \dim$ are introduced by using $S_\beta - \text{open}$ sets. Finally some relations between them are studied and some results relating to these concepts are proved.

1. Preliminaries

Definition (2.1): A subset A of a topological space (X,T) is called :

i. Semi-open (s − open) set in X if $A \subseteq \overline{A^\circ}$ and semi-closed (s−closed), if $\overline{A}^\circ \subseteq A$. **[Das,1973]**

ii. Pre-open if $A \subseteq \overline{A}^\circ$ and pre-closed set if $\overline{A^\circ} \subseteq A$. [**Mashhour *et al.*, 1982**]

iii. Semi-pre-open($= \beta - open$) if $A \subseteq \overline{\overline{A}^\circ}$ and semi-pre-closed ($= \beta - closed$) if $\overline{A^\circ}^\circ \subseteq A$. [**Shareef,2007**]

iv. Regular open if $A = \overline{A}^{\circ}$ and regular closed if $A = \overline{A^{\circ}}$. [**Steen, & Seebach, 1970**]

Definition (2.2)[Alis and Nehmat ,2012]: A semi open subset A of a topological space(X,T) is said to be $S_\beta - open$ if for each x $\in A$ there exists a β-closed set F such that $x \in F \subseteq A$. The complement of an $S_\beta - \text{open}$ set is said to be an $S_\beta - \text{closed}$ set The family of $S_\beta - \text{open}$ *sub*set of X is denoted by $S_\beta O(X)$.

Proposition (2.3)[Alis and Nehmat, 2012]: A subset A of a topological space (X,T) is $S_\beta - \text{open}$ set if and only if A is semi open and it is union of $\beta - closed$ sets.

Example(2.4): It is clear that every $S_\beta - \text{open}$ set is S-open set and every $S_\beta - \text{closed}$ set is S-closed, but the converse is not true in general, see the following example. Let $X = \{a,b,c\}$ and $T = \{\{a\},\emptyset,X\}$ the S-open sets are :$\{a\},\{a,b\},\{a,c\},\emptyset,X$ $\beta O(X) = \{\{b,c\},\{c\},\{b\},\emptyset,X\}$ and $S_\beta - \text{open}$ sets are $\emptyset, X$ thus every $S_\beta - \text{open}$ set is S-open but converse is not true since {a},{a,b},{a,c} is S-open but not $S_\beta - \text{open}$ set.

Corollary (2.5)[Alis and Nehmat, 2012]: Let $A \subseteq Y \subseteq X$, if $A \in S_\beta O(X)$ and Y is clopen subset of X, then $A \cap Y \in S_\beta O(Y)$.

Definition (2.6) [Alis and Nehmat, 2012]: Intersection of all $S_\beta - closed\ sets\ containing$ F is called the $S_\beta - closure$ of F and is clenoted by $\overline{F}^{S_\beta}$.

Theorem(2.7)[Alis and Nehmat, 2012]: For any subset F of a topological space X, the following statements are true .

1- $\overline{F}^{S_\beta}$ is the intersection of all $S_\beta - closed\ sets$ in X containing F.

2- $\overline{F}^{S_\beta}$ is the smallest $S_\beta - closed\ sets$ in x containing F.

3- F is smallest $S_\beta - closed$ if and only if $F = \overline{F}^{S_\beta}$.

Theorem (2.8)[Alis and Nehmat ,2012]: If F and E are any subsets of a topological space X , $If\ F \subseteq$ E ,then $\overline{F}^{S_\beta} \subseteq \overline{E}^{S_\beta}$.

Proposition (2.9)[Alis and Nehmat, 2012]: Let A be any subset of a topological space X. If a point x is in the S_β- interior of A, then there exists a semi-closed set F of X containing x such that F $\subseteq$ A.

Definition (2.10) [Alis and Nehmat, 2012]: A a subset A of a topological space X , is called S_β –boundary of A if $b_{S_\beta}(A) = \bar{A}^{S_\beta} \setminus A^{\circ S_\beta}$ and denoted by $b_{S_\beta}(A)$.

Remark(2.11) [Alis and Nehmat , 2012]: Let A be a subset of a topological space X, then $b_{S_\beta}(A) = \varphi$ if and only if A is both S_β-open and S_β- closed set.

Definition (2.12) [Alis and Nehmat, 2012]: Let A be a subset of a topological space X. A point x $\in$ X is said to be S_β-limit point of A if for each S_β-open set U containingx, $U \cap (A \setminus \{x\}) \neq \varphi$. The set of all Sβ-limit point of A is called S_β-derived set of A and is denoted by A'^{S_β} .

Theorem (2.13) [Alis and Nehmat, 2012]: Let A be a subset of a space X, then $\bar{A}^{S_\beta} = A \cup A'^{S_\beta}$.

Definition (2.14) : A space X is called $S_\beta - T_1\ space$ if and only if , for each x $\neq$ y in X , there exist $S_\beta - open$ sets U and V such that $x \in U, y \notin U$ and $y \in V, x \notin V$.

Proposition (2.15)[Alis and Nehmat, 2012]: If a space X is T_1-space, then S_βO(X) = SO(X).

Proposition (2.16) : Let X be a topological space .Then (X, T) is an $S_\beta - T_1$ *space* if and only if {x} is $S_\beta - closed$ set for each x ∈ X .

Proof : Let X is $S_\beta - T_1$ *space* Let $y \in X$ *such that* $y \notin \{x\}$ .since X is $S_\beta - T_1$ *space* then there exist an $S_\beta - open$ set U such that $y \in U, x \notin U$ .It is clear that (U-y) ∩ {x} = ∅ and hence y $\notin \{x\}'^{S_\beta}$ by definition(1.1.40) Then $\{x\}'^{S_\beta} \subseteq$ {x} for each x ∈ X ,and hence $\{\bar{x}\}'^{S_\beta}$ ={x} ∪ $\{x\}'^{S_\beta} = \{x\}$ by theorem (2.13) , so that {x} is $S_\beta - closed$ set for each x ∈ X by theorem (2.7).

Conversely :Assume that {x} is $S_\beta - closed$ set for each x ∈ X .Let $x \neq y$ in X , then $X - \{x\} = U$ is $S_\beta - open$ set contains y not x .Now $X - \{y\} = V$,hence Vis $S_\beta - open$ set which contains x but not y .Therefore X is $S_\beta - T_1$ *space.*

Definition (2.17): A space X is called S_β −regular space iff for each x in X and a closed set F such that $x \notin F$,there exist disjoint S_β-open sets U, V such that $x \in U, F \subseteq V$.

Definition (2.18): A space X is said to be $S_\beta{}^*$ −regular space if and only if ,for each x ∈ X and an S_β −closed set F such that x ∉ F ,there exist disjoint S_β −open sets and V in X such that $x \in U$ *and* $F \subseteq V$.

Proposition(2.19)[Pervin,1964]:A space X is regular space iff for every x ∈ X and each open set U in X such that x ∈ U there exist an open set w such that x ∈ W ⊆ $\bar{W}$ ⊆ U .

Proposition(2.20): A space X is S_β −regular space iff for every x ∈ X and each open set U in X such that x ∈ U there exist an open set W such that $x \in W \subseteq \bar{W}^{S_\beta} \subseteq U$.

Proof: Let X be an S_β − regular space and x ∈ X , U is open set in X such that x ∈ U thus U^c is closed set in x and x ∉ U^c then there exist disjoint S_β −open sets W,V such that x ∈ W , $U^c \subseteq V$.. Hence $x \in W \subseteq \bar{W}^{S_\beta} \subseteq \overline{V^c}^{S_\beta} = V^c \subseteq U$.By theorem(2.8)we have $\bar{W}^{S_\beta} \subseteq \overline{V^c}^{S_\beta}$,and by theorem (2.7) $\overline{V^c}^{S_\beta} = V^c$

Conversely: Let x ∈ X and F be closed set in X such that x ∉ F ,then F^c is open set and $x \in F^c$,thus there exist an S_β −open set w such that $x \in W \subseteq \bar{W}^{S_\beta} \subseteq F^c$ then $x \in W, F \subseteq \bar{W}^{cS_\beta}$ and W , $\bar{W}^{cS_\beta}$ sre disjoint S_β −open sets Hence X is S_β −regular space.

Proposition(2.21): A topological space X is $S_\beta{}^*$ −regular topological space if and only if, for every x ∈ X and each S_β −open set U in X such that x ∈ U then there exists an S_β −open set W such that $x \in W \subseteq \bar{W}^{S_\beta} \subseteq U$.

.**Proof:** Assume that X is $S_\beta{}^*$ −regular topological space and let x ∈ X is an S_β −open in X such that x ∈ U.then U^c is an S_β −closed in X , $x \notin U^c$.since X is $S_\beta{}^*$ −regular topological space then there exist disjoint S_β −open sets W and V such that $x \in W$ and $U^c \subseteq V$.then by proposition (2.20) $x \in W \subseteq \overline{W}^{S_\beta} \subseteq \overline{V^c}^{S_\beta} = V^c \subseteq U$.

Conversely: Let x ∈ X and C be S_β −closed set such that $x \in C^c$. thus there exists an S_β −open set W such that $x \in W \subseteq \overline{W}^{S_\beta} \subseteq C^c$ .Then x ∈ W , $C \subseteq (\overline{W}^{S_\beta})^c$ and $(\overline{W}^{S_\beta})^c$ is an S_β −open set,$W \cap (\overline{W}^{S_\beta})^c \neq \emptyset$.Hence X is $S_\beta{}^*$ −regular topological space .

Definition (2.22): A space X is said to be $S_\beta - \text{normal}$ space if and only if for every disjoint S_β- closed sets F_1, F_2 there exist disjoint $S_\beta - \text{open}$ subsets V_1, V_2 such that $F_1 \subset V_1, F_2 \subset V_2$.

Definition (2.23): A space X is said to be $S_\beta{}^* - \text{normal}$ space if and only if for every disjoint closed sets F_1, F_2 there exist disjoint $S_\beta - \text{open}\ subsets$ V_1, V_2 such that $F_1 \subset V_1, F_2 \subset V_2$.

Example(2.24): This example show that normal space is not $S_\beta - \text{normal}$ space in general .Let $x = \{a, b, c, d\}$, $T = \{X, \emptyset, \{a, b\}, \{a, b, c\}\}$ $S_\beta O(X) = \{\emptyset, X, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$, $S_\beta C(X) = \{\emptyset, X, \{c, d\}, \{d\}, \{c\}\}$.

It is clear that X is normal space and $S_\beta{}^* - \text{normal}$ space since there exists no disjoint closed sets . but X is not $S_\beta - normal$ space since $\{d\}, \{c\}$ is disjoint $S_\beta - closed$ sets cannot be separated by $S_\beta - open$ in X .

Proposition (2.25): A topological space X is $S_\beta - \text{normal}$ topological space if and only if for every $S_\beta - \text{closed}$ set $F \subseteq X$ and each $S_\beta - \text{open}$ set U in X such that $F \subseteq U$ then there exists an $S_\beta - \text{open}$ set W such that $F \subset W \subset \overline{W}^{S_\beta} \subset U$.

Proof: Let F be an $S_\beta - \text{closed}$ set and U $S_\beta - \text{open}$ set such that $F \subseteq U$. It's clear that F, U^C are disjoint $S_\beta - \text{closed}$ set in X. Thus since X is $S_\beta - \text{normal}$ topological space then there exist disjoint $S_\beta - \text{open}$ sets W, V such that $F \subseteq W, U^C \subseteq V$, by theorem(2.8)we have $\overline{W}^{S_\beta} \subseteq \overline{V^c}^{S_\beta}$,and by theorem (2.7) $\overline{V^c}^{S_\beta} = V^c$ then $F \subseteq W \subseteq \overline{W}^{S_\beta} \subseteq \overline{V^C}^{S_\beta} = V^C \subseteq U$.

Conversely

Let F_1, F_2 be disjoint $S_\beta - \text{closed}$ sets in X then $F_2{}^C$ is an $S_\beta - \text{open}$ set, $F_1 \subseteq F_2{}^C$. Thus there exists an $S_\beta - \text{open}$ set W such that $F_1 \subseteq W \subseteq \overline{W}^{S_\beta} \subseteq F_2{}^C$. Then $F_1 \subseteq W, F \subseteq \left(\overline{W}^{S_\beta}\right)^C$, $W, \overline{W}^{S_\beta}$ are disjoint $S_\beta - \text{open}$ sets. Hence X is $S_\beta - \text{normal}$ topological space.

Proposition (2.26): A topological space X is $S_\beta{}^* - \text{normal}$ if and only if for every closed set $F \subseteq X$ and each $S_\beta - \text{open}$ set U in X such that $F \subseteq U$ then there exists an $S_\beta - \text{open}$ set W such that $F \subset W \subset \overline{W}^{S_\beta} \subset U$.

Proof: Let F be an closed set and U open set such that $F \subseteq U$. It's clear that F, U^C are disjoint closed set in X. Thus since X is $S_\beta{}^* - \text{normal}$ topological space then there exist disjoint $S_\beta - \text{open}$ sets W, V such that $F \subseteq W, U^C \subseteq V$,by theorem(2.8) we have $\overline{W}^{S_\beta} \subseteq \overline{V^c}^{S_\beta}$,and by theorem (2.7) $\overline{V^c}^{S_\beta} = V^c$ then $F \subseteq W \subseteq \overline{W}^{S_\beta} \subseteq \overline{V^C}^{S_\beta} = V^C \subseteq U$.

Conversely: Let F_1, F_2 be disjoint closed sets in X then ${F_2}^C$ is an open set, $F_1 \subseteq {F_2}^C$. Thus there exists an $S_\beta -$ open set W such that $F_1 \subseteq W \subseteq \overline{W}^{S_\beta} \subseteq {F_2}^C$. Then $F_1 \subseteq W, F \subseteq \left(\overline{W}^{S_\beta}\right)^C$, $W, \overline{W}^{S_\beta}$ are disjoint $S_\beta -$ open sets. Hence X is ${S_\beta}^* -$ normal topological space.

2. On small S_β − Inductive Dimension Function

Definition (3.1)[Raad,1992]: Let X be a topological space, we say that $S-\text{ind}X = -1$ iff $X = \emptyset$ and if n is a positive integer or 0 , then we say that $S\text{ind}X \leq n$ iff the following satisfied:

for each $x \in X$ and for each open set G containing x , there exists an S −open set V such that $x \in V \subseteq G$ and $S-\text{ind}b(V) \leq n-1$, there exists no integer n for which $\text{ind}X \leq n$, then we put $S-\text{ind}X = \infty$.

In similar way , we introduce the following :

Definition (3.2): The S_β_small inductive dimension of a space X, S_β_$ind\ X$,is defined inductively as follows . topological space X , $S_\beta - \text{ind}X = -1$,and only if, X is empty. If n is a non-negative integer, then S_β_$indX \leq n$ means that for each point $x \in X$ and each open set G such that $x \in G$ there exists an $S_\beta -$ open set U such that $x \in U \subseteq G$ and $S_\beta - \text{ind}b_{S_\beta}(U) \leq n-1$. We put $S_\beta\text{ind}X = n$ if it is true that $S_\beta - \text{ind}X \leq n$, but it is not true that S_β_$\text{ind}X \leq n-1$. If there exists no integer n for which $S_\beta\text{ind}X \leq n$, then we put $S_\beta - \text{ind}X = \infty$.

Definition (3.3): The S_β^*_small inductive dimension of a space X, $S_\beta^* - \text{ind}X$, is defined inductively as follows .

A space X satisfies $S_\beta^* - \text{ind}X = -1$ if and only if , $X = \emptyset$. If n is a non-negative integer, then $S_\beta^* - \text{ind}X \leq n$ means that for each point $x \in X$ and each S_β −open set G such that $x \in G$ there exists an S_β −open set U such that $x \in U \subseteq G$ and $S_\beta^* - \text{ind}b_{S_\beta}(U) \leq n-1$. We put $S_\beta^* - \text{ind}X = n$ if it is true that $S_\beta^* - \text{ind}X \leq n$, but it is not true that $S_\beta^* - \text{ind}X \leq n1$. If there exists no integer n for which $S_\beta^*\text{ind}X \leq n$, then we put $S_\beta^* - \text{ind}X = \infty$.

Theorem (3.4): Let X be a topological space, if $S_\beta - \text{ind}X = 0$ then X is S_β −regular space.

Proof: Let $x \in X$ and G an open set such that $x \in G$, since $S_\beta - \text{ind}X = 0$, then there exists an $S_\beta -$ open set V such that $x \in V \subseteq G$ and $S_\beta - \text{ind}b_{S_\beta}(V) = -1$. Thus $b_{S_\beta}(V) = \emptyset$ by Remark (2.11) , hence V is an S_β −open and S_β −closed set . Therefore $x \in V \subseteq \overline{V}^{S_\beta} \subseteq G$ by proposition (2.20) , hence X is S_β −regular space.

Theorem (3.5): Let X be a topological space, if $S_\beta^* - \text{ind}X = 0$ then X is S_β^* −regular space.

Proof: Let $x \in X$ and G an$S_\beta -$ open set such that $x \in G$, since $S^*_\beta - \text{ind}X = 0$, then there exists an $S_\beta -$ open set V such that $x \in V \subseteq G$ and $S^*_\beta - \text{ind}b_{S_\beta}(V) = -1$. Thus $b_{S_\beta}(V) = \emptyset$ by Remark (2.11) , hence V is an $S_\beta -$open and $S_\beta -$closed set . Therefore $x \in V \subseteq \bar{V}^{S_\beta} \subseteq G$ by proposition (2.21) , hence X is $S^*_\beta -$regular space.

Proposition (3.6): Let X be a topological space, if $S_\beta - \text{ind}X$ is exists then $S - \text{ind}X \leq S_\beta - \text{ind}X$.

Proof: By induction on n . If $n = -1$, then $S_\beta - \text{ind}X = -1$ and $X = \emptyset$, so that $S - \text{ind}X = -1$. Suppose the statements is true for $n - 1$.

Now, suppose that $S_\beta - \text{ind}X \leq n$, to prove $S - \text{ind}X \leq n$, let $x \in X$ and G is an open set in X such that $x \in G$ since $S_\beta - \text{ind}X \leq n$, then there exist an S_β-open set V in X such that $x \in V \subseteq G$ and S_β-ind$b_{S_\beta}(V) \leq n$-1 and since each S_β-open set is S-open set . Then V is an S-open set such that $x \in V \subseteq G$ and S-ind$b(V) \leq n$-1 . Hence S-ind$X \leq n$.

Theorem (3.7): Let X be a topological space, then S_β-ind$X = 0$ if and only if S-ind$X = 0$.

Proof: By proposition (3.6) if S_β-ind$X = 0$, then S-ind$X \leq 0$ and since $X \neq \emptyset$ then S-ind$X = 0$. Now let S-ind$X = 0$, let $x \in X$ and G an open set in X such that $x \in G$, since S-ind$X = 0$ then there exists an S-open set V such that $x \in V \subseteq G$ and S-ind$b(V) \leq$ -1 , then $b(V) = \emptyset$,hence V is both open and closed set . Since every open (closed) set is S-open an (S-closed) set , therefore $x \in V \subseteq G$ Hence V is an S_β-open and S_β-closed set and S_β-ind$b_{S_\beta}(V) \leq$ -1. So that S_β-ind$X \leq 0$ and since $X \neq \emptyset$, then S_β-ind$X = 0$.

The following example a space X with S_β-ind$X = S$-ind$X = 0$.

Example (3.8): Let $X = \{a, b, c\}$ and $T = \{\emptyset, X\}$ be a topology on X , then $S_\beta O(X) = \{\emptyset, X\}$, let $x \in X$ and only S_β-open set is X ,then $x \in X \subseteq X$ and $b_{S_\beta}(X) = \emptyset$, so S_β-ind $b_{S_\beta}(X) =$ -1. Then S_β-ind$X \leq 0$, since $X \neq \emptyset$ thus S_β-ind$X \neq$ -1, hence S_β-ind$X = 0$ and since S_β-open set is S-open , then by theorem (3.7) , we have S_β-ind$X = S$-ind$X = 0$.

4. On Large $S_\beta -$ Inductive Dimension Function

Definition (4.1)[Raad, 1992]: Let X be a topological space . It is said that S-Ind$X =$ -1 if and only if X is empty. If n is a positive integer or 0 , then we say that S-Ind$X \leq n$ if and only if the following is satisfied:

for each closed set F and each open set G of X such that $F \subset G$ there exists an S-open set U such that $F \subset U \subset G$ and S-Ind$b_S(U) \leq n$-1. We put S-Ind$X = n$ if it is true that S-Ind$X \leq n$, but it is not true that S-Ind$X \leq n$-1. If there exists no integer n for which S-Ind$X \leq n$ then we put S-Ind$X = \infty$.

Definition (4.2) : The S_β-large inductive dimension of a space$X$,S_β-Ind$X$, is defined inductively as follows. A space X satisfies S_β-Ind$X =$ -1 if and only if X is empty. If n is a non-negative integer, then S_β-Ind$X \leq n$ means that for each closed set F and each open set G of X such that $F \subset G$ there exists an S_β-open set U such that $F \subset U \subset G$ and S_β-Ind$b_{S_\beta}(U) \leq n$-1. We put S_β-Ind$X = n$ if it is true that S_β-Ind$X \leq n$, but it is not true that S_β-Ind$X \leq n - 1$. If there exists no integer n for which S_β-Ind$X \leq n$ then we put S_β-Ind$X = \infty$.

Definition (4.3): The $S_\beta{}^*$-large inductive dimension of a space X,$S_\beta{}^*$-Ind$X$, is defined inductively as follows: A space X satisfies $S_\beta{}^*$-Ind$X$ = -1 if and only if X is empty. If n is a non-negative integer , then $S_\beta{}^*$-Ind$X \leq n$ means that for each S_β-closed set F and each S_β-open set G of X such that $F \subset G$ there exists S_β-open set U such that $F \subset U \subset G$ and $S_\beta{}^*$-Ind$b_{S_\beta}(U) \leq n$-1.We put $S_\beta{}^*$-Ind$X = n$ if it is true that $S_\beta{}^*$-Ind$X \leq n$, but it is not true that $S_\beta{}^*$-Ind$X \leq n$-1. If there exists no integer n for which $S_\beta{}^*$-Ind$X \leq n$ then we put $S_\beta{}^*$-Ind$X = \infty$.

Proposition (4.4): Let X be a topological space, if S_β-Ind$X = 0$, then X is $S_\beta{}^*-$normal.

Proof: Let F be a closed set in X and U is an open set such that $F \subseteq U$. Since S_β-Ind$X = 0$, then there exist S_β-open W such that S_β-Ind$b_{S_\beta}(W)$ = -1 , hence W is S_β-open and S_β-closed set therefore $F \subseteq W \subseteq \overline{W}^{S_\beta} \subseteq U$ by proposition (2.26) X is $S_\beta{}^*-$normal.

Proposition (4.5): Let X be a topological space, if $S_\beta{}^*$-Ind$X = 0$, then X is $S_\beta -$normal space.

Proof: Let F be an $S_\beta -$ closed set in X and U is an $S_\beta -$ open set such that $F \subseteq U$ .Since S_β-Ind$X = 0$, then there exist S_β-open W such that S_β-Ind$b_{S_\beta}(W)$ = -1 , hence W is S_β-open and S_β-closed set therefore $F \subseteq W \subseteq \overline{W}^{S_\beta} \subseteq U$ by proposition (2.25) X is $S_\beta -$normal space.

Proposition (4.6): Let X be a topological space. $S_\beta -$ Ind$X$ is exists , then S-Ind$X \leq S_\beta$-Ind$X$.

Proof: By induction on n . It is clear that $n = -1$.

Suppose that it is true for n-1. Now , suppose that S_β-Ind$X \leq n$, to prove that S-Ind$X \leq n$, let F be a closed set in X and G is an open set in X such that $F \subseteq G$, since S_β-Ind$X \leq n$, then there is S_β-open set U in X such that $F \subseteq U \subseteq G$ and S_β-Ind$b_{S_\beta}(U) \leq n - 1$, since each S_β-open set is S-open set . Then U is S-open set such that $F \subseteq U \subseteq G$ and S-Ind$b(U) \leq n - 1$. Hence S-Ind$X \leq n$.

Theorem (4.7) Let X be a topological space, then S_β-Ind$X = 0$ iff S-Ind$X = 0$.

Proof: By proposition (4.6*)* If $S_\beta -$ Ind$X = 0$, then S-Ind$X \leq 0$, and since $X \neq \emptyset$, then S-Ind$X = 0$.

Now,

Let S-Ind$X = 0$ and Let F is closed set in X and each open set G in X such that $F \subseteq G$.Since S-Ind$X = 0$ then there exists an S-open set U in X such that $F \subseteq U \subseteq G$ and S-Ind$b(U) \leq$ -1 . Then $b(U) = \emptyset$, therefore U is both open and closed set , since each open and closed set is S-open and S-closed set . Thus $F \subseteq U \subseteq G$, then U is S_β-open set and S_β-Ind$X \leq 0$ and since $X \neq \emptyset$,then S_β-Ind$X = 0$.

5. On S_β − Covering Dimension Function

Definition (5.1): The S_β-covering dimension (S_β-dim$X$) of a topological X is the least integer n such that every finite S_β-open covering of X has S_β-open refinement of order not exceeding n or ∞ is if there is no such integer. Thus S_β-dim$X$ = -1 if and only if X is empty , and S_β-dim $X \leq n$ if each finite S_β-open covering of X has S_β- open refinement of order not exceeding n that $S_\beta - dim\ X \leq n-1$. Finally S_β-dim $X = \infty$ if for every integer n it is false that $S_\beta - dim\ X \leq n$.

Definition (5.2): The S_β^*-covering dimension (S_β^*-dim$X$) of a topological space X is the least integer n such that every finite open covering of X has S_β-open refinement of order not exceeding n or ∞ if there is no such integer. Thus S_β^*-dim$X$ = -1 if and only if X is empty , and S_β^*-dim$X \leq n$ if each finite open covering of X has S_β-open refinement of order not exceeding n . We have S_β^*-dim$X = n$ if it is true that S_β^*-dim$X \leq n$, but it is not true that S_β^*-dim$X \leq n-1$. Finally S_β^*-dim$X = \infty$ if for every integer n it is false that S_β^*-dim$X \leq n$.

Definition (5.3) [Raad, 1992]: Let X be a topological space then S-dim$X$ = -1 if and only if $X = \emptyset$ and if n is a positive integer or 0 , then we say that S-dim$X \leq n$ if and only if every finite S-open cover of X has S-open refinement of order $\leq n$.

Remark (5.4) [Raad, 1992]: Since each open set is S-open, then it follows that S-dim$X \leq$ dim$X$.

Remark (5.5): Since each S_β-open set is S-open, then it follows that S-dim$X \leq S_\beta$-dim$X$.

Definition (5.6): Let (X,T) be a topological space , the family $\mathbb{C}$ of S_β-open sets is called $S.B$ if and only if each S_β-open set in X is a union of members of β .

Theorem (5.7): Let X be a topological space. If X has a $S.B$ of sets which are both S_β-open and S_β-closed , then S_β^*-dim$X = 0$ for a T_1-space , the converse is true.

Proof: By using proposition (2.15) if a space X is a T_1-space , then $S_\beta O(X) = SO(X)$, the proof see[**Raad, 1992**].

Theorem (5.8): Let X be a topological space. If X has a $S.B$ of sets which are both S_β-open and S_β-closed , then S_β-dim$X = 0$ for an S_β -T_1space , the converse is true .

Proof: Suppose that X has S.B of sets which are both S_β-open and S_β-closed.

Let $\{U_i\}_{i=1}^k$ be a finite S_β-open cover of X, it has an S_β-open refinement $\mathcal{W}$, if $w \in \mathcal{W}$, then $w \subset U_i$ for some i. Let each $w \in \mathcal{W}$ be associated with one of the U_i sets containing it and let U_i be the union of these members of $\mathcal{W}$, thus associated with U_i . Thus V_i is S_β-open set and hence $\{V_i\}_{i=1}^{k}$ forms disjoint S_β-open refinement of $\{U_i\}_{i=1}^{k}$, then S_β-dim$X = 0$

Conversely:

Suppose that X is S_β -T_1space such that S_β-dim$X = 0$ .Let $x \in X$ and G be a S_β-open set in X such that $x \in G$. Then $\{x\}$ is S_β-closed set by using proposition (2.16) and $\{G , X\text{-}\{x\}\}$ is finite S_β-open of X. Since S_β-dim$X = 0$, then there exists S_β-open refinement $\{V ,W \}$ of order 0 such that $V \cap W = \emptyset$, $V \cup W = X$, $V \subset G$ and $W \subset X\text{-}\{x\}$. Then V is S_β-open and S_β-closed set in X such that $x \in W^c \subseteq V \subseteq G$ and hence X has a S.B of S_β-open and S_β-closed .

Remark (5.9) [Pears, 1975]: Let X be a topological space with dim $X = 0$. Then X is normal space.

Theorem (5.10): Let X be a topological space. If S_β^*-dim $X = 0$, then X is S_β^*-normal space .

Proof: Let F_1 and F_2 are disjoint closed sets of X . Then $\{X\text{-}F_1 , X\text{-} F_2\}$ is open cover of X. Since S_β^*-dim $X = 0$, then it is has S_β-open refinement of order 0, hence So that S_β-open sets H and G such that $H \cap G = \emptyset$, $H \cup G = X$, therefore $H \subset X\text{-}F_1$ and $G \subset X\text{-}F_2$. Thus $F_1 \subset H^c = G$, $F_2 \subset G^c = H$ and since $H \cap G = \emptyset$, then is X is S_β^*-normal space .

Theorem (5.11): Let X be a topological space. If S_β-dim $X = 0$, then X is S_β-normal space.

Proof: Let F_1 and F_2 are disjoint S_β-closed sets of X. Then $\{X\text{-}F_1 , X\text{-} F_2\}$ is S_β-open covering of X. Since S_β-dim $X = 0$, then it is has S_β-open refinement of order 0, hence there exists S_β-open sets H and G such that $H \cap G = \emptyset$, $H \cup G = X$, so that $H \subset X\text{-}F_1$ and $G \subset X\text{-}F_2$. Thus $F_1 \subset H^c = G$, $F_2 \subset G^c = H$ and since H $\cap G = \emptyset$, then is X is S_β-normal space.

Proposition(5.12) [Pears,1975]: Let X be a topological space and A closed subset of X then dim $A \leq$ dim X.

Theorem (5.13): If A clopen subset of a topological space X , then S_β^*-dim $A \leq S_\beta^*$-dim X.

Proof: Suppose that S_β^*-dim $X \leq n$, let $\{U_1, U_2, \dots , U_k\}$ be an open cover of A , then for each i , $U_i = A \cap V_i$, where V_i is an open set in X. The finite open covering $\{ V_1 , V_2, \dots , V_k , X\text{-}A \}$ of X has S_β-open refinement $\mathcal{W}$ in X of order $\leq n$. Let $\mathcal{V} = \{ W \cap A : W \in \mathcal{W} \}$ by corollary (2.5) , then $\mathcal{V}$ is S_β-open refinement of $\{U_1, U_2, \dots , U_k\}$ of order $\leq n$. Thus S_β^*-dim $A \leq n$.

6. Relation between the dimensions S_β-ind and S_β-Ind

Proposition (6.1)[Pears, 1975]: Let X be a topological space, if X is a regular space then indX $\leq$ IndX.

Theorem (6.2): Let X be a topological space, if X is regular, then S_β-ind$X \leq S_\beta$-Ind$X$.

Proof: By induction on n , if $n = -1$, then the statement is true.
Suppose that the statement is true for $n - 1$.
Now,
Suppose that S_β-Ind$X \le n$, to prove S_β-ind$X \le n$. Let $x \in X$ and G be an open set such that $x \in G$ since X is regular space then there exists an open set U such that $x \in U \subseteq \bar{U} \subseteq G$ by proposition (2.19) .
Also since S_β-Ind$X \le n$ and $\bar{U}$ is closed , $\bar{U} \subseteq G$ then there exists an S_β-open set V such that $\bar{U} \subseteq V \subseteq G$ and S_β-Ind$b_{S_\beta}(V) \le n - 1$, then S_β-ind$b_{S_\beta}(V) \le n - 1$ [by induction] and S_β-ind$X \le n$, then S_β-ind$X \le S_\beta$-Ind$X$.

Proposition (6.3): Let X be a topological space , if X $S_\beta - T_1$ *space* ,then $S_\beta{}^*$-ind$X \le S_\beta{}^*$-Ind$X$.

Proof: By induction on n , if $n = -1$, then the statement is true.
Suppose that the statement is true for $n - 1$.
Now,
Suppose that $S_\beta{}^*$-Ind$X \le n$, to prove $S_\beta{}^*$-ind$X \le n$. Let $x \in X$ and each S_β −open set $G \subset X$ of the point x , since X is S_β-T_1space, then$\{x\}$ $\subseteq$G such that $\{x\}$ is S_β-closed set by proposition (2.16) . Since $S_\beta{}^*$-Ind$X \le n$, then there exists an S_β-open set V in X such that $\{x\} \subseteq V \subseteq G$ and $S_\beta{}^*$-Ind$b_{S_\beta}(V) \le n - 1$. Hence $S_\beta{}^*$-ind$b_{S_\beta}(V) \le n - 1$ and $x \in V \subseteq G$.
Thus $S_\beta{}^*$-ind$X \le n$.

Proposition (6.4): Let X be a topological space, if X is $S_\beta{}^*$-regular space, then $S_\beta{}^*$-ind$X \le S_\beta{}^*$-Ind$X$.

Proof: By induction on n , if $n = -1$, then the statement is true.
Suppose that the statement is true for $n - 1$.
Now, Suppose that $S_\beta{}^*$-Ind$X \le n$, to prove $S_\beta{}^*$-ind$X \le n$. Let $x \in X$ and each S_β −open set $G \subset X$ of the point x , since X is $S_\beta{}^*$-regular space, then there exists an S_β-open set V in X such that $x \in V \subseteq \bar{V}^{S_\beta} \subseteq G$ .Also since $S_\beta{}^*$-Ind$X \le n$ and $\bar{V}^{S_\beta}$ is S_β-closed set $\bar{V}^{S_\beta} \subseteq G$, then there exists an S_β-open set U in X such that $\bar{V}^{S_\beta} \subseteq U \subseteq G$ and $S_\beta{}^*$-Ind$b_{S_\beta}(U) \le n - 1$.
Hence $S_\beta{}^*$-ind$b_{S_\beta}(U) \le n - 1$ and $x \in U \subseteq G$ (by induction) ,thus $S_\beta{}^*$-ind$X \le n$.

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